数分第七次习题课

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一、求不定积分

(1) 三角/双曲三角换元

$$\begin{cases} \sec^2 t = \tan^2 t + 1 & (\tan t)' = \sec t \\ & (\sec t)' = \frac{\sin t}{\cos^2 t} = \tan t \sec t \\ \cosh t = \frac{e^t + e^{-t}}{2} \geq 1 & \Rightarrow (\cosh t)' = \sinh t \\ \sinh t = \frac{e^t - e^{-t}}{2} \in \mathbb{R} & (\sinh t)' = \cosh t \end{cases}$$

$\cosh(2t) = 2\cosh^2 t - 1$ $\quad (\cosh t)^2 - (\sinh t)^2 = 1$

$\sinh(2t) = 2\sinh t \cosh t$ $\quad x = \cosh t \Rightarrow t = \ln(x \pm \sqrt{x^2-1})$

$+: t>0 \quad -: t<0$

注意：反三角函数的值域

$\arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad \arccos x \in [0, \pi]$ $\quad x = \sinh t \Rightarrow t = \ln(x + \sqrt{x^2+1})$

$\arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

一般尽量用基础的反三角，少用 arccot、arcsec、arccsc

也最好别写 arcsinh、arccosh

1. $\int \sqrt{x^2+a^2}\,dx$ （这种题一般默认 $a>0$，如果纠结符号可以自行调整）

"$\sqrt{1+x^2}$ 型" $\longrightarrow$ 利用 $\tan^2 t + 1 = \sec^2 t$ 故设 $x = a\tan t$

换元原则：① 所有 x 都取到 $\longrightarrow$ 区间长度为 π

② 开根号是正的：$\sqrt{x^2+a^2} = a\sec t > 0 \Rightarrow \cos t > 0$

故令 $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\int \sqrt{x^2+a^2}\,dx = \int a\sec t \, d(a\tan t) = a^2 \int \sec^3 t \, dt$$

$$\overset{见后}{=} \frac{a^2}{2}\left(\tan t \sec t + \ln|\sec t + \tan t|\right) + C$$

$$= \frac{a^2}{2}\left(\frac{x}{a^2}\sqrt{x^2+a^2} + \ln\left|\frac{1}{a}\sqrt{x^2+a^2} + \frac{x}{a}\right|\right) + C$$

$$= \frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\ln|x + \sqrt{x^2+a^2}|\right) + C$$

or $\quad x = a\sinh t \quad \sqrt{x^2+a^2} = a\cosh t \quad (t\in\mathbb{R})$

原式 $=\int a\cosh t\, d(a\sinh t) = \int a^2\cosh^2 t\,dt$

$$=\int a^2 \frac{1+\cosh 2t}{2}\, dt$$

$$= \frac{a^2 t}{2} + \frac{a^2}{4}\sinh(2t)+C$$

$$=\frac{a^2 t}{2} + \frac{a^2}{2}\cosh t\sinh t + C$$

$$=\frac{a^2}{2}\ln\left(\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right) + \frac{a^2}{2}\frac{x}{a}\frac{1}{a}\sqrt{x^2+a^2}+C$$

$$=\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + \frac{x}{2}\sqrt{x^2+a^2}+C.$$

2. $\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$

"$\sqrt{1-x^2}$型" → 利用 $1-\cos^2 t = \sin^2 t$

令 $x=a\cos t$, $\quad t\in(0,\pi) \quad \sin t = \sqrt{1-\frac{x^2}{a^2}} = \frac{1}{a}\sqrt{a^2-x^2}$

原式 $=\int \frac{a^2\cos^2 t}{a\sin t}\, d(a\cos t) = \int -a^2\cos^2 t\,dt$

$$= -\int a^2\frac{1+\cos 2t}{2}\,dt = -\frac{a^2 t}{2} - \frac{a^2}{4}\sin 2t + C$$

$$= -\frac{a^2}{2}\arccos\frac{x}{a} - \frac{a^2}{2}\cdot\frac{1}{a}\sqrt{a^2-x^2}\,\frac{x}{a}+C$$

$$= -\frac{a^2}{2}\arccos\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2}+C$$

(2) 分部积分

3. $\int \sec^3 x\,dx = \int \sec x\, d\tan x$

$$= \tan x\sec x - \int \tan^2 x\sec x\,dx$$

$$= \tan x\sec x - \int \sec^3 x - \sec x\,dx$$

$$= \tan x\sec x + \ln|\sec x+\tan x| - \int \sec^3 x\,dx.$$

$$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\tan x\sec x + \frac{1}{2}\ln|\sec x+\tan x| + C.$$

4. $\int x(\arctan x)^2 dx$

$= \int \frac{1}{2}(\arctan x)^2 dx^2 = \frac{1}{2}x^2(\arctan x)^2 - \int \frac{x^2\arctan x}{1+x^2}dx$

$= \frac{1}{2}x^2(\arctan x)^2 - \int \arctan x\,dx + \int \frac{1}{1+x^2}\arctan x\,dx$

$= \frac{1}{2}(x^2+1)(\arctan x)^2 - x\arctan x + \int \frac{x}{1+x^2}dx$

$= \frac{1}{2}(x^2+1)(\arctan x)^2 - x\arctan x + \frac{1}{2}\ln(1+x^2) + C$

5. $\int \frac{x+\sin x}{1+\cos x}dx$

$= -\ln(1+\cos x) + \int \frac{x}{1+\cos x}dx$

$\int \frac{x}{1+\cos x}dx = \int \frac{x}{2\cos^2\frac{x}{2}}dx \overset{u=\frac{x}{2}}{=} 2\int \frac{u}{\cos^2 u}du = 2\int u\,d\tan u$

$= 2u\tan u - 2\int \tan u\,du$

$= 2u\tan u + 2\ln|\cos u| + C = x\tan\frac{x}{2} + 2\ln|\cos\frac{x}{2}| + C$

$\Rightarrow$ 原式 $= -\ln(1+\cos x) + x\tan\frac{x}{2} + 2\ln|\cos\frac{x}{2}| + C$

$= x\tan\frac{x}{2} + C$

6 $\int x^n e^x dx \triangleq I_n$ $\qquad I_0 = \int e^x dx = e^x + C$

$I_n = \int x^n e^x dx = \int x^n de^x = x^n e^x - n\int e^x x^{n-1}dx = x^n e^x - nI_{n-1}$

$\Rightarrow \frac{I_n}{n!} = \frac{x^n e^x}{n!} - \frac{I_{n-1}}{(n-1)!}$

$\Rightarrow (-1)^n\frac{I_n}{n!} = \frac{(-1)^n x^n e^x}{n!} + (-1)^{n-1}\frac{I_{n-1}}{(n-1)!}$

$\Rightarrow (-1)^n\frac{I_n}{n!} = e^x + \sum_{k=1}^{n}(-1)^k\frac{x\,e}{k!} = \sum_{k=0}^{n}(-1)^k\frac{x^k e^x}{k!}$

$\Rightarrow I_n = \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}x^k e^x + C$

7 $\int \ln(x+\sqrt{x^2+1})\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \int x\frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}dx$

$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C$

(3) "对称性"

8 $\int \frac{1}{1+\tan x}dx$

$= \int \frac{\cos x}{\sin x+\cos x}dx \triangleq I \qquad J = \int \frac{\sin x}{\sin x+\cos x}dx$

$\Rightarrow I+J = x \qquad I-J = \int \frac{\cos x-\sin x}{\cos x+\sin x}dx = \ln|\cos x+\sin x| + C$

$\Rightarrow I = \frac{I+J}{2} + \frac{I-J}{2} = \frac{x}{2} + \frac{1}{2}\ln|\cos x+\sin x| + C$

9 $\int \frac{1}{1+x^4}dx$

$I = \int \frac{1-x^2}{1+x^4}dx \qquad J = \int \frac{1+x^2}{1+x^4}dx$

$= \int \frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}dx \qquad = \int \frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}dx$

$= -\int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} \qquad = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} + C$

$= -\int \frac{du}{(u-\sqrt{2})(u+\sqrt{2})} = -\frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|$

$= -\frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|$

故原式 $= \frac{I+J}{2} = \frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$

(4) 有理函数积分

10. Chebyshev型积分: $\int x^{\alpha}(a+bx^{\beta})^{\gamma}dx \quad (a,b\in\mathbb{R},\ b\neq 0,\ \alpha,\beta,\gamma\in\mathbb{Q})$

定理 上述积分有初等解 $\Leftrightarrow$ γ, $\frac{\alpha+1}{\beta}$, $\frac{\alpha+1}{\beta}+\gamma$ 中有一个整数

$$\int x^{\alpha}(a+bx^{\beta})^{\gamma}dx \xlongequal{t=x^{\beta}} \frac{1}{\beta}\int t^{\frac{\alpha+1}{\beta}-1}(a+bt)^{\gamma}dt$$

Case 1. $\gamma\in\mathbb{Z}$ 设 $\frac{\alpha+1}{\beta}-1=\frac{p}{q}\in\mathbb{Q}$ 则

原式 $\xlongequal{u=\sqrt[q]{t}} \frac{q}{\beta}\int u^{p+q-1}(a+bu^{q})^{\gamma}du$

Case 2. $\frac{\alpha+1}{\beta}\in\mathbb{Z}$ 记为 n $\quad \gamma=\frac{p}{q}\in\mathbb{Q}$ 则

原式 $\xlongequal{u=\sqrt[q]{a+bt}} \frac{q}{\beta b^{n}}\int u^{p+q-1}(u^{q}-a)^{n-1}du$

Case 3. $\gamma+\frac{\alpha+1}{\beta}\in\mathbb{Z}$ 记为 n $\quad \gamma=\frac{p}{q}\in\mathbb{Q}$ 则

原式 $\xlongequal{u=\sqrt[q]{\frac{a}{t}+b}} -\frac{qa^{n}}{\beta}\int u^{p+q-1}(u^{q}-b)^{-n-1}du$

$\longrightarrow$ 有理函数

Ex

$$I=\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx=\int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{\frac{1}{3}}dx$$

$\alpha=-\frac{1}{2}$, $\beta=\frac{1}{4}$, $\gamma=\frac{1}{3}$ $\quad \frac{\alpha+1}{\beta}=2$

$$\xlongequal{t=x^{\frac{1}{4}}} 4\int t(1+t)^{\frac{1}{3}}dt$$

$$\xlongequal{u=(1+t)^{\frac{1}{3}}} 4\int(u^{3}-1)u\,d(u^{3}-1)$$

$$=12\int u^{6}-u^{3}du$$

$$=\frac{12}{7}u^{7}-3u^{4}+C$$

$$=\frac{12}{7}(1+\sqrt[4]{x})^{\frac{7}{3}}-3(1+\sqrt[4]{x})^{\frac{4}{3}}+C$$

二负积分

(1) 计算题

1. $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

$\underset{=}{x=\tan t} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan t)}{\sec^2 t} d\tan t = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt$

"对称性": $\ln(1+\tan(\frac{\pi}{4}-t))$

$=\ln(1+\frac{1-\tan t}{1+\tan t})=\ln 2-\ln(1+\tan t)$

$\Rightarrow I=\int_0^{\frac{\pi}{4}} \ln 2-\ln(1+\tan t)dt=\frac{\pi}{4}\ln 2-I$ $\quad I=\frac{\pi}{8}\ln 2$

2. $I=\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^a x}$

$\frac{1}{1+\tan^a x}+\frac{1}{1+\tan^a(\frac{\pi}{2}-x)}=\frac{1}{1+\tan^a x}+\frac{\tan^a x}{1+\tan^a x}=1$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} 1-I=\frac{\pi}{2}-I \Rightarrow I=\frac{\pi}{4}$

3. $\lim_{n\to\infty}\sum_{k=1}^{n}[(1+\frac{k}{n})\sin\frac{k\pi}{n^2}]$

$\sin\frac{k\pi}{n^2}=\frac{k\pi}{n^2}+o(\frac{1}{n^2}) \Rightarrow$ 原式 $=\lim_{n\to\infty}\sum_{k=1}^{n}(1+\frac{k}{n})(\frac{k\pi}{n^2}+o(\frac{1}{n^2}))$

$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k\pi}{n^2}(1+\frac{k}{n})$

$=\frac{\pi}{n}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}(1+\frac{k}{n})$

$=\pi\int_0^1 x(1+x)dx=\frac{5}{6}\pi$

(2) 估计与不等式

4. $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\cos^n x\,dx=0$

思路 分段估计 x 远离0时 $\cos^n x\to 0$

靠近0时 $\cos^n x\to 1$ (⟶ 控制这一段的长度)

Sol. $\forall\varepsilon>0$ 取 $\delta=\frac{\varepsilon}{2}$ 则 $\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\delta}\cos^n x\,dx+\int_{\delta}^{\frac{\pi}{2}}\cos^n x\,dx$

$\leq\delta+\int_{\delta}^{\frac{\pi}{2}}\cos^n x\,dx\leq\delta+\frac{\pi}{2}\cos^n\delta$

取 $N>0$ s.t. $\cos^N\delta<\frac{\varepsilon}{\pi}$ 则 $\forall n>N$

$\int_0^{\frac{\pi}{2}}\cos^n x\,dx\leq\delta+\frac{\pi}{2}\cos^N\delta$

$<\frac{\varepsilon}{2}+\frac{\pi}{2}\frac{\varepsilon}{\pi}=\varepsilon$

5 $f\geq 0$ 在 $[a,b]$ 连续 $M=\max_{a\leq x\leq b}f(x)$. 证 $\lim_{n\to\infty}\left(\int_a^b(f(x))^n dx\right)^{\frac{1}{n}}=M$

Sol. $\left(\int_a^b(f(x))^n dx\right)^{\frac{1}{n}}\leq\left(\int_a^b M^n dx\right)^{\frac{1}{n}}=M(b-a)^{\frac{1}{n}}\to M$

下界控制?

连续性 $\Rightarrow\forall\varepsilon>0$ $\exists[\alpha,\beta]\subseteq[a,b]$ s.t. $f(x)\in[M-\varepsilon,M]$ on $[\alpha,\beta]$

$\Rightarrow\left(\int_a^b(f(x))^n dx\right)^{\frac{1}{n}}\geq\left(\int_\alpha^\beta(f(x))^n dx\right)^{\frac{1}{n}}$

$=(M-\varepsilon)(\beta-\alpha)^{\frac{1}{n}}\to M-\varepsilon$

再令 $\varepsilon\to 0$ 即可.

6. f在$[-1,1]$可导 $|f'(x)|\le M$ 若$\exists a\in(0,1)$ s.t. $\int_{-a}^{a}f(x)dx=0$

证: $\left|\int_{-1}^{1}f(x)dx\right|\le M(1-a^2)$

思路 $f(x)=f(x_0)+\underbrace{f'(\xi)}_{\text{可控制}}(x-x_0)$ 希望$f(x_0)=0$

pf. 积分中值定理 $\Rightarrow \exists x_0\in(0,1)$ $f(x_0)=0$

$\forall x\in[-1,1]$. $\exists\xi$在x与x_0之间 s.t. $f(x)=f'(\xi)(x-x_0)$

$$\Rightarrow \left|\int_{-1}^{1}f(x)dx\right| \le \int_{-1}^{-a}|f(x)|dx+\int_{a}^{1}|f(x)|dx$$
$$\le M\int_{-1}^{-a}(x_0-x)dx+M\int_{a}^{1}(x-x_0)dx$$
$$=\frac{M}{2}\left((x_0+1)^2-(x_0+a)^2+(1-x_0)^2-(a-x_0)^2\right)$$
$$=M(1-a^2)$$

7 f在$[0,1]$上有二阶连续导. $f(0)=f(1)=0$ 且 $f(x)\ne0$ on $(0,1)$

证: $\int_0^1\left|\frac{f''(x)}{f(x)}\right|dx\ge4$

pf. 不妨 $f(x)\ge0$ 则 设$M=\max\limits_{0\le x\le1}|f(x)|=f(x_0)$

原式 $\ge\frac{1}{M}\int_0^1|f''(x)|dx$

$f(x_0)=f'(\xi)x_0$ $(\xi\in(0,x_0))$

$=f'(\eta)(x_0-1)$ $(\eta\in(x_0,1))$

$$\Rightarrow \int_0^1|f''(x)|dx \ge \left|\int_\xi^\eta f''(x)dx\right|$$
$$=|f'(\eta)-f'(\xi)|$$
$$=\left|\frac{f(x_0)}{x_0-1}-\frac{f(x_0)}{x_0}\right|=\frac{f(x_0)}{x_0(1-x_0)}$$

$\Rightarrow$原式 $\ge\frac{1}{x_0(1-x_0)}\ge4$

8. f 在 $[0,1]$ 上可导. $f(0)=f(1)=0$ 证 $\left(\int_0^1 f(x)dx\right)^2 \le \frac{1}{12}\int_0^1 |f'(x)|^2 dx$

取等 $\Leftrightarrow$ $f(x)=Ax(1-x)$

Pf $\left(\int_0^{\frac12}\int_0^x f'(t)dt\,dx - \int_{\frac12}^1\int_x^1 f'(t)dt\,dx\right)^2$

① $\le 2\left(\int_0^{\frac12}\int_0^x f'(t)dt\,dx\right)^2 + 2\left(\int_{\frac12}^1\int_x^1 f'(t)dt\,dx\right)^2$

② $\le 2\left(\int_0^{\frac12}\sqrt{x\int_0^x f'(t)^2dt}\,dx\right)^2 + 2\left(\int_{\frac12}^1\sqrt{(1-x)\int_x^1 f'(t)^2dt}\,dx\right)^2$

③ $\le 2\left(\int_0^{\frac12}\sqrt{x}\sqrt{\int_0^{\frac12} f'(t)^2dt}\,dx\right)^2 + 2\left(\int_{\frac12}^1\sqrt{1-x}\sqrt{\int_{\frac12}^1 f'(t)^2dt}\,dx\right)^2$

$= 2\left(\int_0^{\frac12}\sqrt{x}dx\right)^2\int_0^{\frac12} f'(t)^2dt + 2\left(\int_{\frac12}^1\sqrt{1-x}\,dx\right)^2\int_{\frac12}^1 f'(t)^2dt$

$= \frac19\int_0^1 f'(t)^2dt$ 放过了?

观察取等条件 ① $\Leftrightarrow \int_0^{\frac12} f(x)dx = \int_{\frac12}^1 f(x)dx$ ✓

② (Cauchy) $\Leftrightarrow f'(t)\equiv k$ ✗

$f(x)=Ax(1-x)$ $f'(x)=A(1-2x) = -2A(\frac12-t)$ ← 看到这个给 Cauchy 取等条件

倒回到此处 $\int_0^{\frac12}\int_0^x f'(t)dt\,dx$

$=\int_0^{\frac12} f'(t)\int_t^{\frac12}dx\,dt = \int_0^{\frac12} f'(t)(\frac12-t)dt$

$\overset{\text{Cauchy}}{\le}\sqrt{\int_0^{\frac12} f'(t)^2dt\int_0^{\frac12}(\frac12-t)^2dt}$

同理 $\int_{\frac12}^1\int_x^1 f'(t)dt\,dx$

$=\int_{\frac12}^1 f'(t)\int_{\frac12}^t x\,dx\,dt = \int_{\frac12}^1 f'(t)(t-\frac12)dt$

$\le\sqrt{\int_{\frac12}^1 f'(t)^2dt\int_{\frac12}^1(t-\frac12)^2dt}$

故原式 $\le 2\int_0^{\frac12} f'(t)^2dt\int_0^{\frac12}(\frac12-t)^2dt + 2\int_{\frac12}^1 f'(t)^2dt\int_{\frac12}^1(t-\frac12)^2dt$

$=2\int_0^{\frac12}t^2dt\int_0^1 f'(t)^2dt = \frac{1}{12}\int_0^1|f'(x)|^2dx$

另解（由某同学提出）　很好的方法

$$\int_0^1 \frac{1}{2} f'(x)\,dx = 0$$

$$\Rightarrow \left(\int_0^1 f(x)\,dx\right)^2 = \left(x f(x)\Big|_0^1 - \int_0^1 x\,d f(x) \right)^2$$

$$= \left(\int_0^1 x f'(x)\,dx\right)^2$$

$$= \left(\int_0^1 \left(x-\frac{1}{2}\right) f'(x)\,dx\right)^2$$

$$\overset{\text{Cauchy}}{\leq} \int_0^1 \left(x-\frac{1}{2}\right)^2 dx \int_0^1 (f'(x))^2 dx$$

$$= \frac{1}{12} \int_0^1 (f'(x))^2 dx$$