Complex Geometry

chapter 1 Rudiments of Several Complex Variables

1.1 Holomorphic Function

I. One Variable

$\Omega \subset \mathbb{C}$ open, $f \in C^1(\Omega, \mathbb{C})$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$dz = dx + \sqrt{-1}\, dy \quad d\bar{z} = dx - \sqrt{-1}\, dy$

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)$

f is holomorphic $\Leftrightarrow \bar{\partial} f = 0 \overset{f = u + iv}{\Longleftrightarrow} \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

Cauchy formula: $\Omega \subset \mathbb{C}$ compact. boundary $\partial\Omega$ C^1.

$f \in C^1(\bar{\Omega}, \mathbb{C})$, then

$$f(w) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \int_{\Omega} \frac{1}{\pi(z-w)} \frac{\partial f}{\partial \bar{z}} dx \wedge dy$$

$$\underline{\frac{\sqrt{-1}}{2} dz \wedge d\bar{z} = dx \wedge dy}$$

II Several Variables

holomorphic $\Leftrightarrow$ complex analytic $\Leftrightarrow \frac{\partial f}{\partial \bar{z}_v} = 0 \quad (\forall v)$

Polydisk: $D(a,R) = D(a_1,R_1) \times \cdots \times D(a_n,R_n)$

"boundary" $S(a,R) = S(a_1,R_1) \times \cdots \times S(a_n,R_n)$

Cauchy formula: If $\bar{\partial} f = 0$ then

$$f(w) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{S(a,R)} \frac{f(z_1,\cdots z_n)}{(z_1-w_1)\cdots(z_n-w_n)} dz_1 \cdots dz_n$$

1.2 Domain of Holomorphy

<u>Envelope</u> of holomorphy of Ω: $D_\Omega = \bigcap_{f\in\mathcal{O}(\Omega)} D_f$

Thm (Hartogs) Ω open in $\mathbb{C}^n$ $(n\geq 2)$. $K\subset\Omega$ compact and $\Omega\setminus K$ is connected. Then every holomorphic function $f\in\mathcal{O}(\Omega\setminus K)$ extends to $\tilde{f}\in\mathcal{O}(\Omega)$

Thm $\Omega\subset\mathbb{C}^n$ open. if Ω is convex, then it's a domain of holomorphy

Def $\Omega\subset\mathbb{C}^n$, for a compact set K in Ω

<u>holomorphic hull</u> $\hat{K} = \{x\in\Omega \mid |f(x)| \leq \sup_K |f| \quad f\in\mathcal{O}(\Omega)\}$

If $\hat{K}$ is compact when K is compact, then Ω is holomorphically convex.

Thm (Cartan-Thullen) Ω a domain. then TFAE.

(i) Ω is a domain of holomophy

(ii) Ω is a holomorphically convex domain

(iii) $\exists f \in \mathcal{O}(\Omega)$ s.t. $D_f = \Omega$

1.3 Pseudoconvex Domain

Def. $f: \Omega \to [-\infty, +\infty)$ is plurisubharmonic if

(i) f is upper continuous

(ii) every complex line $L \subset \mathbb{C}^n$, $f|_{L \cap \Omega}$ is subharmonic

Thm. $f \in C^2(\Omega)$ is plurisubharmonic iff

$$\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \geq 0 \quad \forall z \in \Omega.\ w \in \mathbb{C}^n$$

Def Ω a domain. $y \in \partial\Omega$, if $\exists$ a nbhd U of y and $\varphi \in C^2(U)$, s.t

(i) $\Omega \cap U = \{x \in U \mid \varphi(x) < 0\}$

(ii) $d\varphi|_y \neq 0$

(iii) $\sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\Big|_y \xi_i \bar{\xi}_j \geq 0$ for $\forall \sum_{i=1}^{n} \frac{\partial \varphi}{\partial z_i}\Big|_y \xi_i = 0$

then Ω is pseudoconvex at y

Rmk. This is independent of φ.

Real and Complex Hessian:

$\mathbb{C}^n \cong \mathbb{R}^{2n}$ $\quad z_i = x_i + \sqrt{-1}\, y_i \quad y_i = x_{n+i}$

Real Hessian $\quad \left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta}\right)_{2n \times 2n} = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} & \frac{\partial^2 \varphi}{\partial x_i \partial x_{n+j}} \\ \frac{\partial^2 \varphi}{\partial x_{n+i} \partial x_j} & \frac{\partial^2 \varphi}{\partial x_{n+i} \partial x_{n+j}} \end{pmatrix}$

Complex Hessian $\quad \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right)_{n \times n}$

$J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$

$J_*\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{n+i}} = \frac{\partial}{\partial y_i} \qquad J_*\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$

$J_*\left(\frac{\partial}{\partial z_i}\right) = \sqrt{-1}\frac{\partial}{\partial z_i} \qquad J_*\left(\frac{\partial}{\partial \bar{z}_i}\right) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}_i}$

$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

$\mathrm{Herm}(n) = \{ \bar{H}^T = H \}$

$\check{S} = \{ K \in \mathrm{Sym}(2n) \mid [K, J] = KJ - JK = 0 \}$

$i: \mathrm{Herm}(n) \to \check{S}$ is an isomorphism

$H = A + \sqrt{-1} B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$

Prop ① $\left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right) \geq 0 \Rightarrow \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) \geq 0$

② $\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) = 0 \Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right) \in \check{S}$

Prop. Ω a domain in $\mathbb{C}^n$. if Ω is strongly pseudoconvex at $y \in \partial\Omega$, then we can choose a defining function s.t. its complex Hessian is positive at y

Thm Ω is strongly pseudoconvex at $y \in \partial\Omega$ iff $\exists$ a holomorphic coordinate s.t. Ω is strictly convex at $y \in \partial\Omega$

Chapter 2 Complex Manifold

2.1 Holomorphic Map

$f: \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if $\forall f_\lambda$ is holomorphic.

$(z_1, \cdots z_n) \in \mathbb{C}^n \qquad (w_1, \cdots w_m) \in \mathbb{C}^m$

$z_i = x_i + \sqrt{-1}\, y_i \qquad w_\lambda = u_\lambda + \sqrt{-1}\, v_\lambda$

$$df = \frac{\partial u_\lambda}{\partial x_i} dx_i \otimes \frac{\partial}{\partial u_\lambda} + \frac{\partial u_\lambda}{\partial y_i} dy_i \otimes \frac{\partial}{\partial u_\lambda} + \frac{\partial v_\lambda}{\partial x_i} dx_i \otimes \frac{\partial}{\partial v_\lambda} + \frac{\partial v_\lambda}{\partial y_i} dy_i \otimes \frac{\partial}{\partial v_\lambda}$$

$$\partial f = \frac{\partial f_\lambda}{\partial z_i} dz_i \otimes \frac{\partial}{\partial w_\lambda} \qquad \bar\partial f = \frac{\partial f_\lambda}{\partial \bar z_i} d\bar z_i \otimes \frac{\partial}{\partial w_\lambda}$$

$$\partial \bar f = \frac{\partial \bar f_\lambda}{\partial z_i} dz_i \otimes \frac{\partial}{\partial w_\lambda} \qquad \bar\partial \bar f = \frac{\partial \bar f_\lambda}{\partial \bar z_i} d\bar z_i \otimes \frac{\partial}{\partial w_\lambda}$$

$$df = \partial f + \bar\partial \bar f + \bar\partial f + \partial \bar f$$

f is holomorphic $\Leftrightarrow \bar\partial f = 0 \Leftrightarrow \partial \bar f = 0$

$$\begin{pmatrix} I & 0 \\ -i & I \end{pmatrix} \begin{pmatrix} \frac{\partial u_\lambda}{\partial x_i} & \frac{\partial v_\lambda}{\partial x_i} \\ \frac{\partial u_\lambda}{\partial y_i} & \frac{\partial v_\lambda}{\partial y_i} \end{pmatrix} \begin{pmatrix} I & 0 \\ i & I \end{pmatrix} \overset{\text{check}}{=} \begin{pmatrix} \frac{\partial f_\lambda}{\partial \bar z_i} + \frac{\partial f_\lambda}{\partial z_i} & \frac{\partial v_\lambda}{\partial x_i} \\ -2\sqrt{-1}\frac{\partial f_\lambda}{\partial \bar z_i} & \frac{\partial \bar f_\lambda}{\partial \bar z_i} \; \frac{\partial f_\lambda}{\partial \bar z_i} \end{pmatrix}$$

f holomorphic, then $\det\begin{pmatrix} \frac{\partial u_\lambda}{\partial x_i} & \frac{\partial v_\lambda}{\partial x_i} \\ \frac{\partial u_\lambda}{\partial y_i} & \frac{\partial v_\lambda}{\partial y_i} \end{pmatrix} = \left|\det\left(\frac{\partial f_\lambda}{\partial z_i}\right)\right|^2 \geq 0$

2.2 Complex Manifold. Pseudogroup Structure

Def Complex Manifold

e.g. $\mathbb{C}P^{n-1} = \mathbb{C}^n - \{0\}/_n$

Def. D a domain of $\mathbb{R}^n$ or $\mathbb{C}^n$. A pseudogroup of transformations in D is a set Γ of local transformations of D s.t.

(i) $f \in \Gamma \Rightarrow f^{-1} \in \Gamma$

(ii) $f \in \Gamma, g \in \Gamma \Rightarrow g \circ f \in \Gamma$

(iii) $f \in \Gamma \Rightarrow f|_W \in \Gamma \quad \forall W$ open

(iv) $Id \in \Gamma$

(v) f: any local diffeomorphism of D. $D = \cup U_j$, $f|_{U_j} \in \Gamma (\forall j)$. then $f \in \Gamma$

Def. Γ as above. X paracompact Hausdorff. By a system of local Γ-coordinates we mean a set $\{z_j\}_{j \in I}$ of local homeomorphism z_j of X into D s.t. $z_j \circ z_k^{-1} \in \Gamma$

$\{w_\lambda\}$ $\{z_j\}$ are equivalent if $w_\lambda \circ z_j^{-1} \in \Gamma$

A Γ-structure on X is an equivalent class of systems of local Γ-coordinates. X with a Γ-structure is a Γ-manifold submanifold:

$f: M \to N$ smooth. if f_* is injective, then f is an immersin

If f is injective, then M is an immersed submanifold.

$f(M)$ has the relative topology.

If $f: M \to f(M)$ is a homeomorphism, then it's an embedding

Prop. M a connected compact complex manifold and f be a holomorphic function on M, then f is constant.

Cor There are no compact complex submanifold of $\mathbb{C}^n$

2.3 Almost Complex Manifold

Def. A $(1,1)$-tensor $J \in \Gamma(T^*M \otimes TM)$ on a differential manifold M satisfying $J^2 = -\mathrm{Id}$ is called an almost complex structure.

Rmk. Complex Manifold $\overset{\Rightarrow}{\not\Leftarrow}$ Almost Complex Manifold

$J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $\quad (x_1, \cdots x_n, y_1, \cdots y_n) \longmapsto (-y_1, \cdots -y_n, x_1, \cdots x_n)$

$$= (x_1 \cdots x_n, y_1 \cdots y_n) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$(J)_* = J_{\mathbb{R}^{2n}} \qquad J_{\mathbb{R}^{2n}} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \\ \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}$$

$f: \mathbb{C}^n \to \mathbb{C}^m$ $\quad$ f holomorphic $\Leftrightarrow \bar{\partial} f = 0$

$$\Leftrightarrow f_* \circ J_{\mathbb{R}^{2n}} = J_{\mathbb{R}^{2m}} \circ f_*$$

J above is local, needs to check it can be extended globally.

$\{\varphi_j, U_j\}_{j \in I}$ $\quad \varphi_j : U_j \to \mathbb{C}^n = \mathbb{R}^{2n}$

$J_j = (\varphi_j)_*^{-1} \cdot J_{\mathbb{R}^{2n}} (\varphi_j)_*$ $\qquad J_j^2 = -Id$

$U_j \cap U_k \neq \emptyset$ $\quad J_k = (\varphi_j)_*^{-1} \circ (\varphi_k \circ \varphi_j^{-1})_*^{-1} \circ J_{\mathbb{R}^{2n}} \circ \underbrace{(\varphi_k \circ \varphi_j^{-1})_*}_{\text{holomorphic}} (\varphi_j)_*$

$$= J_j \quad \checkmark$$

Def (Nijenhuis Tensor) A (2,1)-tensor

$$N^J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

Thm (Newlander - Nirenberg) (M, J) almost complex then J comes from a complex structure iff $N^J = 0$

Lemma. A real finite-dimensional vector space V which admits endomorphism $J: V \to V$ s.t. $J^2 = -\mathrm{id}$. is necessarily even dimensional.

Cor. Almost complex manifold must be even dimensional.

$f: M \to N$ holomorphic $\Leftrightarrow \psi \circ f \circ \varphi^{-1}: \mathbb{C}^m \to \mathbb{C}^n$ holomorphic

$$\Leftrightarrow (\psi \circ f \circ \varphi^{-1})_* \circ J_{\mathbb{R}^{2m}} = J_{\mathbb{R}^{2n}} \circ (\psi \circ f \circ \varphi^{-1})_*$$

$$\Leftrightarrow f_* \circ \varphi_*^{-1} \circ J_{\mathbb{R}^{2m}} \circ \varphi_* = \psi_*^{-1} \circ J_{\mathbb{R}^{2n}} \circ \psi_* \circ f_*$$

$$\Leftrightarrow f_* \circ J_M = J_N \circ f_*$$

Lemma Almost complex manifold must be oriented

pf. $\forall p \in M$ $T_pM = \mathrm{span}\{x_1, Jx_1, \cdots x_n, Jx_n\}$

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ JX_1 \\ \vdots \\ JX_n \end{pmatrix} = \begin{pmatrix} A & B \\ D & C \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \\ JY_1 \\ \vdots \\ JY_n \end{pmatrix}$$

$$X = AY + BJY$$
$$JX = DY + CJY$$
$$= AJY - BY$$

Thus $D = -B$. $A = C$

$$\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A+iB)^2| > 0 \quad (*)$$

Take $(x_p^1, \cdots x_p^n, y_p^1, \cdots y_p^n)$ s.t. $\frac{\partial}{\partial x_p^i} = X_{i,p}$ $\frac{\partial}{\partial y_p^i} = JX_{i,p}$

$(x_q^1, \cdots x_q^n, y_q^1, \cdots y_q^n)$ $\frac{\partial}{\partial x_q^i} = X_{i,q}$. $\frac{\partial}{\partial y_q^i} = JX_{i,q}$

Then $\forall$ basis $(x^1, \cdots x^n, y^1 \cdots y^n)$ at p

$(u^1, \cdots u^n, v^1, \cdots v^n)$ at q

$$(x^1, \cdots x^n, y^1, \cdots y^n) \xrightarrow[\text{change}]{\text{coordinate}} (x_p^1, \cdots y_p^n) \xrightarrow{(*)} (x_q^1, \cdots y_q^n)$$

$$\xrightarrow[\text{change}]{\text{coordinate}} (u^1, \cdots u^n, v^1, \cdots v^n)$$

then the Jacobi should be positive. #

2.4 The Complexified Tangent Bundle

(M, J) almost complex

$TM^{\mathbb{C}} \triangleq TM \otimes_{\mathbb{R}} \mathbb{C}$ extend J to $TM^{\mathbb{C}}$

$$J(aX) = aJ(X) \quad (a \in \mathbb{C})$$

$T^{1,0}M$ (resp. $T^{0,1}M$) $\triangleq$ the eigenbundle of $TM^{\mathbb{C}}$ w.r.t. the eigenvalue $\sqrt{-1}$ (resp $-\sqrt{-1}$) of J

$Z = X + \sqrt{-1}Y \in T^{1,0}M \Rightarrow \quad JX + \sqrt{-1}JY = \sqrt{-1}Z = \sqrt{-1}X - Y$

$\Rightarrow \quad Y = -JX. \quad X = JY$

$Z = X - \sqrt{-1}JX$

Thus $T^{1,0}M = \{X - \sqrt{-1}JX \mid X \in TM\}$

$T^{0,1}M = \{X + \sqrt{-1}JX \mid X \in TM\}$

and $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$

On T^*M. (still use J)

$(J\theta)(X) \triangleq \theta(JX)$. $\quad \theta \in T^*M \quad X \in TM$.

$$\Lambda^1_{\mathbb{C}}M \triangleq T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$$

$$\Lambda^{1,0}M \triangleq \{\theta \in \Lambda^1_{\mathbb{C}}M \mid \theta(X)=0 \ \forall X \in T^{0,1}M\}$$
$$= \{\theta \in \Lambda^1_{\mathbb{C}}M \mid J\theta = \sqrt{-1}\theta\} = \{\theta - \sqrt{-1}J\theta \mid \theta \in \Lambda^1 M\}$$

$$\Lambda^{0,1}M = \{\theta \in \Lambda^1_{\mathbb{C}}M \mid J\theta = -\sqrt{-1}\theta\} = \{\theta + \sqrt{-1}J\theta \mid \theta \in \Lambda^1 M\}$$

$$\Lambda^{1,0}M = \overline{\Lambda^{0,1}M}$$

More generally. $\Lambda^k_{\mathbb{C}}M = \{\theta_1 \wedge \cdots \wedge \theta_k \mid \theta_i \in \Lambda^1_{\mathbb{C}}M\}$

$$\Lambda^{p,q}M = \underbrace{\Lambda^{1,0}M \wedge \cdots \wedge \Lambda^{1,0}M}_{p} \wedge \underbrace{\Lambda^{0,1}M \cdots \Lambda^{0,1}M}_{q}$$

Lemma ① ω is $(k,0)$ iff $X \lrcorner \omega = 0$ $(\forall X \in T^{0,1}M)$

② ω is $(1,1)$ iff $\omega(X,Y) = \omega(JX, JY)$ $\forall X, Y \in TM$

Pf. ②. $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$.

$\omega \in \Lambda^{1,1}M \Leftrightarrow \omega^{2,0} = 0 = \omega^{0,2}$

$[\omega(X,Y) = \omega(X^{1,0}, Y^{1,0}) + \omega(X^{0,1}, Y^{0,1}) + \omega(X^{1,0}, Y^{0,1})$
$+ \omega(X^{0,1}, Y^{1,0})]$

$\Leftrightarrow \omega(X - \sqrt{-1}JX, Y - \sqrt{-1}JY) = \omega(X + \sqrt{-1}JX, Y + \sqrt{-1}JY) = 0$

$\Leftrightarrow \omega(X,Y) - \omega(JX, JY) - \sqrt{-1}\underbrace{(\omega(JX,Y) + \omega(X,JY))}_{=0}$

$\Leftrightarrow \omega(X,Y) = \omega(JX, JY)$

We assumed ω is real above. By taking conjugation it still holds when ω is complex #

Prop (M,J) almost complex, TFAE:

(1) J is a complex structure.

(2) $N^J=0$ (3) $T^{0,1}M$ is integrable

(4) $d\Gamma(\Lambda^{1,0}M)\subset\Gamma(\Lambda^{2,0}M\oplus\Lambda^{1,1}M)$

(5) $d\Gamma(\Lambda^{p,q}M)\subset\Gamma(\Lambda^{p+1,q}M\oplus\Lambda^{p,q+1}M)$

Pf. (1)$\Leftrightarrow$(2) Newlander-Nirenberg

(2)$\Leftrightarrow$(3) $Z_1=X_1+\sqrt{-1}JX_1$, $Z_2=X_2+\sqrt{-1}JX_2\in\Gamma(T^{0,1}M)$

Check: $[Z_1,Z_2]-\sqrt{-1}J[Z_1,Z_2]=N^J(X_1,X_2)-\sqrt{-1}JN^J(X_1,X_2)$

$T^{0,1}M$ integrable $\Leftrightarrow[Z_1,Z_2]\in T^{0,1}M$

$\Leftrightarrow[Z_1,Z_2]=\sqrt{-1}J[Z_1,Z_2]$

$\Leftrightarrow N^J(X_1,X_2)=0$

(3)$\Leftrightarrow$(4) $\theta\in\Gamma(\Lambda^{1,0}M)$

$d\theta\in\Gamma(\Lambda^{2,0}M\oplus\Lambda^{1,1}M)\Leftrightarrow(d\theta)^{0,2}=0$

$\Leftrightarrow d\theta(X,Y)=0\quad\forall X,Y\in T^{0,1}M$

$[\,d\theta(X,Y)=\underbrace{X(\theta(Y))}_{0}-\underbrace{Y(\theta(X))}_{0}-\theta([X,Y])\,]$

$\Leftrightarrow\theta([X,Y])=0\Leftrightarrow[X,Y]\in T^{0,1}M\overset{\text{Frobineus}}{\Leftrightarrow}T^{0,1}M$ integrable

(4)$\Leftrightarrow$(5) Just some trivial calculation

#.

2.5 Holomorphic Objects on Complex Manifold

$(M, \mathcal{G})$ complex manifold. J

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}. \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

$$J(dx^i) = -dy^i \qquad J(dy^i) = dx^i$$

By the last prop. $d = \partial + \bar\partial$

$$d^2 = 0 \Leftrightarrow \partial^2 = \bar\partial^2 = 0. \quad \partial\bar\partial = -\bar\partial\partial$$

Def. A vector field $Z \in \Gamma(T^{1,0}M)$ is called holomorphic if $Z(f)$ is holomorphic for every locally defined holomorphic function f

A $(p,0)$ form θ is holomorphic if $\bar\partial\theta = 0$

Def. A real vector field X is called real holomorphic if $X - \sqrt{-1}JX$ is holomorphic

Lemma X is a real vector field. TFAE:

(1) X is real holomorphic

(2) $L_X J = 0$

(3) The flow of X consists of holomorphic transformations

pf. $\{\varphi_t\}$ is generated by X

$Y \in \Gamma(TM)$ $\quad (\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{1}{t}[Y_p - (\varphi_t)_*(Y_{\varphi_t^{-1}(p)})]$

$= [X, Y]$

$\theta \in \Gamma(T^*M)$ $\quad (\mathcal{L}_X \theta)_p = \lim_{t\to 0} \frac{1}{t}[\theta_p - (\varphi_t^{-1})^* \theta_{\varphi_t^{-1}(p)}]$

(2) $\Leftrightarrow$ (3)

$\mathcal{L}_X (Y \otimes \theta) = \lim_{t\to 0} \frac{1}{t}[Y_p \otimes \theta_p - (\varphi_t)_*(Y_{\varphi_t^{-1}(p)}) \otimes (\varphi_t^{-1})^*(\theta_{\varphi_t^{-1}(p)})]$

$\Rightarrow (\mathcal{L}_X J)_p = \lim_{t\to 0} \frac{1}{t}[J_p - (\varphi_t)_* \circ J_{\varphi_t^{-1}(p)} \circ (\varphi_t^{-1})_*]$

$(\varphi_s)_* \circ (\mathcal{L}_X J) \circ (\varphi_s^{-1})_* = \mathcal{L}_X ((\varphi_s)_* \circ J \circ (\varphi_s^{-1})_*)$

$= -\frac{d}{dt}[(\varphi_t)_* \circ J \circ (\varphi_t^{-1})_*]_p\Big|_{t=s}$

$\mathcal{L}_X J = 0 \Leftrightarrow (\varphi_t)_* \circ J \circ (\varphi_t^{-1})_* = J \Leftrightarrow \varphi_t$ is holomorphic

#.

Lemma (Poincaré) ① M is a smooth manifold.

θ is closed r-form. Then $\forall p \in M$. $\exists\, U \ni p$ and $(r-1)$ form

ψ on U s.t. $\theta = d\psi$.

② M is a complex manifold, θ is a $\bar\partial$-closed

(p,q) form on M. then $\forall p \in M$. $\exists\, U \ni p$ and $(p,q-1)$ form

ψ on U s.t. $\theta = \bar\partial\psi$

Lemma ($\partial\bar\partial$ Lemma): M is a complex manifold. $\omega \in \Omega^{1,1}(M) \cap \Omega^2(M,\mathbb{R})$. Then ω is closed iff $\forall p \in M$, $\exists U \ni p$ s.t. $\omega|_U = \sqrt{-1}\partial\bar\partial u$ for some real function u on U

Pf. $\Leftarrow$ $\omega|_U = \sqrt{-1}\partial\bar\partial u$, then $d\omega = \sqrt{-1}(\partial+\bar\partial)(\partial\bar\partial u)$

$$= \sqrt{-1}(\partial^2\bar\partial u - \partial\bar\partial^2 u) = 0$$

$\Rightarrow$ $\exists$ locally real 1-form θ s.t. $\omega = d\theta$

$\theta = \theta^{1,0} + \theta^{0,1}$, $\theta^{1,0} = \overline{\theta^{0,1}}$

$$\omega = d\theta = \partial\theta^{1,0} + (\bar\partial\theta^{1,0} + \partial\theta^{0,1}) + \bar\partial\theta^{0,1}$$

$\omega \in (1,1)$, thus $\partial\theta^{1,0} = \bar\partial\theta^{0,1} = 0$

$\omega = \partial\theta^{0,1} + \bar\partial\theta^{1,0}$ $\quad$ $\exists f$, $\theta^{0,1} = \bar\partial f \Rightarrow \theta^{1,0} = \partial\bar f$

$$\omega = \partial\theta^{0,1} + \bar\partial\theta^{1,0} = \partial\bar\partial(f - \bar f) = \sqrt{-1}\partial\bar\partial(2\,\mathrm{Im} f) \quad \#$$

2.6 Complex and holomorphic vector Bundle

(M, E, π) is a holomorphic bundle

$\Lambda^{p,q}E = \Lambda^{p,q}M \otimes E$ (E valued (p,q) form)

$\bar{\partial}_E : \Gamma(\Lambda^{p,q}E) \to \Gamma(\Lambda^{p,q+1}E)$.

$e_i = \varphi_u^{-1}(p, (0, \cdots 1, \cdots 0))$ $p \in U$

$\theta = \theta^i \otimes e_i$ $(\theta^i \in \Gamma(\Lambda^{p,q}M))$ then $\bar{\partial}_E \theta = \bar{\partial}\theta^i \otimes e_i$

Leibniz rule: $\bar{\partial}_E(\omega \wedge \sigma) = \bar{\partial}_E \omega \wedge \sigma + (-1)^{p+q} \omega \wedge \bar{\partial}_E \sigma$

$\forall \omega \in \Omega^{p,q}(E),\ \sigma \in \Omega^{r,s}(E)$ $(*)$

Def (Pseudo-holomorphic structure)

(E, M, π) complex bundle. M complex manifold

An operator $\bar{\partial}_E$ $\Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying

$(*)$ is called a pseudo-holomorphic structure.

If moreover, $\bar{\partial}_E^2 = 0$, then $\bar{\partial}_E$ is called a holomorphic

structure

A section σ in a pseudo-holomorphic bundle $(E, \bar\partial_E)$ is called holomorphic if $\bar\partial_E \sigma = 0$

Lemma A pseudo-holomorphic bundle $(E, \bar\partial_E)$ of rank r is holomorphic $\Leftrightarrow$ $\forall p \in M$. $\exists U \ni p$ $\exists$ holomorphic basis $\{\sigma_i(x)\}_{i=1}^{k}$

Thm. A complex bundle (E, M, π) is holomorphic iff it has a holomorphic structure $\bar\partial_E$.

pf. $\Leftarrow$ $\{\sigma_1 \cdots \sigma_k\}$ local basis of E on U

$$\bar\partial_E \sigma_i = \tau_{ij} \otimes \sigma_j \qquad \tau_{ij}: (0,1) \text{ form}$$

$$\bar\partial_E^2 = 0 \Rightarrow 0 = \bar\partial_E(\tau_{ij} \otimes \sigma_j) = \bar\partial \tau_{ij} \otimes \sigma_j - \tau_{il} \wedge \tau_{lj} \otimes \sigma_j$$

$$\Rightarrow \bar\partial \tau_{ij} = \tau_{il} \wedge \tau_{lj}. \quad \text{in simplicity } \bar\partial \tau = \tau \wedge \tau.$$

[Lemma $\tau = (\tau_{ij})$ is a $GL_k(\mathbb{C})$ valued $(0,1)$-form on U

$\bar\partial \tau = \tau \wedge \tau$, then $\forall p \in U$, $\exists U' \subset U$ $f: U' \to GL_k(\mathbb{C})$

s.t. $\bar\partial f + f\tau = 0$]

Then on U', define $s_j = f_{jl} \sigma_l$

$$\bar\partial s_j = \bar\partial f_{jl} \otimes \sigma_l + f_{jy} \tau_{yl} \otimes \sigma_l$$

$$= (\bar\partial f + f\tau)_{jl} \otimes \sigma_l = 0$$

pf of Lemma: $N = U \times \mathbb{C}^k$ $\{z_\alpha\}_{\alpha=1}^n$ a local coordinate of U. $\{w^i\}_{i=1}^k$ the complex coordinate in $\mathbb{C}^k$

we want to fix an almost complex structure $\tilde{J}$

$\Leftrightarrow$ a subbundle of $\Lambda^1_{\mathbb{C}} M$ as a $(1,0)$-subbundle

$\Lambda^{1,0}_{\mathbb{C}} N = \{dz^\alpha, dw^i - T_{il} w_l \mid 1 \le \alpha \le n, 1 \le i \le k\}$ (check independence)

J is integrable $\Leftrightarrow$ $d\Gamma(\Lambda^{1,0}_{\mathbb{C}} N) \subset \Gamma(\Lambda^{1,0} N \wedge \Lambda^1_{\mathbb{C}} N)$

$d(dz^\alpha) = 0$

$$
\begin{aligned}
d(dw^i - T_{il} w_l) &= -dT_{il} w_l + T_{il} \wedge dw_l \\
&= -\partial T_{il} w_l - \bar{\partial} T_{il} w_l + T_{il} \wedge dw_l \\
&\overset{\bar{\partial} T = T \wedge T}{=} -\partial T_{il} w_l - (T_{is} \wedge T_{sl}) w_l + T_{il} \wedge dw_l \\
&= -\partial T_{il} w_l + T_{il} \wedge (dw_l - T_{ls} w_s) \\
&\in \Gamma(\Lambda^{1,0} N \wedge \Lambda^1_{\mathbb{C}} N)
\end{aligned}
$$

$\{z^\alpha, u^j\}$ on $U' \subset U$ $\qquad du^j = \psi_{ji}(dw_i - T_{il} w_l) + \psi_{j\alpha} dz^\alpha$

$$
0 = d^2 u^j = d\psi_{ji} \wedge (dw_i - T_{il} w_l) + \psi_{ij}(-dT_{il} w_l + T_{il} \wedge dw_l) + d\psi_{j\alpha} \wedge dz^\alpha
$$

restrict to $\{w_i = 0\}$ $\qquad f_{ij}(z) \triangleq \psi_{ij}(z, 0)$

$$
\bar{\partial} f_{ji} \wedge dw_i + f_{ji} T_{il} \wedge dw_l = 0
$$

then $\bar{\partial} f_{ji} + f_{il} T_{li} = 0$

#

Chapter 3 Vector Bundle

3.1 Connections on Complex Vector Bundle

Def (E, M, π) complex vector bundle. A connection on E is $\mathbb{C}$-linear map $D: \Gamma(E) \to \Gamma(\Lambda^1 E)$

s.t. $D(f\sigma) = df \otimes \sigma + f \cdot D\sigma$ $\quad f \in C^1(M)$, $\sigma \in \Gamma(E)$

Rmk. (1) D can also be extended to

$D: \Gamma(\Lambda^k E) \to \Gamma(\Lambda^{k+1} E)$

$D(\theta \otimes \sigma) = d\theta \otimes \sigma + (-1)^k \theta \wedge D\sigma \quad \theta \in \Gamma(\Lambda^k M)$

(2) $(D_1 - D_2)(f\sigma) = f \cdot (D_1 - D_2)\sigma \Rightarrow D_1 - D_2 \in \Gamma(\Lambda^1(E^* \otimes E))$
$= \Gamma(\Lambda^1(\mathrm{End}(E)))$

(3) M complex manifold. $D = D^{1,0} + D^{0,1}$

$D^{1,0}: \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$

$D^{0,1}: \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$

Def. The curvature F_D of D is $\mathrm{End}(E)$-valued 2 form defined by $F_D = D \circ D$

$F_D(X, Y)\sigma = D(D\sigma)(X, Y)$
$= D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X,Y]}\sigma$

$De_\beta = e_\alpha A^\alpha_\beta$ $\quad A^\alpha_\beta \in \Lambda^1 M$: connection 1-form

$A = (A^\alpha_\beta)_{k\times k}$

$D(e_1, \cdots e_k) = (e_1, \cdots e_k)(A^\alpha_\beta)$

$x \in E$ $\quad x = (e_1, \cdots e_k)\begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix} \Rightarrow DX = (e_1, \cdots e_k)(d+A)\begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}$

locally. $D = d + A$

$F_D X = D\left((e_1, \cdots e_k)(d+A)\begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}\right) = (e_1, \cdots e_k)\{A \wedge [(d+A)X] + d[(d+A)X]\}$

$= (e_1, \cdots e_k)(dA + A\wedge A)X$

$F_D = dA + A \wedge A$

Def (Hermitian Structure) $\pi: E \to M$ complex rank k vector bundle. A Hermitian structure H on E is a smooth field of Hermitian inner products on the fibers of E

Rmk Every complex manifold admits Hermitian structure

local $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ $\{f_\alpha\}_{\alpha \in I}$ is the P.O.U.

$\forall p \in U_\alpha$ define $H_\alpha(X,Y) = \pi_\alpha(\psi_\alpha(X)) \overline{\pi_\alpha(\psi_\alpha(Y))}^T$

$$H = \sum_{\alpha\in I} f_\alpha H_\alpha$$

Now given (E, H). we say D is H-connection.

i.e. $X(H(e_1, e_2)) = H(D_X e_1, e_2) + H(e_1, D_X e_2)$

Def (chern connection) $(E, \bar{\partial}_E)$ holomorphic vector bundle over complex manifold M. H a Hermitian metric

A chern connection D is a H-connection which is compatible with $\bar{\partial}_E$. i.e. $D^{0,1} = \bar{\partial}_E$

Prop $(E, \bar{\partial}_E)$ is a holomorphic vector bundle with a Hermitian metric, then $\exists !$ chern connection $D_{H, \bar{\partial}_E}$

Pf Define $I_H : E \to E^*$

$$I_H(e_1)(e_2) = H(e_2, e_1)$$

$\bar{\partial}_E$ induces a holomorphic structure on E^*

$$\bar{\partial}_{E^*} \theta(e) = \bar{\partial}(\theta(e)) - \theta(\bar{\partial} e)$$

set $D_X^{1,0} e = I_H^{-1} \circ (\bar{\partial}_{E^*})_{\bar{X}} \circ I_H(e)$ $\quad D_X e = (\bar{\partial}_E)_X e + D_X^{1,0} e$

is a chern connection

Uniqueness: D is a chern connection, then $D^{0,1} = \bar{\partial}_E$

then $\forall X \in T^{1,0}M$. $e_1 . e_2 \in \Gamma(E)$

$$\bar{X} H(e_1, e_2) = H(D_{\bar{X}} e_1, e_2) + H(e_1, D_X e_2)$$

$$= H((\bar{\partial}_E)_{\bar{X}} e_1, e_2) + \underbrace{H(e_1, D_X^{1,0} e_2)}_{I_H(D_X^{1,0} e_2)(e_1)}$$

$$\Rightarrow I_H(D_X^{1,0} e_2)(e_1) = \bar{X}(I_H(e_2)(e_1)) - I_H(e_2)(\bar{\partial}_{\bar{X}} e_1)$$

$$= (\bar{\partial}_{\bar{X}} I_H(e_2)) e_1 \qquad \#$$

Now for locally holomorphic basis $\{e_1, \cdots e_k\}$ $\quad \bar{\partial}_E e_\alpha = 0$

$$D_H e_\alpha = e_\beta A_\alpha^\beta$$

$$H_{\alpha\beta} = \langle e_\beta, e_\alpha \rangle_H = \overline{\langle e_\alpha, e_\beta \rangle}_H \qquad H = (H_{\alpha\beta})$$

$$d\bar{H} = d\left\langle \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1, \cdots e_k) \right\rangle_H = \left\langle D\begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1, \cdots e_k) \right\rangle_H$$

$$+ \left\langle \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, D(e_1, \cdots e_k) \right\rangle_H$$

$$= \underbrace{A^T \bar{H}}_{(1,0)} + \underbrace{\bar{H} \bar{A}}_{(0,1)} \qquad \partial \bar{H} = A^T \bar{H}$$

$$\Rightarrow A = H^{-1} \circ \partial H$$

$$F_H \triangleq F_{D_H} = dA_H + A_H \wedge A_H$$

$$= \underbrace{d}_{\partial + \bar{\partial}}(H^{-1} \partial H) + H^{-1} \partial H \wedge H^{-1} \partial H \qquad \partial H^{-1} = -H^{-1} \partial H \cdot H^{-1}$$

$$= \bar{\partial}(H^{-1} \partial H)$$

Given two hermitian H, K $\quad h = K^{-1}H \in \mathrm{End}(E)$

where $\langle he_1, e_2\rangle_K = \langle e_1, e_2\rangle_H$

[$h(e_1 \cdots e_k) = (e_1 \cdots e_k)(h^\alpha_\beta)$

$\langle h\begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1 \cdots e_k)\rangle_K = \bar{H}$ $\Rightarrow h^T = \bar{H}\cdot\bar{K}^{-1}$

$\quad \| $

$h^T \bar{K}$ $\qquad h = K^{-1}H$]

Chern connection: $D_H = d + A_H \quad D_K = d + A_K$

$$A_H = H^{-1}\partial H = h^{-1}K^{-1}\partial(Kh)$$

$$= A_K + h^{-1}(\partial h + K^{-1}\partial K\, h - h\circ K^{-1}\partial K)$$

$$D_H - D_K = h^{-1}D_K^{1,0}h = h^{-1}\partial_K h$$

$$F_H = \bar{\partial}(H^{-1}\partial H) = \bar{\partial}(K^{-1}\partial K + h^{-1}\partial_K h)$$

$$= F_K + \bar{\partial}(h^{-1}\partial_K h)$$

$$\mathrm{tr}\, F_H = \mathrm{tr}\, F_K + \bar{\partial}\partial \log \det h$$

Chapter 4 Kähler Manifold

4.1 Almost Hermitian Manifold

Def. A Hermitian metric g on an almost complex manifold (M, J) is a Riemannian metric g

s.t. $g(X, Y) = g(JX, JY) \quad \forall X, Y \in TM$

The fundamental form is defined $\omega(X, Y) = g(JX, Y)$

If $d\omega = 0$, (M, J, g) will be called an almost Kähler manifold

Rmk. ① Extend g to $T^{\mathbb{C}}M$ satisfies

(i) $g(\overline{Z_1}, \overline{Z_2}) = \overline{g(Z_1, Z_2)} \quad \forall Z_1, Z_2 \in T^{\mathbb{C}}M$

(ii) $g(Z, \overline{Z}) > 0 \quad \forall Z \in T^{\mathbb{C}}M - \{0\}$

(iii) $g(Z_1, Z_2) = 0 \quad \forall Z_1, Z_2 \in T^{1,0}M$

② TM is in particular a complex vector bundle:

$\forall p \in M$, nbhd $U \ni p$, frame $e_1, \cdots e_m, Je_1, \cdots Je_m$

$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^m$

$X^i e_i + Y^j Je_j = X \mapsto (p, (X^1 + \sqrt{-1}Y^1, \cdots, X^m + \sqrt{-1}Y^m))$

$T^{1,0}M \simeq TM$: $\quad e_i \mapsto \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)$

$\quad Je_i \mapsto \frac{\sqrt{-1}}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)$

$\forall X \in TM \quad X \mapsto \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$

③ $H(X,Y) \triangleq (g - \sqrt{-1}\omega)(X,Y)$: Hermitian structure on the complex vector bundle TM

$Z = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX) \quad W = \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)$

$$H(Z,W) = g(Z, \overline{W})$$

Conversely, if a Hermitian structure H on $T^{1,0}M \simeq TM$.

we have a hermitian metric g on TM

$$g(X,Y) \triangleq \mathrm{Re}\left(H\left(\frac{1}{\sqrt{2}}(X - \sqrt{-1}JX), \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)\right)\right)$$

$$g(JX, JY) = g(X,Y)$$

$$\omega(X,Y) = \mathrm{Re}\left(H\left(\frac{\sqrt{-1}}{\sqrt{2}}(X - \sqrt{-1}JX), \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)\right)\right)$$

④ Every almost complex manifold admits Hermitian metric.

Volume form:

$z^\alpha = x^\alpha + \sqrt{-1}\, y^\alpha \quad dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge dy^1 \cdots \wedge dx^m \wedge dy^m$

$\overset{\text{☆}}{=}$

$$\frac{\omega^m}{m!} = \frac{1}{m!}(\sqrt{-1}\, g_{\alpha_1\bar\beta_1} dz^{\alpha_1} \wedge d\bar z^{\beta_1}) \wedge \cdots \wedge (\sqrt{-1}\, g_{\alpha_m\bar\beta_m} dz^{\alpha_m} \wedge d\bar z^{\beta_m})$$

$$= \frac{(\sqrt{-1})^m}{m!}\, g_{\alpha_1\bar\beta_1} \cdots g_{\alpha_m\bar\beta_m}\, dz^{\alpha_1} \wedge d\bar z^{\beta_1} \cdots \wedge dz^{\alpha_m} \wedge d\bar z^{\beta_m}$$

$$= \frac{(\sqrt{-1})^m}{m!} \sum_{\alpha_1 \cdots \alpha_m} (-1)^{\operatorname{sgn}(\alpha_1 \cdots \alpha_m)} \quad (-1)^{\frac{m(m-1)}{2}} \det(g_{\alpha\bar\beta})$$

$$= (\sqrt{-1})^m \det(g_{\alpha\bar\beta})\, dz^1 \wedge d\bar z^1 \cdots \wedge dz^m \wedge d\bar z^m \quad dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_m} \wedge d\bar z^1 \cdots \wedge d\bar z^m$$

$$= 2^m \underbrace{\det(g_{\alpha\bar\beta})}_{\sqrt{\det(g_{ij})}}\, dx^1 \wedge dy^1 \cdots \wedge dx^m \wedge dy^m$$

4.2 Kähler Metric

Def Let g be a Hermitian metric on a complex manifold (M, J). If $d\omega = 0$, then g is called a Kähler metric. (M, J, g) is called a Kähler manifold.

Levi-Civita connection: metric g

$$\nabla g : \Gamma(TM) \to \Omega^1(TM)$$

(1) $\nabla g = 0$

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$

Lemma (M, J, g) an almost Hermitian manifold and ∇^g. then (M, J, g) is Kähler $\Leftrightarrow$ $\nabla J = 0$

pf. $\Leftarrow$. $\forall X, Y \in TM$

$$N^J(X,Y) = \nabla_X Y - \nabla_Y X + J(\nabla_{JX} Y - \nabla_Y JX) + J(\nabla_X JY - \nabla_{JY} X) - \nabla_{JX} JY + \nabla_{JY} JX$$

$$= -(\nabla_{JX} J)Y + (\nabla_{JY} J)X - J((\nabla_Y J)X) + J((\nabla_X J)Y)$$

$\nabla J = 0 \Rightarrow N^J = 0$ i.e. J is integrable

$$(\nabla_X \omega)(Y,Z) = \nabla_X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)$$

$$= \nabla_X(g(JY,Z)) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z)$$

$$= g((\nabla_X J)Y, Z) = 0.$$

$$d\omega(X,Y,Z) = (\nabla_X \omega)(Y,Z) + (\nabla_Y \omega)(Z,X) + (\nabla_Z \omega)(X,Y)$$

$$= g((\nabla_X J)Y, Z) - g((\nabla_Y J)X, Z) + g((\nabla_Z J)X, Y) \quad (*)$$

$\Rightarrow$: Replace X by JX, Y by JY.

and add up the two equations

#

(M, J, g) Kähler $\Leftrightarrow \nabla J = 0$.

ω Kähler form: $\sqrt{-1}\, g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$

real $(1,1)$ form $\Rightarrow \omega = \sqrt{-1}\,\partial\bar\partial u$ $\qquad g_{\alpha\bar\beta} = \frac{\partial^2 u}{\partial z^\alpha \partial \bar z^\beta}$

Prop. (M, J, g) Kähler $\forall x \in M$. $\exists$ local complex coordinate $(z^1, \cdots z^n)$ s.t. $g_{\alpha\bar\beta}(x) = \frac{1}{2}\delta_{\alpha\beta}$ $\quad dg_{\alpha\bar\beta}(x) = 0$.

pf. Diagonalize $(g_{\alpha\bar\beta}) = \frac{1}{2}(\delta_{\alpha\beta})$ at x $\qquad (z^1, \cdots z^n)$

$$\omega = \sqrt{-1}\, g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta \qquad g_{\alpha\bar\beta}(x) = \frac{1}{2}\delta_{\alpha\beta}$$

Near x. $\quad g_{\alpha\bar\beta} = \frac{1}{2}\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} z^\gamma + \psi_{\alpha\beta\bar\gamma} \bar z^\gamma + o(|z|)$

$$g_{\alpha\bar\beta} = \overline{g_{\beta\bar\alpha}} \Rightarrow \psi_{\alpha\beta\bar\gamma} = \overline{\psi_{\beta\alpha\gamma}}$$

$$0 = d\omega = \sqrt{-1}\left(\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma} dz^\gamma + \frac{\partial g_{\alpha\bar\beta}}{\partial \bar z^\gamma} d\bar z^\gamma\right) \wedge dz^\alpha \wedge d\bar z^\beta$$

$$= \sqrt{-1}\left(\psi_{\alpha\beta\gamma}\, dz^\gamma \wedge dz^\alpha \wedge d\bar z^\beta + \psi_{\alpha\beta\bar\gamma}\, d\bar z^\gamma \wedge dz^\alpha \wedge d\bar z^\beta\right) + o(1)\theta$$

$$\psi_{\alpha\beta\gamma} = \psi_{\gamma\beta\alpha} \qquad \psi_{\alpha\beta\bar\gamma} = \psi_{\alpha\gamma\beta}$$

$$z^\alpha = w^\alpha - \frac{1}{2}\sum \psi_{\gamma\alpha\beta} w^\beta w^\gamma \qquad \text{inverse function theorem}$$

$$dz^\alpha = dw^\alpha - \frac{1}{2}\left(\psi_{\gamma\alpha\beta}\, dw^\beta w^\gamma + \psi_{\gamma\alpha\beta} w^\beta dw^\gamma\right) = dw^\alpha - \psi_{\gamma\alpha\beta} w^\beta dw^\gamma$$

Then $\quad \omega = \sqrt{-1}\left(\frac{1}{2}\delta_{\alpha\beta} + o(|w|)\right) dw^\alpha \wedge d\bar w^\beta$

$\qquad d\omega = 0$

#

4.3 Chern connection on Hermitian Manifold

(M, J, g) $\quad TM \simeq T^{1,0}M$ $\quad \bar\partial$: holomorphic structure on $T^{1,0}M$

$X \mapsto \frac{1}{2}(X - \sqrt{-1}JX)$ $\quad \bar\partial\left(\frac{\partial}{\partial z_\alpha}\right) = 0$

$\bar\partial'$: holomorphic structure on TM

$\bar\partial'(X) \mapsto \bar\partial\left(\frac{1}{2}(X - \sqrt{-1}JX)\right)$

Define $\bar\partial^\nabla Y(X) = \bar\partial_X^\nabla Y = \frac{1}{2}\left(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X\right)$

One can check: (i) Leibniz rule

(ii) $\bar\partial Y = 0 \Leftrightarrow L_Y J = 0$

$\Leftrightarrow (L_Y J)(X) = 0 \overset{\text{check}}{\Longleftrightarrow} J(\bar\partial_X^\nabla Y) = 0 \Leftrightarrow \bar\partial_X^\nabla Y = 0$

Thus $\bar\partial = \bar\partial^\nabla$

$f: TM \to T^{1,0}M$ $\quad f(JX) = \sqrt{-1}f(X)$

$X \mapsto \frac{1}{2}(X - \sqrt{-1}JX)$ $\quad f^{-1}(\sqrt{-1}Z) = Jf^{-1}(Z)$

On TM. $\quad H'(X,Y) = H(f(X), f(Y)) = g(f(X), \overline{f(Y)})$

$= g(X,Y) - \sqrt{-1}g(JX, Y)$

$\bar\partial'(Y) = f^{-1}\bar\partial(f(Y))$

D_H : Chern connection on $T^{1,0}M$. $\quad D_H H = 0$

$(D_H^{0,1}) = \bar\partial$

∇' : Chern connection on TM $\quad \nabla' Y = f^{-1}(D_H f(Y))$

One has: $\nabla' g = 0$, $\nabla' J = 0$ $\quad (T^{\nabla'})^{(1,1)} = 0$

Check: $(\nabla'_X J)(Y) = \nabla'_X(JY) - J(\nabla'_X Y)$

$= f^{-1}(D_{H_X} f(JY)) - J(f^{-1} D_{H_X} f(Y))$

$= f^{-1}(\sqrt{-1} D_{H_X} f(Y)) - f^{-1}(\sqrt{-1} D_{H_X} f(Y)) = 0.$

$$(T^{\nabla'})^{1,1}(X,Y) = \frac{1}{4} T^{\nabla'}(X - \sqrt{-1}JX,\ Y + \sqrt{-1}JY)$$

$$= T^{D_H}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\gamma}\right)$$

$$= \underbrace{D_{H\frac{\partial}{\partial \bar z^\gamma}} \frac{\partial}{\partial z^\alpha}}_{0} - \underbrace{D_{H\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial \bar z^\gamma}}_{0} - \underbrace{\left[\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\gamma}\right]}_{0} = 0$$

Prop g is Kähler iff $\nabla' = \nabla$

pf. $\Leftarrow$ $\nabla' = \nabla$. $\nabla' J = 0$ then $\nabla J = 0$

$\Rightarrow$: $\nabla J = 0$. $\nabla g = 0$ $\Rightarrow$ $\nabla \omega = 0$

Thus $\nabla H = 0$, $T^\nabla = 0$ $\Rightarrow$ $\nabla = \nabla'$ #.

4.4 Curvature of Kähler Manifold

Let D be a connection on E.

$$DFD \overset{\text{locally}}{=\!=} D(dA + A\wedge A)$$

$$\overset{D=d+A}{=\!=} d(dA + A\wedge A) + A\wedge(dA + A\wedge A) - (dA + A\wedge A)\wedge A$$

$$= dA\wedge A - A\wedge dA$$

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}\{W \mapsto R(W,X)Y\}$$

$$= \langle R(e_i, X)Y, e_j\rangle g^{ij}$$

$$R(X,Y,JZ,JW) = R(JX,JY,Z,W) = R(JX,JY,JZ,JW)$$

$$R(JX,JY,Z) = R(X,Y)Z$$

$$\mathrm{Ric}(JX,JY) = \sum_{i=1}^{2n} R(Je_i, X, Je_i, Y)$$

Def. Ricci form ρ is defined by $\rho(X,Y) = \mathrm{Ric}(JX,Y)$ $(\forall X, Y \in TM)$

Rmk. ρ is closed $\quad 2\rho(X,Y) \overset{\text{check}}{=} \mathrm{Tr}(R(X,Y)\circ J)$

$$d\rho = \frac{1}{2} d\,\mathrm{Tr}(R(X,Y)\circ J)$$

$$= \frac{1}{2} d\,\mathrm{Tr}\left(\nabla(R(X,Y)\circ J)\right) = 0$$

Holomorphic Sectional Curvature

$\sigma : \{X, JX\} \to J$-invariant plane

$$H(\sigma) = \frac{R(X, JX, X, JX)}{|X|^4}$$

Def. Given two J-invariant planes σ, σ'

$$H(\sigma, \sigma') = R(X, JX, Y, JY)$$

Rmk (i) One can check it's well defined

(ii) $H(\sigma, \sigma) = H(\sigma)$

(iii) $R(X, JX, Y, JY) \overset{\text{Bianchi I}}{=} -R(JX, Y, X, JY) - R(Y, X, JX, JY)$

acted by J

$= R(X, JY, X, JY) + R(X, Y, JX, JY)$

$(e_1, \cdots e_n, Je_1, \cdots Je_n)$ orthonormal basis for T_pM

$$Ric(X, Y) = R(e_\alpha, X, e_\alpha, Y) + R(Je_\alpha, X, Je_\alpha, Y)$$

$$\overset{\text{Bianchi I}}{=} R(e_\alpha, Je_\alpha, X, Y)$$

$Ric(X, X)$ can be decided by the holomorphic bisectional curvature.

Moreover, bisectional curvature positive $\Rightarrow$ Ricci curvature positive

4.5 In Local Coordinates

(M, g, J) Kähler

$\{z^1, \cdots z^n\}$ $\qquad \nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \frac{\partial}{\partial z^\alpha} = \nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = 0$

$$\nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\beta} = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial z^\gamma} + \Gamma^{\bar{\gamma}}_{\alpha\beta} \frac{\partial}{\partial \bar{z}^\gamma}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$

[$\nabla J = 0$. Thus ∇ maps $(1,0)$ to $(1,0)$]

$$\nabla_{\frac{\partial}{\partial \bar{z}^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = \Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} \frac{\partial}{\partial \bar{z}^\gamma}$$

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \langle \nabla_{\frac{\partial}{\partial z^\gamma}} \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \rangle \Rightarrow \Gamma^\gamma_{\alpha\beta} = g^{\gamma\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha}$$

$$\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} = \overline{\Gamma^\gamma_{\alpha\beta}}$$

$$R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right)X = R\left(\frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right)X = 0$$

$$R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right)\frac{\partial}{\partial z^\gamma} = R^\delta_{\alpha\bar{\beta}\gamma} \frac{\partial}{\partial z^\delta}$$

$$= \underbrace{\nabla_{\frac{\partial}{\partial z^\alpha}} \nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \frac{\partial}{\partial z^\gamma}}_{0} - \nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\gamma}$$

$$= -\nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \left(\Gamma^\delta_{\alpha\gamma} \frac{\partial}{\partial z^\delta}\right)$$

$$= -\frac{\partial}{\partial \bar{z}^\beta}\left(g^{\delta\bar{\tau}} \frac{\partial g_{\gamma\bar{\tau}}}{\partial z^\alpha}\right) \frac{\partial}{\partial z^\delta}$$

$$R_{\alpha\bar{\beta}\bar{\eta}\gamma} = R^\delta_{\alpha\bar{\beta}\gamma} g_{\delta\bar{\eta}} = -g_{\delta\bar{\eta}} \frac{\partial}{\partial \bar{z}^\beta}\left(g^{\delta\bar{\tau}} \frac{\partial g_{\gamma\bar{\tau}}}{\partial z^\alpha}\right)$$

$$R_{\alpha\bar\beta\delta}^{\ \ \ \ \delta} = R_{\alpha\bar\beta\bar\gamma\delta}\, g^{\delta\bar\gamma} = -\frac{\partial}{\partial \bar z^\beta}\left(g^{\delta\bar\tau} \frac{\partial g_{\delta\bar\tau}}{\partial z^\alpha}\right)$$

$$= -\frac{\partial}{\partial \bar z^\beta}\frac{\partial}{\partial z^\alpha}\left(\log\det(g_{s\bar\gamma})\right)$$

$$R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}\right) = R\left(\frac{\partial}{\partial z^\alpha}, e_\gamma, \frac{\partial}{\partial \bar z^\beta}, e_\gamma\right) + R\left(\frac{\partial}{\partial z^\alpha}, Je_\gamma, \frac{\partial}{\partial \bar z^\beta}, Je_\gamma\right)$$

$$= -\sqrt{-1}\, R\left(\frac{\partial}{\partial z^\alpha}, e_\gamma, \frac{\partial}{\partial \bar z^\beta}, Je_\gamma\right) + \sqrt{-1}\, R\left(\frac{\partial}{\partial z^\alpha}, Je_\gamma, \frac{\partial}{\partial \bar z^\beta}, e_\gamma\right)$$

$$= -\sqrt{-1}\, R\left(\frac{\partial}{\partial z^\alpha}, e_\gamma, \frac{\partial}{\partial \bar z^\beta}, Je_\gamma\right) - \sqrt{-1}\, R\left(e_\gamma, \frac{\partial}{\partial \bar z^\beta}, \frac{\partial}{\partial z^\alpha}, Je_\gamma\right)$$

$$\overset{\text{Bianchi I}}{=} \sqrt{-1}\, R\left(\frac{\partial}{\partial \bar z^\beta}, \frac{\partial}{\partial z^\alpha}, e_\gamma, Je_\gamma\right)$$

$$= -R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}, \frac{1}{\sqrt2}(e_\gamma - \sqrt{-1}Je_\gamma), \frac{1}{\sqrt2}(e_\gamma + \sqrt{-1}Je_\gamma)\right)$$

$T^{1,0}M$ unitary basis

$$\begin{pmatrix} \frac{1}{\sqrt2}(e_1 - \sqrt{-1}Je_1) \\ \vdots \\ \frac{1}{\sqrt2}(e_m - \sqrt{-1}Je_m) \end{pmatrix} = (\psi_{st}) \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \vdots \\ \frac{\partial}{\partial z^m} \end{pmatrix}$$

$$\Rightarrow (g_{\alpha\bar\beta}) = (\psi_{st})^{-1} (\overline{\psi_{st}})^{-T}$$

$$R_{\alpha\bar\beta} = \mathrm{Ric}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}\right) = -R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}, \frac{\partial}{\partial z^s}, \frac{\partial}{\partial \bar z^t}\right)\ \psi_{\gamma s}\, \overline{\psi_{\gamma t}}$$

$$= R_{\alpha\bar\beta\bar t s}\, g^{s\bar t}$$

$$= R_{\alpha\bar\beta s}^{\ \ \ \ s}$$

$$= -\frac{\partial^2 \log\det(g_{s\bar t})}{\partial z^\alpha \partial \bar z^\beta}$$

$\rho(JX, JY) = \rho(X, Y) \Rightarrow \rho$ is $(1,1)$.

$$\rho = \rho\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) dz^\alpha \wedge d\bar{z}^\beta = Ric\left(J\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) dz^\alpha \wedge d\bar{z}^\beta$$

$$= -\sqrt{-1} \frac{\partial^2 \log\det(g_{s\bar{t}})}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta$$

$$= -\sqrt{-1}\, \partial\bar{\partial} \log\det(g_{s\bar{t}})$$

Holomorphic Bisectional Curvature:

σ_α: J-invariant plane $\left\{\frac{\partial}{\partial x^\alpha}, J\frac{\partial}{\partial x^\alpha}\right\}$

$$H(\sigma_\alpha, \sigma_\beta) = R\left(\frac{\partial}{\partial x^\alpha}, J\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, J\frac{\partial}{\partial x^\beta}\right)$$

$$= -4R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial z^\beta}, \frac{\partial}{\partial \bar{z}^\beta}\right)$$

Def. A Kähler manifold (M, J, g) is said to be of constant bisectional curvature if

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\lambda\left(g_{\alpha\bar{\beta}}\, g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}\, g_{\gamma\bar{\beta}}\right)$$

4.6 Examples

1. $M=\mathbb{C}^n$ $\quad \omega=\frac{\sqrt{-1}}{2}dz^\alpha\wedge d\bar{z}^\alpha$ $\quad$ flat.

2. $M=\mathbb{CP}^n$ $\quad U_0=\{[1,z^1,\cdots z^n]\}\subset\mathbb{CP}^n$

$$\omega_g=\frac{\sqrt{-1}}{2}\partial\bar\partial\log(1+|z|^2)$$

$$=\frac{\sqrt{-1}}{2}\left(\frac{dz^\alpha\wedge d\bar{z}^\beta}{1+|z|^2}\delta_{\alpha\beta}-\frac{\bar{z}^\alpha z^\beta dz^\alpha\wedge d\bar{z}^\beta}{(1+|z|^2)^2}\right)$$

$SU(n+1)$ acts transitively on $\mathbb{CP}^n$. $\quad T\in SU(n+1)$

$T^*g=g$.

$$\omega_g^n=\left(\frac{\sqrt{-1}}{2}\right)^n\frac{(dz^\alpha\wedge d\bar{z}^\alpha)^n}{(1+|z|^2)^{n+1}}$$

$$R_{\alpha\bar\beta}=-\frac{\partial^2}{\partial z^\alpha\partial\bar{z}^\beta}\log\left(\frac{1}{(1+|z|^2)^{n+1}}\right)=(n+1)g_{\alpha\bar\beta}$$ $\boxed{\text{Einstein}}$

$$R_{\alpha\bar\beta\gamma\bar\eta}=-(g_{\alpha\bar\beta}g_{\gamma\bar\eta}+g_{\alpha\bar\eta}g_{\gamma\bar\beta})$$ constant bisectional curvature $+1$

3. $M=B^n=\{z\in\mathbb{C}^n \mid |z|<1\}$ $\quad \omega_g=\frac{\sqrt{-1}}{2}\partial\bar\partial\log(1-|z|^2)$

constant bisectional curvature -1

Chapter 5 Laplace Operator

§1 Hodge $*$-operator

Def. The Hodge $*_g$-operator

$*: \Lambda^r M \to \Lambda^{m-r} M$ is defined by

$$\omega \wedge * \tau = \langle \omega, \tau \rangle_g \, dV_g \qquad \forall \omega\ \tau \in \Lambda^r M$$

[$\theta, \eta \in \Lambda^1 M$ $\quad \langle \theta, \eta \rangle_g = \theta(\frac{\partial}{\partial x^i}) \eta(\frac{\partial}{\partial x^j}) g^{ij}$

$$\langle \theta_1 \otimes \cdots \otimes \theta_r, \eta_1 \otimes \cdots \otimes \eta_r \rangle_g = \langle \theta_1, \eta_1 \rangle_g \cdots \langle \theta_r, \eta_r \rangle_g$$

$$\theta_1 \wedge \theta_2 = \theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1$$

$$\theta_1 \wedge \theta_2 (X, Y) = \det \begin{pmatrix} \theta_1(X) & \theta_1(Y) \\ \theta_2(X) & \theta_2(Y) \end{pmatrix}$$

$$\langle \omega, \tau \rangle_g = \sum_{i_1 < \cdots < i_r} \omega^{i_1 \cdots i_r} \tau_{i_1 \cdots i_r} \qquad \tau = \tau_{i_1 \cdots i_r} dx^{i_1} \cdots \wedge dx^{i_r}$$

$$= \frac{1}{r!} \sum \omega^{i_1 \cdots i_r} \tau_{i_1 \cdots i_r}. \qquad \omega^{i_1 \cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} \omega_{j_1 \cdots j_r}$$

Rmk. (1) $\{\frac{\partial}{\partial x^i}\}$ $\{dx^i\}$

$$* \tau = \sum_{j_{r+1} < \cdots < j_m} * \tau_{j_{r+1} \cdots j_m} dx^{j_{r+1}} \cdots \wedge dx^m$$

$$* \tau_{j_{r+1} \cdots j_m} = \sum_{i_1 < \cdots < i_r} \eta_{i_1 \cdots i_r j_{r+1} \cdots j_m} \tau^{i_1 \cdots i_r}$$

(2) $\{e_i\}$ orthonormal basis of TM

$\{\theta^i\}$ dual basis

$$\tau=\sum_{i_1<\cdots<i_r}\hat{\tau}_{i_1\cdots i_r}\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}\qquad 1=\theta^1\wedge\cdots\wedge\theta^m$$

$$*\tau=\sum_{j_{r+1}<j_m}\widehat{*\tau}_{j_{r+1}\cdots j_m}\theta^{j_{r+1}}\wedge\cdots\wedge\theta^{j_m}$$

$$\widehat{*\tau}_{j_{r+1}\cdots j_m}=\sum_{i_1<\cdots<i_r}\delta^{1\cdots m}_{i_1\cdots i_r j_{r+1}\cdots j_m}\tilde{\tau}^{i_1\cdots i_r}$$

(3) $*1=dV_g\quad *dV_g=1.$

$$**\alpha=(-1)^{r(m-r)}\alpha$$

$$*(\alpha+\beta)=*\alpha+*\beta\qquad *(f\alpha)=f*\alpha$$

$$\langle *\omega,*\tau\rangle_g=\langle\omega,\tau\rangle_g$$

Def $d:\Gamma(\Lambda^r M)\to\Gamma(\Lambda^{r+1}M)$ formal adjoint

$$d^*:\Gamma(\Lambda^r M)\to\Gamma(\Lambda^{r-1}M)$$

$$d^*\omega=(-1)^{m(r-1)+1}*d*\omega=-g^{ij}(\nabla_{\frac{\partial}{\partial x^i}}\omega)(\tfrac{\partial}{\partial x^j},\cdots)$$

$\tau\in\Gamma(\Lambda^{r-1}M)\quad\omega\in\Gamma(\Lambda^r M)$

$$\langle d\tau,\omega\rangle_g dV_g=d\tau\wedge*\omega=d(\tau\wedge*\omega)-(-1)^{r-1}\tau\wedge d*\omega$$

$$=d(\tau\wedge*\omega)-(-1)^{r-1+(m-r+1)(r-1)}\tau\wedge**d*\omega$$

$$=d(\tau\wedge*\omega)-(-1)^{m(r-1)}\langle\tau,*d*\omega\rangle dV_g$$

$$=d(\tau\wedge*\omega)+\langle\tau,d^*\omega\rangle dV_g$$

$$\Rightarrow\int_M\langle d\tau,\omega\rangle dV_g=\int_M\langle\tau,d^*\omega\rangle dV_g$$

E.g. α 1-form X dual vector field

$$\alpha = \alpha_i dx^i \qquad X = \alpha^i \frac{\partial}{\partial x^i} \qquad \alpha^i = g^{ij}\alpha_j.$$

$$\langle X, Y\rangle_g = \alpha(Y) \qquad \forall Y \in TM$$

$$d^*\alpha = -g^{ij}\left(\nabla_{\frac{\partial}{\partial x^i}}\alpha\right)\left(\frac{\partial}{\partial x^j}\right) = -g^{ij}\alpha_{j,i}$$

$$= -\mathrm{div}(X).$$

$$\Delta_g : \Gamma(\Lambda^r M) \to \Gamma(\Lambda^r M)$$

$$\Delta_g = -dd^* - d^*d$$

Rmk (1) $d \circ \Delta = \Delta \circ d.$ $\quad d^* \circ \Delta = \Delta \circ d^*$

$$*\Delta = \Delta *$$

(2) $\int_M \langle \Delta\omega, \tau\rangle dV_g = \int_M \langle \omega, \Delta\tau\rangle dV_g$

(3) M closed (i.e. compact, without boundary)

$$\Delta\omega = 0 \Leftrightarrow d\omega = 0.\ d^*\omega = 0$$

(4) f is a C^2 function.

$$\Delta f = -d^*df = g^{ij}\left(\nabla_{\frac{\partial}{\partial x^i}} df\right)\left(\frac{\partial}{\partial x^j}\right)$$

$$= g^{ij}\nabla df\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

$$= g^{ij} f_{,ji} = \frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial x^i}\right)$$

(5) trace Laplace operator

$$\operatorname{tr}_g \nabla^2 : \Gamma(\Lambda^r M) \to \Gamma(\Lambda^r M)$$

$$\operatorname{tr}_g \nabla^2 \alpha = g^{ij}\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \alpha - \nabla_{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}} \alpha \right)$$

Weitenböck Formula:

$$\Delta \alpha (X_1, \cdots X_r) = \operatorname{tr}_g \nabla^2 \alpha (X_1, \cdots X_r) - \sum_{t=1}^{r} (-1)^t \sum_{i=1}^{m} (R(e_i, X_t)\alpha)(e_i, X_1, \cdots \hat{X_t} \cdots)$$

$\alpha \in A^1 M \Rightarrow \Delta \alpha (Y) = (\operatorname{tr}_g \nabla^2 \alpha)(Y) - \operatorname{Ric}(X, Y)$

X is dual of α

α harmonic form $\Leftrightarrow \Delta \alpha = 0$

$$\Delta \underbrace{|\alpha|_g^2}_{f} = g^{ij} (\nabla_{\frac{\partial}{\partial x^i}} df)\left(\frac{\partial}{\partial x^j}\right)$$

$$= g^{ij}\left[\left(\frac{\partial}{\partial x^i} \left(df\left(\frac{\partial}{\partial x^j}\right)\right)\right) - df\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)\right]$$

$$= g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) |\alpha|_g^2$$

$$\overset{\text{compute}}{=} 2|\nabla \alpha|^2 + 2 \operatorname{Ric}(\alpha^\#, \alpha^\#)$$

$\operatorname{Ric} > 0$. M closed. then $\alpha^\# = 0 \Leftrightarrow \alpha = 0$

5.2 Complex Laplace Operator on Hermitian Manifold

(M^{2n}, J, g) $\quad \omega(X,Y)=g(JX,Y)$

$$\theta \wedge * \tau = \langle \theta, \tau \rangle_g \frac{\omega^n}{n!}$$

$$*\omega^p = \frac{p!}{(n-p)!}\omega^{n-p}$$

$$*: \Lambda^{p,q}(M) \to \Lambda^{n-q,n-p}(M)$$

Def. The Lefschetz operator L_ω associate to ω is $L_\omega: \Lambda^p M \to \Lambda^{p+2} M$

$$\theta \mapsto \theta \wedge \omega.$$

The contraction operator

$$\Lambda_\omega: \Lambda^p M \to \Lambda^{p-2} M. \quad \text{adjoint of } L_\omega$$

$$\langle \Lambda_\omega \theta, \tau \rangle = \langle \theta, L_\omega \tau \rangle$$

$$\Lambda_\omega \theta = (-1)^p * L_\omega * \theta$$

Rmk. ① Λ_ω maps $\Lambda^{p,q} M$ to $\Lambda^{p-1,q-1} M$

② L_ω. Λ_ω are real operators

$$\overline{L_\omega(\theta)} = L_\omega(\bar\theta). \qquad \overline{\Lambda_\omega(\theta)} = \Lambda_\omega \bar\theta$$

③ $\theta \in \Lambda^2(M)$. $\quad \Lambda_\omega \theta \cdot 1 = \langle \theta, \omega \rangle_g$

$$\Lambda_\omega \theta \frac{\omega^n}{n!} = \langle \theta, \omega \rangle_g \frac{\omega^n}{n!}$$

$$= \theta \wedge * \omega = \theta \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$\Lambda_\omega \omega = n$

$L_\omega^{(n-1)}\theta = \theta \wedge \omega^{n-1} = \frac{1}{n}\Lambda_\omega\theta\, \omega^n = (n-1)!\, \Lambda_\omega\theta \frac{\omega^n}{n!}$

$\Lambda_0^p(M) = \{\theta\in\Lambda^p(M) \mid \Lambda_\omega\theta = 0\}$: primitive forms

$\Lambda_0^{p,q}(M) = \Lambda_0^{p+q}(M) \cap \Lambda^{p,q}(M)$

$\theta\in\Lambda_0^{2p}(M) \qquad g(\theta,\omega^p)=0 \Leftrightarrow g(\theta, L_\omega^p)=0$

$\Leftrightarrow g(\Lambda_\omega^p\theta, 1) = 0 \Leftrightarrow L_\omega^{n-p}\theta = 0$

$\theta\wedge *\omega^p = g(\theta,\omega^p)\frac{\omega^n}{n!}$

$= \frac{p!}{(n-p)!}\theta\wedge\omega^{n-p} = \frac{p!}{(n-p)!} L_\omega^{n-p}(\theta)$

$\Lambda^2(M) = \Lambda_0^2(M)\oplus \mathbb{C}\omega = \Lambda^{2,0}(M)\oplus\Lambda^{1,1}(M)\oplus\Lambda^{0,2}(M)\oplus \mathbb{C}\cdot\omega$

$\theta\in\Lambda^1(M) \qquad *\theta = \frac{-1}{(n-1)!} L_\omega^{n-1}(J(\theta))$

$*\theta^{1,0} = -\frac{\sqrt{-1}}{(n-1)!}\theta^{1,0}\wedge\omega^{n-1} \qquad *\theta^{0,1} = \frac{\sqrt{-1}}{(n-1)!}\theta^{0,1}\wedge\omega^{n-1}$

$\theta^{1,0}\wedge *\overline{\theta^{1,0}} = g(\theta^{1,0},\overline{\theta^{1,0}})\frac{\omega^n}{n!} = |\theta^{1,0}|_H^2 \frac{\omega^n}{n!}$

$\parallel$

$\frac{\sqrt{-1}}{(n-1)!}\theta^{1,0}\wedge\overline{\theta^{1,0}}\wedge\omega^{n-1} \qquad \Rightarrow \sqrt{-1}\,\Lambda_\omega\theta^{1,0}\wedge\overline{\theta^{1,0}} = |\theta^{1,0}|_H^2$

$\sqrt{-1}\,\Lambda_\omega\theta^{0,1}\wedge\overline{\theta^{0,1}} = -|\theta^{0,1}|_H^2$

$d=\partial+\bar\partial$ $\qquad$ $\partial^*: \Lambda^{p,q}\to\Lambda^{p-1,q}$ $\qquad$ $\bar\partial^*: \Lambda^{p,q}\to\Lambda^{p,q-1}$

$\partial^* = -*\bar\partial*$ $\qquad$ $\bar\partial^* = -*\partial*$

$(\bar\partial\theta,\tau)=(\theta,\partial^*\tau)$ $\qquad$ $(\partial\theta,\tau)=(\theta,\bar\partial^*\tau)$

$\Delta^{\partial} = -\partial\partial^*-\partial^*\partial$ $\qquad$ $\Delta^{\bar\partial} = -\bar\partial\bar\partial^*-\bar\partial^*\bar\partial$

Lemma $\quad \partial^*\theta = \sqrt{-1}\Lambda_\omega\bar\partial\theta + \frac{\sqrt{-1}}{(n-1)!}*(\bar\partial(\omega_g^{n-1})\wedge\theta)$ $\quad \theta\in\Lambda^{1,0}M$

$\bar\partial^*\theta = -\sqrt{-1}\Lambda_\omega\partial\theta - \frac{\sqrt{-1}}{(n-1)!}*(\partial(\omega_g^{n-1})\wedge\theta)$ $\quad \theta\in\Lambda^{0,1}M$

Prop (Kähler Identities) $\quad (M,J,g)$ Kähler

(1) $[\bar\partial, L_\omega]=[\partial,L_\omega]=0$ $\qquad$ $[\bar\partial^*,\Lambda_\omega]=[\partial^*,\Lambda_\omega]=0$

(2) $[\bar\partial^*, L_\omega]=\sqrt{-1}\partial$. $\qquad$ $[\partial^*, L_\omega]=-\sqrt{-1}\bar\partial$

$[\Lambda_\omega,\bar\partial]=-\sqrt{-1}\partial^*$ $\qquad$ $[\Lambda_\omega,\partial]=\sqrt{-1}\bar\partial^*$.

(3) $\Delta^{\partial}=\Delta^{\bar\partial}=\frac{1}{2}\Delta$. $\quad$ Δ commutes with $\partial,\bar\partial,\partial^*,\bar\partial^*, L_\omega,\Lambda_\omega$

5.3 Generalization to Bundle Valued Form

E complex bundle $\quad (M, J, g) \quad \omega$

1. H: Hermitian metric on E

extends to $\quad H: \Lambda^p(E) \times \Lambda^q(E) \to \Lambda^{p+q}(E)$

$$H(\theta \otimes S, \eta \otimes T) = H(S,T)\, \theta \wedge \bar{\eta}$$

$$g(\theta \otimes S, \eta \otimes T) = g(\theta, \bar{\eta}) \cdot H(S,T)$$

$$H(a, * b) = g(a,b)\, dV_g$$

$$g(\Lambda_\omega c, a) = g(c, L_\omega a)$$

$F \in A^p(\mathrm{End}\, E)$. $\quad F^{*H}$: H-adjoint of F

$$|F|^2 dV_g = \mathrm{tr}(F \wedge * F^{*H})$$

$D_A = d + A \quad$ H-unitary connection in E

$$D_A = \partial_A + \bar{\partial}_A \qquad D_A^* = -* D_A * \qquad \bar{\partial}_A^* = -* \partial_A *$$

$$\Delta_A = -D_A^* D_A - D_A D_A^* \qquad \partial_A^* = -* \bar{\partial}_A *$$

In Kähler case. $\quad a \in A^{1,0}(E) \quad \partial_A^* a = \sqrt{-1}\, \Lambda_\omega \bar{\partial}_A a$

$a \in A^{0,1}(E) \quad \bar{\partial}_A^* a = -\sqrt{-1}\, \Lambda_\omega \partial_A a$

$a \in A^0(E)$
$$\Delta_A^{\bar{\partial}} = -\bar{\partial}_A^* \bar{\partial}_A = \sqrt{-1}\, \Lambda_\omega \partial_A \bar{\partial}_A$$
$$\Delta_A^{\partial} = -\sqrt{-1}\, \Lambda_\omega \bar{\partial}_A \partial_A$$

$$F_A = \bar\partial_A^2 + \partial_A^2 + \partial_A\bar\partial_A + \bar\partial_A\partial_A$$

$$\Rightarrow \Delta_A^{\bar\partial} - \Delta_A^{\partial} = \underbrace{\sqrt{-1}\Lambda_\omega F_A} \rightarrow \text{mean curvature}$$

$$\Delta_A = \Delta_A^{\bar\partial} + \Delta_A^{\partial} = \sqrt{-1}\Lambda_\omega(\partial_A\bar\partial_A - \bar\partial_A\partial_A)$$

$$= 2\Delta_A^{\partial} + \sqrt{-1}\Lambda_\omega F_A$$

$$= 2\Delta_A^{\bar\partial} - \sqrt{-1}\Lambda_\omega F_A$$