Chapter 6 Hodge Theory

6.1 De Rham Cohomology Group

(M, g) compact oriented Riemannian manifold

$Z^r(M,\mathbb{C})$ $\quad B^r(M,\mathbb{C})$ $\quad H^r_{DR}(M,\mathbb{C}) = \dfrac{Z^r(M,\mathbb{C})}{B^r(M,\mathbb{C})}$

Thm. (De Rham) $H^k(M,\mathbb{C}) \simeq H^k_{DR}(M,\mathbb{C})$

Def. $\mathcal{H}^r(M,g) = \{\eta \in A^r(M) : \Delta\eta = 0\}$

Thm (Hodge Decomposition) $\forall\, 0 \le r \le n$.

$\dim \mathcal{H}^r(M,g) < \infty$. and we have the orthonormal direct sum

$$\begin{aligned} A^r(M) &= \Delta(A^r(M)) \oplus \mathcal{H}^r \\ &= dd^*(A^r(M)) \oplus d^*d(A^r(M)) \oplus \mathcal{H}^r \\ &= d(A^{r-1}(M)) \oplus d^*(A^{r+1}(M)) \oplus \mathcal{H}^n \end{aligned}$$

Consequently. $\Delta\omega = \alpha$ has solution $\Leftrightarrow \alpha$ is orthogonal to $\mathcal{H}^r$

Pf global inner product $(\eta_1, \eta_2) = \int_M \eta_1 \wedge * \eta_2$.

Δ self-adjoint $\Rightarrow \Delta(A^r(M)) \perp \mathcal{H}^r$

Functional Analysis viewpoint:

A solution $\Delta\omega=\eta \Leftrightarrow$ linear functional on $A^r(M)$

$$l(\beta)=(\omega,\beta)$$

then $l(\Delta\theta)=(\omega,\Delta\theta)=(\eta,\theta)$

A weak solution is a bounded linear functional

$l: \Lambda^r(M)\to\mathbb{R}$ s.t. $l(\Delta\theta)=(\eta,\theta)$

Thm (Regularity Thm) $\eta\in A^r(M)$ l a weak solution of $\Delta\omega=\eta$. Then $\exists\,\omega\in A^r(M)$ s.t. $l(\beta)=(\omega,\beta)$
i.e. $\Delta\omega=\eta$

Thm (Compactness Thm) $\{\eta_n\}$ a sequence in $A^r(M)$
s.t. $\|\eta_n\|\le c$, $\|\Delta\eta_n\|\le c$.
Then $\exists$ subsequence of $\{\eta_n\}$ that is a Cauchy sequence.

(1) $\mathcal{H}^r(M,g)$ is finite dimensional

If not. choose orthonormal sequence $\{\eta_n\}$

$$\sqrt{2}=\lim_{n\to\infty}\|\eta_{n+1}-\eta_n\|=0 \text{ (by compactness Thm)}$$

Contradiction!

(2) $\omega_1,\cdots\omega_k$ orthonormal basis of $\mathcal{H}^r(M)$

$$\omega=\omega^{\perp}+\sum_{i=1}^{k}\langle\omega,\omega_i\rangle\omega_i \qquad \omega^{\perp}\in(\mathcal{H}^r)^{\perp}$$

$$A^r(M)=(\mathcal{H}^r)^{\perp}\oplus\mathcal{H}^r$$

(3) $\exists c>0$ s.t. $\|\eta\|\le c\|\Delta\eta\|$ $(\forall\eta\in(\mathcal{H}^r)^{\perp})$

If not. $\exists\eta_i\in(\mathcal{H}^r)^{\perp}$ s.t. $\|\eta_i\|=1$ $\|\Delta\eta_i\|\to 0$

Then $\exists$subsequence $\{\eta_i\}$ is Cauchy

Then (η_j,ψ) is Cauchy $(\psi\in A^r(M))$

$l(\psi)=\lim_{j\to\infty}(\eta_j,\psi)$ is bounded. $\|l(\psi)\|\le\|\psi\|$

$l(\Delta\psi)=\lim_{j\to\infty}(\Delta\eta_j,\psi)=0 \Rightarrow l$ is a weak solution of $\Delta\omega=0$

By regularity Thm. $\exists\omega\in A^r(M)$ s.t. $l(\psi)=(\omega,\psi)$, $\Delta\omega=0$

$\eta_j\xrightarrow{\text{Cauchy}}\eta\in L^2$

Then $\forall\psi\in A^r(M)$ $(\eta-\omega,\psi)=0 \Rightarrow \eta=\omega$

$\|\omega\|=1$ $\omega\in(\mathcal{H}^r(M))^{\perp}$ Contradiction!

(4) $\Delta(A^r(M))=(\mathcal{H}^r)^{\perp}$

(i) $\Delta(A^r(M))\subset(\mathcal{H}^r)^{\perp}$. trivial

(ii) Given $\eta\in(\mathcal{H}^r)^{\perp}$

Def $l:\Delta(A^r(M))\to\mathbb{R}$

$l(\Delta\psi)=(\eta,\psi)$ $(\eta\in(\mathcal{H}^r)^{\perp} \Rightarrow$ well defined$)$

Bounded: $|l(\Delta\psi)| = \| l(\Delta(\psi^\perp + \pi(\psi)))\|$

$= \| l(\Delta\psi^\perp)\| = |(\eta, \psi^\perp)|$

$\leq \|\eta\|\|\psi^\perp\| \leq C\|\eta\|\|\Delta\psi\|$

Hahn-Banach Thm $\Rightarrow \exists$ continuous extension of l to $A^r(M)$

$\Rightarrow l$ is a weak solution $\Delta\omega = \eta$.

Then by regularity Thm $\exists \omega \in A^r(M)$ $d\omega = \eta$

#

For closed form ω. i.e. $d\omega = 0$

$$\omega = d\omega_1 + \omega^H + d^*\omega_2$$

$$(d^*\omega_2, d^*\omega_2) = (d^*\omega, \omega) = (\omega, d\omega) = 0$$

Thus $d^*\omega_2 = 0 \Rightarrow \omega = d\omega_1 + \omega^H$

If $d^*\omega = 0 \Rightarrow \omega = d^*\omega_2 + \omega^H$.

Prop (Hodge Isomorphism) $\psi: H^r \to H^r_{DR}(M, \mathbb{C})$

$\omega \mapsto [\omega]$

is an isomorphism.

Cor. $\dim H^k_{DR}(M, \mathbb{C}) < \infty$. ($M$ compact orientable)

$* : A^r(M) \to A^{m-r}(M)$ $\quad *^2 = (-1)^{m(m-r)}$

$*\Delta = \Delta *$ $\quad \omega$ is harmonic $\Leftrightarrow *\omega$ is harmonic

Prop (Poincaré duality) $\mathcal{H}^r(M,g) \cong \mathcal{H}^{m-r}(M,g)$

$$b_r(M) = \dim H^r_{DR}(M,\mathbb{C}) = b_{m-r}(M)$$

6.2 Dolbeault Theory

(M,J,g) compact Hermitian manifold

$$\bar\partial : A^{p,q}(M) \to A^{p,q+1}(M)$$

Dolbeault cohomology group $H^{p,q}(M,J) = \dfrac{Z^{p,q}(M,J)}{B^{p,q}(M,J)}$

It depends on the complex structure J.

$$\mathcal{H}^{p,q}(M) = \{\omega \in A^{p,q}(M) \mid \Delta^{\bar\partial}\omega = 0\}$$

ω is $\bar\partial$-harmonic $\Leftrightarrow \bar\partial\omega = \bar\partial^*\omega = 0$

Thm (Dolbeault Decomposition)

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \bar\partial^* A^{p,q+1}(M) \oplus \bar\partial A^{p,q-1}(M)$$

$\bar\partial\omega = 0 \Rightarrow \omega = \bar\partial\omega_1 + \omega^H$

Cor. ① A $(p,0)$ form is holomorphic $\Leftrightarrow$ $\bar\partial$ harmonic

② $\psi: \mathcal{H}^{p,q}(M) \to H^{p,q}(M,\mathbb{J})$ is isomorphism
$\omega \mapsto [\omega]$

Def (Hodge number) $h^{p,q} = \dim^{\mathbb{C}} \mathcal{H}^{p,q}(M)$

Prop (Serre Duality) $\mathcal{H}^{p,q}(M) \simeq \mathcal{H}^{n-p,n-q}(M)$

Pf. $\bar{*}: A^{p,q}(M) \to A^{n-p,n-q}(M)$

$\bar{*}\omega = *\bar\omega$

$\bar{*}\Delta^{\bar\partial}(\omega) = *\,\overline{(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\omega} = *(\partial\partial^* + \partial^*\partial)\bar\omega$

$= -*(\partial*\bar\partial* + *\bar\partial*\partial)\bar\omega$

$= \bar\partial^*\bar\partial(\bar{*}\omega) - *^2\bar\partial*\partial\bar\omega = \Delta^{\bar\partial}(\bar{*}\omega)$

Thus $\bar{*}: \mathcal{H}^{p,q}(M) \to \mathcal{H}^{n-p,n-q}(M)$ is isomorphism #.

In Kähler case $\Delta = 2\Delta^{\bar\partial}$

$\mathcal{H}^{p,q}(M) \subset \mathcal{H}^{p+q}(M)$

$\mathcal{H}^{k}(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$

$\Delta(\bar\omega) = \overline{\Delta\omega}$ "$-$": $\mathcal{H}^{p,q}(M) \to \mathcal{H}^{q,p}(M)$.

Prop. $b_k = \sum_{p+q=k} h^{p,q}$ $\quad h^{p,q} = h^{q,p}$ $\quad \underbrace{h^{p,p} \geq 1 \quad (1 \leq p \leq n)}$

for kähler manifold

$\downarrow$

kähler form $\omega \in \mathcal{H}^{1,1}$

$(\nabla J = 0 \Rightarrow \nabla \omega = 0)$

k is odd $\Rightarrow b_k$ is even

k is even $\Rightarrow b_k \geq 1$.

e.g Hopf manifold $S^1 \times S^{2n+1}$ is non-kähler

Lemma (Global $\partial\bar{\partial}$) (M, J, g) kähler. $\phi_1, \phi_2 \in A^{1,1}(M, \mathbb{C}) \cap A^2(M, \mathbb{R})$

ϕ_1 is cohomologous to ϕ_2 ($\phi_1 = \phi_2 + d\psi$, ψ real)

Then $\exists f \in C^\infty(M, \mathbb{R})$ s.t. $\phi_1 - \phi_2 = \sqrt{-1}\partial\bar{\partial} f$

Pf $\psi = \psi^{1,0} + \psi^{0,1}$ $\qquad \partial\psi^{1,0} = \bar{\partial}\psi^{0,1} = 0$

Consider equation $-\frac{1}{2}\Delta u = \bar{\partial}^* \bar{\partial} u = \bar{\partial}^* \psi^{0,1}$

$\bar{\partial}^*(\psi^{0,1} - \bar{\partial} u) = 0$ $\quad = \sqrt{-1}\Lambda_\omega \partial(\psi^{0,1} - \bar{\partial} u)$ $\qquad [\int_M \bar{\partial}^* \psi^{0,1} \omega^n = 0$

$\Rightarrow$ has solution$]$

$\bar{\partial}(\psi^{0,1} - \bar{\partial} u) = 0$

One can prove that $\partial(\psi^{0,1} - \bar{\partial} u) = 0$

Then $\phi_1 - \phi_2 = d\psi = \partial\psi^{0,1} + \bar{\partial}\psi^{1,0} = \partial\bar{\partial} u + \bar{\partial}\partial\bar{u}$

$= 2\sqrt{-1}\partial\bar{\partial}\,\mathrm{Im}\,u = \sqrt{-1}\partial\bar{\partial} f$ #

Chapter 7 Chern Classes

[Chern-weil Theory]

V complex vector subspace

k-multilinear symmetric map $\tilde{p}: V\times V \cdots \times V \to \mathbb{C}$ $\quad S^k(V)^*$

$B\in V$. $\tilde{p}\in S^k(V)^*$ $\quad P(B)=\tilde{p}(B,B,\cdots B)$

homogeneous polynomial of degree k: $S_k(V)$

$$P\in S_k(V) \quad \tilde{p}(B_1\cdots B_k)=\frac{(-1)^k}{k!}\sum_{j=1}\sum_{i_1<\cdots<i_j}(-1)^j P(B_{i_1}+\cdots B_{i_j})\in S^k(V)^*$$

Consider $V=gl(r,\mathbb{C})$

Def. $\tilde{p}\in S^k(gl(r,\mathbb{C}))^*$ is called invariant if $\to \tilde{I}_k(gl(r,\mathbb{C}))$

$\forall g\in GL(r,\mathbb{C})$ $\quad A_i\in gl(r,\mathbb{C})$

$$\tilde{p}(gA_1g^{-1},\cdots,gA_kg^{-1})=\tilde{p}(A_1,\cdots A_k)$$

$$P(gAg^{-1})=P(A)\longrightarrow I_k(gl(r,\mathbb{C}))$$

Eg. E complex vector bundle. $\operatorname{rank}(E)=r$.

extend $\tilde{p}\in\tilde{I}_k(gl(r,\mathbb{C}))$ to $A^*(M,\operatorname{End}(E))$

Prop. $\tilde{\varphi} \in \tilde{I}_k(gl(r,\mathbb{C}))$ then for any vector bundle E of rank r and any partition $m = i_1 + \cdots i_k$

$\exists$ an induced k-linear map

$$\hat{\varphi} = (\Lambda^{i_1}_M \otimes \mathrm{End}(E)) \times \cdots (\Lambda^{i_k}_M \otimes \mathrm{End}(E)) \to \Lambda^m_{\mathbb{C}} M$$

$$\hat{\varphi}(\alpha_1 \otimes \theta_1, \cdots, \alpha_k \otimes \theta_k) = (\alpha_1 \wedge \cdots \wedge \alpha_k)\, \tilde{\varphi}(\theta_1, \cdots \theta_k)$$

Lemma $\forall \eta_j \in A^{i_j}(M, \mathrm{End}(E))$

$$d\hat{\varphi}(\eta_1, \cdots \eta_k) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l}\, \hat{\varphi}(\eta_1, \cdots D\eta_j, \cdots \eta_k)$$

Pf. $D = d + A$ $\qquad D\eta = d\eta + [A, \eta]$

$$d\hat{\varphi}(\eta, \cdots \eta) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l}\, \hat{\varphi}(\eta_1, \cdots, D\eta_j - [A, \eta_j], \cdots \eta_k)$$

Let $\eta_j = \alpha_j \otimes \theta_j$ $\qquad A = \beta \otimes B$

$$\sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l}\, \hat{\varphi}(\eta_1, \cdots [A, \eta_j] \cdots \eta_k) \qquad [A,\eta_j] \to \beta \wedge \alpha_j \otimes [B, \theta_j]$$

$$= \sum_{j=1}^{k} \beta \wedge \alpha_1 \cdots \wedge \alpha_{j-1} \wedge \alpha_j, \cdots \wedge \alpha_k\, \tilde{\varphi}(\theta_1, \cdots [B, \theta_j], \theta_k) \qquad [B,\theta_j] = \frac{d}{dt}(e^{tB}\theta_j e^{-tB})\Big|_{t=0}$$

$$= \beta \wedge \cdots \wedge \alpha_k \frac{d}{dt}\Big|_{t=0} \tilde{\varphi}(e^{tB}\theta_1 e^{-tB} \cdots e^{tB}\theta_k e^{-tB})$$

$\overset{\text{invariant}}{=} 0$

#

$\varphi(F_D) = \tilde{\varphi}(F_D, \cdots F_D) \in A_{\mathbb{C}}^{2k}(M)$ is closed

$$d\varphi(F_D) = \sum_j \tilde{\varphi}(F_D, \cdots DF_D \cdots F_D) = 0$$

Lemma $\varphi \in I_k(gl(r,\mathbb{C}))$ D, D' are two connections on E, then $[\varphi(F_D)] = [\varphi(F_{D'})] \in H_{DR}^{2k}(M,\mathbb{C})$

pf. $D' = D + A$. $A \in A^1(\mathrm{End}\, E)$

$$F_{D'} = (D+A)^2 = D^2 + D\circ A + A\circ D + A\wedge A$$
$$= F_D + D(A) + A\wedge A$$

$$\varphi(F_{D+tA}) = \varphi(F_D) + kt\tilde{\varphi}(F_D, \cdots F_D, D(A)) + o(t^2)$$

$$\frac{d}{dt}\varphi(F_{D+tA})\Big|_{t=t_0} = \frac{d}{ds}\varphi(F_{D+t_0A+sA})\Big|_{s=0} = k\tilde{\varphi}(F_{D_{t_0}}, \cdots F_{D_{t_0}}, D_{t_0}(A))$$
$$= kd\tilde{\varphi}(F_{D_{t_0}}, \cdots F_{D_{t_0}}, A)$$

$$\varphi(F_{D'}) - \varphi(F_D) = \int_0^1 \frac{d}{dt}\varphi(F_{D+tA})\, dt$$

#

Chern-weil homomorphism $I_k(gl(n,\mathbb{C})) \to H_{DR}^{2k}(M,\mathbb{C})$

$\varphi \mapsto [\varphi(F_D)]$

$$\det(I + tB) = 1 + \sum_{k=1} t^k \Phi_k(B)$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^k \frac{1}{k!} \sum \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} F^{i_1}_{j_1} \wedge \cdots F^{i_k}_{j_k}$$

chern form: $C_k(E,D) = \Phi_k\left(\frac{\sqrt{-1}}{2\pi} F_D\right) \in A_{\mathbb{C}}^{2k}(M)$

chern class: $C_k(E) = [C_k(E,D)] \in H_{DR}^{2k}(M,\mathbb{C})$

Chern character $\operatorname{tr}(e^{tB}) = P_0(B) + tP_1(B) + \cdots$

k-th chern: $ch_k(E) = P_k\left(\frac{\sqrt{-1}}{2\pi} F_D\right) \in A_{\mathbb{C}}^{2k}(M)$

$= \operatorname{tr}\left(\frac{\sqrt{-1}}{2\pi} F_D \wedge \cdots \wedge \frac{\sqrt{-1}}{2\pi} F_D\right)$

Rmk. ① All the characteristic classes are real.

complex vector bundle. H. Hermitian connection D s.t. $DH=0$

$$D = d + A \qquad A^{*H} = -A$$

$$(F_A)^* = (dA + A\wedge A)^{*H} = \bar{H}^{-1}(dA)\bar{H} - \bar{H}^{-1}\bar{A}^T \wedge \bar{A}^T \bar{H}$$

$$dH = \bar{A}^T H + H\bar{A} \quad \Rightarrow dH \cdot H^{-1} = A^T + H\bar{A}H^{-1}$$

$$\bar{H}^{-1} d\bar{H} = \bar{H}^{-1}\bar{A}^T\bar{H} + A$$

$$\Rightarrow (F_A)^* = -F_A \qquad (\sqrt{-1}F_A)^{*H} = \sqrt{-1}F_A$$

By invariance, choose orthonormal basis of E

$$\bar{F}_A^T = -F_A$$

then $c(E,D) = \det\left(I + \frac{\sqrt{-1}}{2\pi} F_D\right) = \overline{\det\left(I + \frac{\sqrt{-1}}{2\pi} F_D\right)}$

$= \overline{c(E,D)}$.

thus $c_k(E) \in H_{DR}^*(M, \mathbb{R})$.

② If E is a holomorphic bundle, one can choose

chern connection D. $F_D \in A^{1,1}(M, \operatorname{End}(E))$ $c_k(E,D) \in A^{1,1}(M)$

$\Rightarrow c_k(E) \in H^{2k}(M, \mathbb{R}) \cap H^{k,k}(M)$

Prop E_1, E_2 are complex vector bundles, then

(a) $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$

(b) $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$

$E_1 \otimes E_2$: $D^{\otimes}(s_1 \otimes s_2) = D_1 s_1 \otimes s_2 + s_1 \otimes D_2 s_2$

$$D^{\otimes} = D_1 \otimes Id_{E_2} + Id_{E_1} \otimes D_2$$

$$\Rightarrow F_{D^{\otimes}} = F_{D_1} \otimes Id_{E_2} + Id_{E_1} \otimes F_{D_2}$$

E^*: $D(e_1, \cdots e_r) = (e_1, \cdots e_r) A$. $\quad D^*(\theta^1, \cdots \theta^r) = (\theta^1 \cdots \theta^r) A^*$

$$D^* \theta^i(e_j) = -\theta^i(De_j) = -\theta^i(A_j^k e_k) = -A_j^i$$

$$A^{*\,i}_{\ k} \theta^k(e_j) \Rightarrow A^{*\,j}_{\ i} = -A^i_j \qquad A^* = -A^T$$

$$F_{D^*} = dA^* + A^* \wedge A^* = -dA^T + A^T \wedge A^T = -F_D^T$$

$$\Rightarrow c_j(E^*) = (-1)^j c_j(E)$$

Prop $f: M \to N$ (E, D) on N. f^*E is pull back bundle. Then locally the connection $f^*D = d + f^*(A)$

$$F_{f^*D} = f^* F_D. \qquad c(f^*E) = f^* c(E)$$

One can checks:

(i) $ch(E) = \mathrm{rank}(E) + ch_1(E) + ch_2(E) + ch_3(E) + \cdots$

$$= \mathrm{rank}(E) + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)}{6}$$

(ii) $C_i(E \otimes L) = \sum_{j=0}^{i} \binom{\mathrm{rank}(E)-j}{i-j} C_j(E) C_1(L)^{i-j}$ L is a line bundle

$$ch(E_1 \otimes E_2) = ch(E_1)\, ch(E_2)$$

Def. The chern class of a complex manifold (M, J, g) is $C_k(M) \triangleq C_k(TM) \in H^{2k}(M, \mathbb{R})$

Rmk. (M, J, g) kähler. ∇ Levi-Civita

$$C_1(M) = \left[\frac{\sqrt{-1}}{2\pi} \mathrm{tr} F_\nabla\right]$$

$\{z_\alpha\}$ $g_{\alpha\bar\beta} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}\right)$

$$\nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\beta} = \Gamma^\gamma_{\beta\alpha} \frac{\partial}{\partial z^\gamma}. \qquad \Gamma^\gamma_{\beta\alpha} = g^{\gamma\bar\delta} \frac{\partial g_{\beta\bar\delta}}{\partial z^\alpha}$$

$$\sqrt{-1}\, \mathrm{tr} F_\nabla = \sqrt{-1}\, \bar\partial \partial \log\det(g_{\bar\alpha\beta}) = \rho \quad (\text{Ricci form})$$

$$\Rightarrow \rho_\omega \in 2\pi C_1(M)$$

Chapter 8 Sheaf Theory

8.1 Presheaves and Sheaves

Def. Presheaf, sheaf, module-sheaf, analytic sheaf

Thm $(M, \mathcal{S})$ be a connected $\mathcal{S}$-manifold, then

(isomophism class of) $\mathcal{S}$-bundle of M

$\Longleftrightarrow$ (isomorphism class of) locally free sheaves of $\mathcal{S}$-modules

Def. Coherent sheaf

8.2 Čech Cohomology

8.3 Resolution of Sheaf

8.4 Cohomology Theory

Chapter 9 Stability of Holomorphic Vector Bundles

9.1 Determinant Bundle

E holomorphic bundle on Kähler manifold (M, ω)

ω-degree $\deg_\omega(E) = \int_M c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = c_1(E)[\omega]^{n-1}$

ω-slope $\mu_\omega(E) = \dfrac{\deg_\omega(E)}{\operatorname{rank}(E)}$

$\det(E) = \Lambda^r E$: line bundle

Prop $c_1(E) = c_1(\det(E))$

pf. D a connection on E. it induces $\hat{D}$ on $\det(E)$

$\hat{D}(e_1, \cdots e_r) = (De_1) \wedge \cdots \wedge e_r + \cdots + e_1 \wedge \cdots \wedge (De_r)$

$F_{\hat{D}}(e_1, \cdots e_r) = (F_D e_1) \wedge \cdots \wedge e_r + \cdots + e_1 \wedge \cdots \wedge (F_D e_r)$

$= \operatorname{tr} F_D (e_1 \wedge \cdots \wedge e_r) \Rightarrow F_{\hat{D}} = \operatorname{tr} F_D$. #

For Riemann surface M. E is stable if

for every proper subbundle E' of E. $\operatorname{rank} E' < \operatorname{rank} E$

we have $\mu(E') < \mu(E)$

$\mathcal{F}$ coherent sheaf $\qquad \mathcal{F}^* = Hom(\mathcal{F}, \mathcal{O})$

$\sigma : \mathcal{F} \to \mathcal{F}^{**}$ natural homomorphism

$\operatorname{Ker}\sigma \hookrightarrow \mathcal{F}$: coherent subsheaf. $\mathcal{T}(\mathcal{F})$: torsion subsheaf

$\mathcal{T}(\mathcal{F}) = 0$: torsion free

σ bijective: reflexive

Prop (1) locally free $\to$ reflexive $\to$ torsion free

(2) coherent subsheaf of a torsion-free sheaf is also torsion free

(3) For any coherent sheaf $\mathcal{F}$. $\mathcal{F}^*$ is reflexive

(4) $\mathcal{F}$ torsion free, then $\mathcal{F}$ is locally free on $M \setminus \Sigma$

Σ is an analytic set of codim at least 2

(5) $\mathcal{F}$ reflexive. . . . . _

. . _ . . . _ 3

Def. $\mathcal{F}$ is normal. if every open set $U \subset M$ and any analytic subset $\Sigma \subset U$ of codim at least 2.

$\Gamma(U, \mathcal{F}) \to \Gamma(U - \Sigma, \mathcal{F})$ is an isomorphism

Prop. (1) reflexive $\Leftrightarrow$ torsion free + normal

(2) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact.

$\mathcal{F}$ reflexive, $\mathcal{F}''$ torsion free, then $\mathcal{F}'$ normal torsion free

Def. $0 \to \mathcal{E}_n \to \cdots \mathcal{E}_0 \to \mathcal{F}_v \to 0$ is a locally free coherent resolution. E_i be the bundle corresponding to $\mathcal{E}_i$, then define $\det \mathcal{F}_v = \bigotimes_{i=0}^{n} (\det E_i)^{(-1)^i}$

Rmk. $\mathcal{F}$ torsion-free coherent, then $\det \mathcal{F} = (\wedge^r \mathcal{F})^{**}$

$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ $\quad$ ω-degree $\deg_\omega(\mathcal{F}) = \deg_\omega(\det \mathcal{F})$

ω-slope $\mu_\omega(\mathcal{F}) = \dfrac{\deg_\omega(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}$

Def. $(E, \bar\partial_E)$ holomorphic bundle on kähler manifold (M, J, g). $(E, \bar\partial_E)$ is ω-stable if for every proper coherent subsheaf $\mathcal{F}$ with torsion free quotient

$\mu_\omega(\mathcal{F}) < \mu_\omega(E)$

9.2 Degree of Coherent Sheaves

$\underset{\substack{\text{subsheaf} \\ \text{with torsion} \\ \text{free quotient}}}{\mathcal{F}} \subset E^{\to \text{bundle}}$

$$0 \to \mathcal{F} \to E \to Q \to 0$$

$\mathcal{F}, Q$ locally free in $M \setminus \Sigma$

Σ: closed analytic subset. codim ≥ 2

$$0 \to \mathcal{F}|_{M\setminus\Sigma} \to E|_{M\setminus\Sigma} \to Q|_{M\setminus\Sigma} \to 0$$

H: Hermitian metric on E

$$\pi_H : E|_{M\setminus\Sigma} \to \mathcal{F}$$

$$\pi \in \Gamma(\mathrm{End}(E')) \qquad \pi^2 = \pi = \pi^{*H}$$

$$(\mathrm{Id} - \pi)\bar\partial_E \pi = 0 \qquad \pi_H \in L_1^2(M, \mathrm{End}(E'))$$

H_F: induced metric on $\mathcal{F}|_{M\setminus\Sigma}$.

$$\deg_\omega(\mathcal{F}) = \int_M c_1(\det \mathcal{F}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\overset{?}{=} \int_{M\setminus\Sigma} \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_{H_\mathcal{F}}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\overset{\text{Gauss-Codazzi: }(*)}{=} \int_{M\setminus\Sigma} \operatorname{tr}\left(\pi_H \frac{\sqrt{-1}}{2\pi} F_H\right) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$- \frac{1}{2\pi} \int_{M\setminus\Sigma} |\bar\partial_E \pi_H|^2 \frac{\omega^n}{n!}$$

$$= \int_{M\setminus\Sigma} \frac{1}{2\pi}\left[\operatorname{tr}(\pi_H \sqrt{-1}\Lambda_\omega F_H) - |\bar\partial_E \pi_H|^2\right] \frac{\omega^n}{n!}$$

Gauss-Codazzi equation:

$$S = \mathcal{F}|_{M\setminus\Sigma}$$

on $M\setminus\Sigma$

$$0 \to S \xrightarrow{i} E \xrightarrow{p} Q \to 0$$

$$X \mapsto [X]$$

H: metric

H_S: induce metric

$$0 \leftarrow S \xleftarrow{\pi_H} E \xleftarrow{p^{*H}} Q \leftarrow 0$$

$$X^{\perp} \leftarrow\!\shortmid [X]$$

H_Q: $H_Q([X],[Y]) = H(p^*(X), p^*(Y)) = H(X^{\perp}, Y^{\perp})$

$\bar\partial_S$: induce holomorphic structure

$\bar\partial_Q$: $Y \in \Gamma(E)$ $\bar\partial_Q[Y] = [\bar\partial_E Y]$

D: H-connection on E

induce connection D^T on subbundle S

$$D^T X = (DX)^T$$

$B(X) = (DX)^{\perp}$ is a tensor [check $B(fX) = fB(X)$]

$B \in \Gamma(\Lambda^1 M \otimes S^* \otimes S^{\perp})$: second fundamental form

For $\xi \in \Gamma(S^{\perp})$. $D^{\perp}\xi = (D\xi)^{\perp}$ induce connection on $S^{\perp}$

$A(\xi) = (D\xi)^T$. $A(f\xi) = Af(\xi)$

$A \in \Gamma(\Lambda^1(M) \otimes (S^{\perp})^* \otimes S)$

$$0 = d(H(X,\xi)) = H(DX,\xi) + H(X, D\xi)$$
$$= H((DX)^{\perp}, \xi) + H(X, (D\xi)^{T})$$
$$= H(B(X), \xi) + H(X, A(\xi))$$

$$A = -B^{*H}$$

$$f_H : S \oplus Q \to E$$
$$(X, [Y]) \mapsto X + Y^{\perp H}$$

$$f_H^*(D) = \begin{pmatrix} D_S & -B^* \\ B & D_Q \end{pmatrix} \Rightarrow f_H^*(F_D) = f^*(D^2) = f^*(D) \circ f^*(D)$$

$$f^*(F_D) = \begin{pmatrix} D_S & -B^* \\ B & D_Q \end{pmatrix}^2 = \begin{pmatrix} D_S^2 - B^* \wedge B & -D_S B^* - B^* \circ D_Q \\ B \circ D_S + D_Q \circ B & D_Q^2 - B \wedge B^* \end{pmatrix}$$
$$= \begin{pmatrix} F_{D_S} - B^* \wedge B & -D_{Q^* \otimes S} B^* \\ D_{S^* \otimes Q} B & F_{D_Q} - B \wedge B^* \end{pmatrix}$$

$$f^*(D)(\overbrace{e_1, \cdots e_s}^{S}, \overbrace{e_{s+1}, \cdots e_r}^{Q}) = (e_1, \cdots e_s, e_{s+1}, \cdots e_r)\begin{pmatrix} A_S & -B^* \\ B & A_Q \end{pmatrix}$$
$$D_S = d + A_S \quad D_Q = d + A_Q$$

$$f^*(\bar{\partial}_E) = \begin{pmatrix} \bar{\partial}_S & \gamma \\ 0 & \bar{\partial}_Q \end{pmatrix} \qquad (\bar{\partial}_E Y^{\perp})^T = \gamma(Y^{\perp}) \in S$$

$\gamma = 0$: we say $(S, \bar{\partial}_S) \oplus (Q, \bar{\partial}_Q) = (E, \bar{\partial}_E)$

$\gamma = A^{0,1}$ $\quad \bar\partial_E = D_H^{0,1}$ $\qquad B^{0,1} = 0 \Rightarrow A^{1,0} = 0$

D_S is the chern connection on S with $\bar\partial_E$, H_E
(a) $\qquad$ (a) $\quad$ (a)

Recall: $f^*(F_{D_H}) = \begin{pmatrix} F_{D_S} - B^* \wedge B & -DB^* \\ DB & F_{D_Q} - B \wedge B^* \end{pmatrix}$

$B = (Id - \pi)\partial_H \pi$

Pf. $B(X) = (I-\pi)(D_H X) = (I-\pi)(\partial_H + \bar\partial_E)X$

$= (I-\pi)\partial_H X$

$\partial_H(\pi X) - \pi(\partial_H X)$ $\quad = (I-\pi)\partial_H \pi X$

A computation gives $F_{D_S} = f^*(\pi \circ F_{D_H}) + \bar\partial_E \pi \wedge \partial_H \pi$

$\sqrt{-1}\,\mathrm{tr} F_{D_S} = \sqrt{-1}\,\mathrm{tr}(\pi \circ F_H) - |\bar\partial_E \pi|_H^2$

Thus we're proved $(*)$.

$(E, \bar{\partial}_E) = (E_1, \bar{\partial}_{E_1}) \oplus (E_2, \bar{\partial}_{E_2})$.

E_1, E_2: stable with same slope

then we say E is polystable

Prop. Any stable bundle is simple.

(i.e. E admits no global holomorphic endomorphism other than homotheties)

pf: $\sigma: E \to E$ holomorphic endomorphism is not a multiple of identity.

Then $\exists p \in M$. $\exists$ an eigenvalue λ

$\sigma - \lambda Id$. kernel E' $\longrightarrow$ proper subbundle

image E'' $\nearrow$

$E = E' \oplus E''$ $\Rightarrow \underline{\deg(E) = \deg(E') + \deg(E'')}$

$$\mu(E) = \frac{\operatorname{rank} E'}{\operatorname{rank} E} \mu(E') + \frac{\operatorname{rank} E''}{\operatorname{rank} E} \mu(E'') < \mu(E) \quad \times$$

#.

Hermitian - Einstein metric H.

$$\underbrace{\sqrt{-1}\Lambda_\omega F_H}_{\text{mean curvature}} = \lambda \mathrm{Id}_E \qquad (*)$$

[Hermitian - Yang-Mills functional $\int_M |F_D|_H^2 \frac{\omega^n}{n!}$

In kähler case $(*)$ gives the critical point]

Donaldson-Uhlenbeck-Yau's Thm.

$\exists$ Hermitian-Einstein metric $\Leftrightarrow (E, \bar\partial_E)$ polystable

Thm. (Kobayashi-Lübke) $(E, \bar\partial_E, H)$ is H-E bundle. then $(E, \bar\partial_E)$ is polystable

Pf $\sqrt{-1}\Lambda_\omega F_H = \lambda \mathrm{Id}$ $\qquad \lambda = \mu(E) \cdot \frac{2\pi}{\int_M \frac{\omega^n}{n!}}$

Any $\mathcal{F} \to E$ coherent subsheaf

$$\mu(\mathcal{F}) = \frac{1}{\mathrm{rank}(\mathcal{F})} \int_M \frac{\sqrt{-1}}{2\pi} \mathrm{tr}\, F_{H_\mathcal{F}} \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$= \frac{1}{\mathrm{rank}(\mathcal{F})} \int_M \left(\frac{i}{2\pi} \operatorname{tr} \pi_H \cdot \Lambda_\omega F_H - |\bar\partial_E \pi_H|^2 \right) \frac{\omega^n}{n!}$$

$$\leq \frac{1}{\mathrm{rank}(\mathcal{F})} \int_M \frac{\lambda}{2\pi} \operatorname{tr} \pi_H = \mu(E) \Rightarrow \text{semistable}$$

Choose the smallest subsheaf $\mathcal{F} \to E$ s.t. $\mu(\mathcal{F}) = \mu(E)$

$\mathcal{F}$ stable $\quad |\bar\partial_E \pi_H| = 0 \Rightarrow |\partial_H \pi_H| = 0$

$\Rightarrow D_H \pi_H = 0$. i.e. π_H is parallel on $M \setminus \Sigma$

By Hartog's Thm. π_H can be extended to M.

$\overset{(*)}{\Rightarrow}$ $\mathcal{F}$ is subbundle of E, then repeat the process

$(*)$: It suffices to show $\mathrm{rank}(\pi_H) = \mathrm{rank}\, \mathcal{F}$

We prove that the eigenvalues of π_H are constant.

$$d\tilde\varphi(\pi_H, \cdots \pi_H) = \tilde\varphi(\overset{= D\pi_H - [A_H, \pi_H]}{d(\pi_H)}, \cdots \pi_H) + \cdots + \tilde\varphi(\pi_H, \cdots d(\pi_H))$$

$$= \tilde\varphi(D\pi_H, \cdots \pi_H) + \cdots \tilde\varphi(\pi, \cdots D\pi_H)$$

$$= 0.$$

\#

$\mathcal{F} \hookrightarrow E$ (bundle) $\Rightarrow$ (*) $\begin{cases} \pi_H \in L_1^2(M, \mathrm{End}(E)) \\ C^\infty \text{ in the complement of an analytic set with codim} \geq 2 \\ \pi = \pi^2 = \pi^* \quad (\mathrm{Id} - \pi)\bar{\partial}_E \pi = 0 \end{cases}$

Lemma (Uhlenbeck–Yau) (E, H) holomorphic bundle with C^∞ Hermitian metric (M, ω) compact Kähler

And π satisfies (*) almost everywhere

Then $\exists$ coherent sheaf $\mathcal{F} \subset \mathcal{O}(E)$ and an analytic set $\Sigma \subset M$ with codim ≥ 2, s.t.

(i) $\pi|_{M \setminus \Sigma}$ is C^∞

(ii) $\pi = \pi^* = \pi^2$. $(\mathrm{Id}_E - \pi) \circ \bar{\partial}_E \pi = 0$ on $M \setminus \Sigma$

(iii) $\mathcal{F}|_{M \setminus \Sigma} = \mathrm{Im}\, \pi|_{M \setminus \Sigma}$ is a holomorphic subbundle of $E_{M \setminus \Sigma}$

Prop E admits an approximate H-E metric.

i.e. $\forall \varepsilon > 0$ $\exists H_\varepsilon$ s.t.

$$\max_M |\sqrt{-1}\Lambda_\omega F_{H_\varepsilon} - \lambda \mathrm{Id}_E|_{H_\varepsilon} < \varepsilon$$

then we have the following Bogomolov type inequality

$$\int_M \left(2c_2(E) - \frac{r-1}{r} c_1(E) \wedge c_1(E)\right) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0.$$

9.3 Bogomolov Type Inequality

$$4\pi^2\int_M\left(2C_2(E)-\frac{r-1}{r}C_1(E)\wedge C_1(E)\right)\wedge\frac{\omega^{n-2}}{(n-2)!}$$

$$\overset{(*)}{=}\int_M\left(\operatorname{tr}(F_H\wedge F_H)-\frac{1}{r}\operatorname{tr}F_H\wedge\operatorname{tr}F_H\right)\wedge\frac{\omega^{n-2}}{(n-2)!}$$

$$\left[\ Ch_2(E)=\frac{C_1^2(E)-2C_2(E)}{2}=\frac{1}{2}\operatorname{tr}\left(\left(\frac{\sqrt{-1}}{2\pi}F_H\right)^2\right)\ \right]$$

$$=\int_M\operatorname{tr}(F_H^{\perp}\wedge F_H^{\perp})\wedge\frac{\omega^{n-2}}{(n-2)!}\qquad F_H^{\perp}=F_H-\frac{1}{r}\operatorname{tr}F_H\otimes Id_E$$

$$\overset{(*)}{=}\int_M\left(|F_H^{\perp}|^2-|\Lambda_\omega F_H^{\perp}|^2\right)\frac{\omega^n}{n!}$$

$(*)$: $\operatorname{tr}(F_H\wedge F_H)\wedge\frac{\omega^{n-2}}{(n-2)!}$

take coordinate $H(e_\alpha,e_\beta)=\delta_{\alpha\beta}$, $g_{i\bar j}=\delta_{ij}$, $\omega=\sum dz^i\wedge d\bar z^j$

$$=F_{\alpha i\bar j}^{\beta}F_{\beta k\bar l}^{\alpha}dz^i\wedge d\bar z^j\wedge dz^k\wedge d\bar z^l\wedge\frac{\left(\sum_{s=1}^{n}dz^s\wedge d\bar z^s\right)^{n-2}}{(n-2)!}$$

$i=j$. then $k\ l\neq i$. $k=l$

$$=\sum_{i\neq k}F_{\alpha i\bar i}^{\beta}F_{\beta k\bar k}^{\alpha}dz^i\wedge d\bar z^i\wedge dz^k\wedge d\bar z^k\wedge\frac{\omega^{n-2}}{(n-2)!}$$

$$+\sum_{i\neq j}F_{\alpha i\bar j}^{\beta}F_{\beta j\bar i}^{\alpha}dz^i\wedge d\bar z^j\wedge dz^j\wedge d\bar z^i\wedge\frac{\omega^{n-2}}{(n-2)!}$$

$i\neq j$. then $k=j$, $l=i$

$$=\left(\sum_{i,j}|F_{\alpha i\bar j}^{\beta}|^2-\left|\sum_i F_{\alpha i\bar i}^{\beta}\right|^2\right)\frac{\omega^n}{n!}$$

$$=\left(|F_H|^2-|\Lambda_\omega F_H|^2\right)\frac{\omega^n}{n!}$$

Miyaoka-Yau Inequality:

(M,ω) Kähler. $2\pi C_1(M) = \lambda[\omega]$ $\lambda = \pm 1$ $\dim_{\mathbb{C}} M = n$

If M admits Kähler-Einstein metric $\rho(\omega) = \lambda\omega$.

then $$4\pi^2 \int_M \left(2C_2(M) - \frac{n}{n+1} C_1(M) \wedge C_2(M)\right) \cdot \frac{[\lambda^{-1} C_1(M)]^{n-2}}{(n-2)!} \geq 0$$

Chapter 10 Line Bundle

10.1 The Picard Group

Def The set of holomophic line bundle on (M, J) forms a group under tensor product, duality. this is called the Picard group $Pic(M)$.

Prop isomorphism $\pi \quad Pic(M) \to H^1(M, \mathcal{O}^*)$

Pf. Line bundle L transition $\{g_{\alpha\beta}\}$ with

$g_{\alpha\alpha} = 1. \quad g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \quad g_{\alpha\beta}\, g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$

$\{g_{\alpha\beta}\}$ define 1-cocycle in $C^1(\mathcal{U}, \mathcal{O}^*)$

$$\pi(L) = [\{g_{\alpha\beta}\}] \in H^1(\mathcal{U}, \mathcal{O}^*)$$

(1) well-defined. $\{g'_{\alpha\beta}\}$ $\quad e_\alpha = f_\alpha e'_\alpha \quad e_\beta = f_\beta e'_\beta$

then $\begin{matrix} e_\alpha = g_{\alpha\beta} e_\beta \\ e'_\alpha = g'_{\alpha\beta} e'_\beta \end{matrix} \Rightarrow \dfrac{g_{\alpha\beta}}{g'_{\alpha\beta}} = \dfrac{f_\alpha}{f_\beta} \qquad \dfrac{g}{g'} = \delta f$

(2) homeomorphic

(3) surjective: $[\{g_{\alpha\beta}\}] \in H^1(\mathcal{U}, \mathcal{O}^*)$

define a line bundle $\coprod_\alpha U_\alpha \times \mathbb{C} / \sim$.

$\underset{U_\alpha \times \mathbb{C}}{(x, \xi)} \sim \underset{U_\beta \times \mathbb{C}}{(y, \zeta)} \iff \begin{matrix} x = y \\ g_{\alpha\beta}\zeta = \xi \end{matrix}$

(4) injective. If $\pi(L)=0$

$[\{g_{\alpha\beta}\}]=0 \Leftrightarrow g=\delta f$. $\quad g_{\alpha\beta}=\frac{f_\alpha}{f_\beta}$ (*)

$\bigsqcup_\alpha U_\alpha\times\mathbb{C}/\sim$.

$\sigma_\alpha(x)=[(x,f_\alpha(x))]\in L|_{U_\alpha}$.

$\sigma_\beta(x)=[(x,f_\beta(x))]\in L|_{U_\beta}$

$\forall x\in U_\alpha\cap U_\beta$ (*) gives $\sigma_\beta(x)=\sigma_\alpha(x)$

$\Rightarrow\exists$ a global nonzero section $\Rightarrow L\cong M\times\mathbb{C}$

#.

10.2 Line Bundle, first Chern Class

Prop L is a holomorphic line bundle. then

$$c_1(L)\in H^2(M,\mathbb{Z})\subset H^2(M,\mathbb{R})$$

Ex. $K_M=\Lambda^n T^{*(1,0)}_M$ (canonical line bundle)

$=\Lambda^{n,0}M=\det(\Lambda^{1,0}M)$

$K_M^{-1}=\Lambda^n T^{1,0}M$ (anti-canonical)

$c_1(K_M^{-1})=c_1(M)$

Def: A line bundle L is positive if $\exists$ a metric

H on L with curvature F_H is positive (1,1)-form

Prop. M kähler η is real closed $(1,1)$ form

with $\frac{1}{2\pi}[\eta] = c_1(L) \in H^2_{dR}(M,\mathbb{R})$

then $\exists$ a metric on L with curvature form $\sqrt{-1}F_H = \eta$.

pf Let H be a Hermitian metric on L

s.t. $[\sqrt{-1}F_H] = [\eta]$

$\Rightarrow \sqrt{-1}F_H = \eta + \sqrt{-1}\partial\bar{\partial}\varphi$

Let $H' = e^{\varphi}H$. then $\sqrt{-1}F_{H'} = \sqrt{-1}F_H + \sqrt{-1}\bar{\partial}(e^{-\varphi}\partial e^{\varphi})$

$= \sqrt{-1}F_H + \sqrt{-1}\bar{\partial}\partial\varphi = \eta$. #

10.3 Kodaira Vanishing Theorem

(M,J,g) Kähler E holomorphic bundle with Hermitian metric H.

Let $\phi, \psi \in \Omega^{p,q}(M,E) = \Gamma(\Lambda^{p,q}M \otimes E)$ $\quad (z^1, \cdots z^n)$ $\quad (e_1, \cdots e_r)$

$$\phi = \phi^i_{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \cdots \bar{\beta}_q} e_i \otimes dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots d\bar{z}^{\beta_q}.$$

$$g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) \qquad H_{ij} = H(e_i, e_j)$$

$$\langle\varphi,\psi\rangle = p!\,q!\,\varphi^i_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}\overline{\psi^j_{\gamma_1\cdots\gamma_p\bar\delta_1\cdots\bar\delta_q}}H_{i\bar j}\,g^{\alpha_1\bar\gamma_1}\cdots g^{\alpha_p\bar\gamma_p}g^{\delta_1\bar\beta_1}\cdots g^{\delta_q\bar\beta_q}$$

$$(\varphi,\psi) = \int_M \langle\varphi,\psi\rangle\frac{\omega^n}{n!}$$

∇: induced connection on $\Lambda^{p,q}M\otimes E$ by $\begin{cases}\text{Levi-Civita of } (M,g)\\ \text{Chern connection } (E,H)\end{cases}$

Lemma For $\varphi\in\Gamma(\Lambda^{p,q}(M)\otimes L)$ e holomorphic basis

$$\bar\partial_L\varphi = (-1)^p\nabla_{\bar\beta}\varphi_{A\bar\beta_1\cdots\bar\beta_q}\,e\otimes dz^A\wedge d\bar z^\beta\wedge d\bar z^{\beta_1}\cdots\wedge d\bar z^{\beta_q}$$

$$(\bar\partial_L^{*H}\varphi)_{A\bar\beta_2\cdots\bar\beta_q} = (-1)^{p+1}q\,g^{\alpha\bar\beta}(\nabla_\alpha + H^{-1}\partial_\alpha H)\varphi_{A\bar\beta\bar\beta_2\cdots\bar\beta_q}$$

Pf. $$\bar\partial_L\varphi = (-1)\underbrace{\frac{\partial\varphi_{A\bar\beta_1\cdots\bar\beta_q}}{\partial\bar z^\beta}\,e\otimes d\bar z^\beta\wedge d\bar z^{\beta_1}\cdots\wedge d\bar z^{\beta_q}}_{\shortparallel}$$

$$\nabla_{\bar\beta}\varphi_{A\bar\beta_1\cdots\bar\beta_q} + \underbrace{\sum_k \Gamma^{\bar\alpha}_{\bar\beta\bar\beta_k}\varphi_{A\bar\beta_1\cdots\overset{k\text{-th}}{\bar\alpha}\cdots\bar\beta_q}\,e\otimes d\bar z^\beta\cdots\wedge d\bar z^{\beta_q}}_{0}$$

($\Gamma^{\alpha}_{\beta\beta_k}$ = "$\Gamma^{\bar\alpha}_{\bar\beta\bar\beta_k}$")

$p!(q-1)!$

$$(\bar\partial_L^{*H}\varphi,\psi) = (\varphi,\bar\partial_L\psi)\qquad H(e,e) = e^{-h}$$

$$\langle\varphi,\bar\partial_L\psi\rangle = (-1)^{p+1}p!\,q!\,\varphi_{A\bar\beta_1\cdots\bar\beta_q}\overline{\nabla_{\bar\beta}\psi_{B\bar\delta_2\cdots\bar\delta_q}}\,g^{A\bar B}g^{\beta\bar\beta_1}g^{\delta_2\bar\beta_2}\cdots g^{\delta_q\bar\beta_q}e^{-h}$$

$$=(-1)^{p+1}p!\,q!\int\nabla_\beta\Big(\underbrace{\varphi_{A\bar\beta_1\cdots\bar\beta_q}\overline{\psi_{B\bar\delta_2\cdots\bar\delta_q}}\,g^{A\bar B}g^{\beta\bar\beta_1}\cdots g^{\delta_q\bar\beta_q}e^{-h}}_{\operatorname{div}(X)}\Big)$$

$\int_M \operatorname{div}(X) = 0.$

$$-g^{\beta\bar{\beta}_1}\left(\nabla_\beta \psi_{A\bar{\beta}_1\cdots\bar{\beta}_q} - \partial_\beta h\, \psi_{A\bar{\beta}_1\cdots\bar{\beta}_q}\right)\overline{\psi_{B\delta_1\cdots\delta_q}}\, g^{AB} g^{\delta_2\bar{\beta}_2}\cdots g^{\delta_q\bar{\beta}_q} e^{-h}$$

compare with $(\bar{\partial}_L^{*H}\phi.\psi)$ #

Rmk. $$(\bar{\partial}_L\phi)_{A\bar{\beta}\bar{\beta}_1\cdots\bar{\beta}_q} = \frac{1}{q+1}(-1)^p\left(\nabla_{\bar{\beta}}\phi_{A\bar{\beta}_1\cdots\bar{\beta}_q} - \sum_{k=1}^{q}\nabla_{\bar{\beta}_k}\phi_{A\bar{\beta}_1\cdots\overset{k\text{-th}}{\bar{\beta}}\cdots\bar{\beta}_q}\right)$$

In the following $\phi\in\Omega^{0q}(M,L)$

$$(\bar{\partial}_L^*\bar{\partial}_L\phi)_{\bar{\beta}_1\cdots\bar{\beta}_q} = -(q+1)\, g^{\alpha\bar{\beta}}(\nabla_\alpha + H^{-1}\partial H)(\bar{\partial}_L\phi)_{\bar{\beta}\bar{\beta}_1\cdots\bar{\beta}_q}$$

$$= -g^{\alpha\bar{\beta}}\left\{(\nabla_\alpha + H^{-1}\partial_\alpha H)\nabla_{\bar{\beta}}\phi_{\bar{\beta}_1\cdots\bar{\beta}_q} - \sum_{k=1}^{q}(\nabla_\alpha + H^{-1}\partial H)\nabla_{\bar{\beta}_k}\phi_{\bar{\beta}_1\cdots\bar{\beta}\cdots\bar{\beta}_q}\right\}$$

$$(\bar{\partial}_L\bar{\partial}_L^*\phi)_{\bar{\beta}_1\cdots\bar{\beta}_q} = \frac{1}{q}\sum_{k=1}^{q}(-1)^{k-1}\nabla_{\bar{\beta}_k}(\bar{\partial}_L^*\phi)_{\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}$$

$$= -\sum_{k=1}^{q}(-1)^{k-1}\nabla_{\bar{\beta}_k}\left(g^{\alpha\bar{\beta}}(\nabla_\alpha + H^{-1}\partial_\alpha H)\phi_{\bar{\beta}\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}\right)$$

$$= -\sum_{k=1}^{q} g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}_k}\nabla_\alpha\phi_{\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}$$

$$- \sum_{k=1}^{q} g^{\alpha\bar{\beta}}\,\bar{\partial}_{\beta_k}(H^{-1}\partial_\alpha H)\,\phi_{\bar{\beta}_1\cdots\bar{\beta}\cdots\bar{\beta}_q}$$

$$- \sum_{k=1}^{q} g^{\alpha\bar{\beta}} H^{-1}\partial_\alpha H\, \nabla_{\bar{\beta}_k}\phi_{\bar{\beta}_1\cdots\bar{\beta}\cdots\bar{\beta}_q}$$

$$\Delta^{\bar\partial} = -(\bar\partial_L \bar\partial_L^* + \bar\partial_L^* \bar\partial_L)$$

$$-(\Delta^{\bar\partial}\phi)_{\bar\beta_1\cdots\bar\beta_q} = -g^{\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta_q} - g^{\alpha\bar\beta}H^{-1}\partial_\alpha H\,\nabla_{\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta_q}$$

$$+\underbrace{g^{\alpha\bar\beta}\sum_{k=1}^{q}[\nabla_\alpha,\nabla_{\bar\beta_k}]\phi_{\bar\beta_1\bar\beta\bar\beta_q}}_{(*)} + \sum_{k=1}^{q} g^{\alpha\bar\beta}(F_H)_{\alpha\bar\beta}\,\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q}$$

$$\nabla_\alpha\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q} = \partial_\alpha\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q}$$

$$\nabla_{\bar\beta_k}\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q} = \bar\partial_{\beta_k}\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q} - \overline{\Gamma^{\gamma}_{\beta_k\beta}}\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta_q} - \overline{\Gamma^{\gamma}_{\beta_k\beta_l}}\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta\cdots}$$

Take normal coordinate. $\Gamma^{\gamma}_{\alpha\beta}(0)=0$

$$(*) = \sum_{k=1}^{q} g^{\alpha\bar\beta}\Big(\underbrace{-\partial_\alpha(\overline{\Gamma^{\gamma}_{\beta_k\beta}})}_{= R_{\bar\beta\alpha\bar\gamma\beta_k}}\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta_q} \boxed{-\underbrace{\partial_\alpha(\overline{\Gamma^{\gamma}_{\beta_k\beta_l}})}_{= R_{\bar\beta_l\alpha\bar\gamma\beta_k}}\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta\cdots\bar\beta_q}}\Big) \quad (\star)$$

$$= \sum_{k=1}^{q} R_{\bar\beta\beta_k} g^{\beta\bar\gamma}\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta_q}$$

$$(\star) = \sum_{k=1}^{q}\sum_{l\neq k} g^{\alpha\bar\beta}g^{\eta\bar\gamma}\,\overline{R_{\beta_l\bar\alpha\eta\bar\beta_k}}\,\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta\cdots\bar\beta_q}$$

$$= \sum_{k=1}^{q}\sum_{l<k} g^{\alpha\bar\beta}g^{\eta\bar\gamma}\,\overline{R_{\beta_l\bar\alpha\eta\bar\beta_k}}\,\phi_{\bar\beta_1\cdots\bar\gamma\cdots\bar\beta\cdots\bar\beta_q}$$

$$+ \sum_{k=1}^{q}\sum_{k<l} g^{\alpha\bar\beta}g^{\eta\bar\gamma}\,\overline{R_{\beta_l\bar\alpha\eta\bar\beta_k}}\,\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\gamma\cdots\bar\beta_q} = 0.$$

In general

$$-\Delta^{\bar\partial}\phi_{\bar\beta_1\cdots\bar\beta_q} = -g^{\alpha\bar\beta}(\nabla_\alpha + H^{-1}\partial_\alpha H)\nabla_{\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta_q} + \sum_{k=1}^{q}\underbrace{(R_{\alpha\bar\beta_k} + (F_H)_{\alpha\bar\beta_k})\,g^{\alpha\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta\cdots\bar\beta_q}}_{(*)}$$

(assume ϕ $\bar\partial$-harmonic)

$$\Rightarrow (-\Delta^{\bar\partial}\phi,\phi) = q!\int_M -\Delta^{\bar\partial}\phi_{\bar\beta_1\cdots\bar\beta_q}\,\phi^{\bar\beta_1\cdots\bar\beta_q}e^{-h}\frac{\omega^n}{n!}$$

$$\| \qquad = \underbrace{\int_M -g^{\alpha\bar\beta}\nabla_\alpha(\nabla_{\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta_q}\phi^{\bar\beta_1\cdots\bar\beta_q}e^{-h})\frac{\omega^n}{n!}}_{\mathrm{div}(X)} + \int_M g^{\alpha\bar\beta}\nabla_{\bar\beta}\phi_{\bar\beta_1\cdots\bar\beta_q}\nabla_\alpha\phi^{\bar\beta_1\cdots\bar\beta_q}e^{-h} + \int_M *\frac{\omega^n}{n!}$$

0

$$R_{\alpha\bar\beta} + (F_H)_{\alpha\bar\beta_k} > 0$$

$$= \int_M |\nabla^{0,1}\phi|^2 + *\ \frac{\omega^n}{n!}$$

$$\Rightarrow \phi = 0.$$

Thm (Kodaira) Let $L\to M$ be a holomorphic bundle with a Hermitian metric, s.t.

$$\sqrt{-1}\,(Ric_{\alpha\bar\beta} + (F_H)_{\alpha\bar\beta})\,dz^\alpha\wedge d\bar z^\beta > 0$$

then $H^q(M, \mathcal{O}(L)) = 0 \quad (q\geq 1)$

Cor (Kodaira Vanishing Theorem) Let L be a positive bundle of M then

$$H^q(M, \mathcal{O}(K\otimes L)) = H^{n,q}(M, L) = 0 \quad \forall q \geq 1$$

Pf. $c_1(K\otimes L, \omega\otimes H) = \rho_\omega + \sqrt{-1}F_H$ #

Cor (1) If $L\otimes K^{-1}$ is positive, then $H^q(M, \mathcal{O}(L)) = 0$ $(q>0)$

(2) If L^* is positive, then $H^q(M, \mathcal{O}(L)) = 0$, $q \leq n-1$

Serre Duality:

$$H^{0,q}(M, L) \cong H^{n,n-q}(M, L) \cong H^{n-q}(M, \Omega^{n,0}(L^*)) \cong H^{n-q}(M, \mathcal{O}(K\otimes L^*))$$

Thm (Nakano) L positive, then $H^{p,q}(M, L) = 0$ $p+q \geq n+1$

Nakano Identity $\varphi \in \Omega^{p,q}(M, L)$, H: Hermitian metric.

then

$$\left((\sqrt{-1}\Lambda_\omega F_H - \sqrt{-1}F_H\wedge\omega)\varphi, \varphi\right) = \|\partial_H\varphi\|^2_{L^2} + \|\partial_H^*\varphi\|^2_{L^2} - \|\bar\partial_L\varphi\|^2 - \|\bar\partial_L^*\varphi\|^2$$

Pf. Kähler's identity $[\Lambda_\omega, \bar\partial_L] = -\sqrt{-1}\partial_H^*$

$[\Lambda_\omega, \partial_H] = \sqrt{-1}\bar\partial_H^*$

$$F_H\varphi = (\partial_H\bar\partial_L + \bar\partial_L\partial_H)\varphi$$

$$(\sqrt{-1}\Lambda_\omega F_H\varphi,\varphi) = (\sqrt{-1}\Lambda_\omega(\partial_H\bar\partial_L + \bar\partial_L\partial_H)\varphi,\varphi)$$

$$= (\sqrt{-1}\underbrace{(\Lambda_\omega\partial_H - \partial_H\Lambda_\omega)}_{}\overset{-\bar\partial_L^*}{}\bar\partial_L\varphi,\varphi) + (\sqrt{-1}\partial_H\Lambda_\omega\bar\partial_L\varphi,\varphi)$$

$$+ (\sqrt{-1}\underbrace{(\Lambda_\omega\bar\partial_L - \bar\partial_L\Lambda_\omega)}_{\partial_H^*}\partial_H\varphi,\varphi) + (\sqrt{-1}\bar\partial_L\Lambda_\omega\partial_H\varphi,\varphi)$$

$$(-\sqrt{-1}\Lambda_\omega F_H\varphi,\varphi) \overset{\text{similar}}{=\!=\!=} (\sqrt{-1}\partial_H\partial_H^*\varphi,\varphi) - (\sqrt{-1}\partial_H\Lambda_\omega\bar\partial_L\varphi,\varphi)$$

$$-(\sqrt{-1}\bar\partial_L\bar\partial_L^*\varphi,\varphi) - (\sqrt{-1}\bar\partial_L\Lambda_\omega\partial_H\varphi,\varphi) \quad \#.$$

If $\Delta^{\bar\partial_L}\varphi = 0$, then $\bar\partial_L\varphi = \bar\partial_L^*\varphi = 0 \Rightarrow (\sqrt{-1}\Lambda_\omega F_H - \sqrt{-1}F_H\Lambda_\omega\varphi,\varphi) \geq 0$

Let $\omega = \sqrt{-1}F_H \Rightarrow ((\Lambda_\omega L_\omega - L_\omega\Lambda_\omega)\varphi,\varphi)$

$$= ((n-p-q)\varphi,\varphi) \geq 0 \Rightarrow \text{for } p+q \geq n+1 \quad \varphi = 0$$

Thus we proved Nakano's Thm.

Rmk. M Kähler, $c_1(L) \geq 0$, $\operatorname{rank} c_1(L) \geq k$.

then $H^{p,q}(M,L) = 0$ for $p+q \geq 2n-k+1$

1.4 Line Bundle from Divisors

analytic hypersurface V of M (M compact complex manifold)

(1) $V \subset M$ compact: $\exists$ finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$

U_α is biholomorphic to the unit ball of $\mathbb{C}^n$

(2) $U_\alpha \cap V$ is defined by the zeros of a holomorphic function f_α

If V can't be written as the union of two nonempty hypersurfaces

and f_α can be chosen to vanish to order 1.

then V is called irreducible.

Def. A divisor D on M is a formal finite linear

combination $D = \sum a_i V_i$ $\quad a_i \in \mathbb{Z}$

V_i : irreducible hypersurface

$\mathrm{Div}(M) \to \mathrm{Pic}(M)$

$D = \sum a_i V_i \mapsto \bigotimes_i (L_{V_i})^{a_i}$

10.5 Kodaira Embedding Theorem

M compact complex L positive line bundle

M must be an algebraic manifold.

$$H^0(M, \mathcal{O}(L^k)) = \Gamma(M, L^k)$$

$$\dim \Gamma(M, L^k) = N(k)+1$$

$\varphi = \{\varphi_0 \cdots \varphi_N\}$ form a basis of the holomorphic sections

e is a frame of L_x.

$$\eta_k : x \mapsto [\varphi_0(e), \cdots \varphi_N(e)] \in \mathbb{CP}^N$$

Thm: $\exists k_0 \in \mathbb{Z}^+$, s.t. $\forall k \geq k_0$. η_k is a well-defined embedding

Generalization:

Thm $f: X \to Y$ smooth holomorphic fibration. all the fibers compact Y Kähler If $\exists$ a line bundle on X s.t. $\forall y \in Y$ $L|_{f^{-1}(y)}$ is positive. Then $\exists$ a holomorphic vector bundle $\pi_E : E \to Y$ and an embedding $j: X \to P(E^*)$ s.t. $f = \pi_{P(E^*)} \circ j$.

Chapter 11 H-E metrics on holomorphic Bundle

11.1 H-E connection and Yang-Mills connection

(M,ω) kähler E complex vector bundle

H_0: Hermitian metric on E.

$\mathcal{A}_{H_0} = \{$connections competible with $H_0\}$

$\mathcal{A}^{1,1}_{H_0}$: $F_D^{0,2} = F_D^{2,0} = 0$ $\quad (F_D^{*H_0} = -F_D)$

"unitary integrable connections"

Rmk. $(E, \bar{\partial}_E)$ holomorphic bundle. $\begin{cases} D_A \text{ is } H_0\text{-unitary} \\ D_A^{0,1} = \bar{\partial}_E \end{cases}$

conversely given $D_A \in \mathcal{A}^{1,1}_{H_0}$. $D_A^{0,1}$ defines a holomorphic structure on E. $\quad (D_A^{0,1})^{*H_0} = D_A^{1,0}$

?

$\mathcal{C}^{1,1}(E) = \{$holomorphic bundle structures on $E\}$

$\quad \cong$

$\mathcal{A}^{1,1}_{H_0}$

$GL(E) = Aut(E) =$ group of C^∞ bundle automorphisms of E

$gl(E) = \{C^\infty$ section of $End(E) = E \otimes E^*\}$

unitary gauge group $\mathcal{G} = \{ G \in GL(E) \mid G^{*H_0} \circ G = Id \}$

$\mathcal{G}$ acts on $\mathcal{A}_{H_0}$

$$\mathcal{G} \times \mathcal{A}_{H_0} \to \mathcal{A}_{H_0}$$
$$(G, D_A) \mapsto D_{G(A)} \overset{\circ}{=} G \circ D_A \circ G^{-1}$$
$$= (G \circ D_A - D_A \circ G) G^{-1} + D_A$$
$$= D_A - (D_A G) G^{-1}$$

locally $D_A = d + A$ $\quad D_{G(A)} = d + \boxed{G(A)}$
$$= G \circ (d + A) \circ G^{-1}$$
$$= d - (dG) G^{-1} + G A \circ G^{-1}$$

Check $D_{G(A)}$ compatible with H_0

$$(D_{G(A)})^{*H_0} = (G^{-1})^{*H_0} \circ (D_A)^{*H_0} \circ (G)^{*H_0}$$
$$= G \circ D_A \circ G^{-1} = D_{G(A)}$$

$$\langle D_{G(A)} X, Y \rangle_{H_0} + \langle X, D_{G(A)} Y \rangle_{H_0}$$
$$= \langle G \circ D_A \circ G^{-1} X, Y \rangle_{H_0} + \langle X, G \circ D_A \circ G^{-1} Y \rangle$$

$G^{*H_0} = G^{-1}$

$$= \langle D_A \circ G^{-1} X, G^{-1} Y \rangle + \langle G^{-1} X, D_A \circ G^{-1} Y \rangle = d \langle X, Y \rangle$$

Complexification $\mathcal{G}^{\mathbb{C}}_{(E, H_0)}$ $\mathcal{G}^{\mathbb{C}} \times \mathcal{A}^{1,1}_{H_0} \to \mathcal{A}^{1,1}_{H_0}$

not necessarily unitary $(G, D_A) \mapsto D_{G(A)}$

$$D_A = \bar\partial_A + \partial_A \qquad D_{G(A)} = \bar\partial_{G(A)} + \partial_{G(A)}$$

$$\bar\partial_{G(A)} = G \circ \bar\partial_A \circ G^{-1}$$

$G(A)$ is H_0-unitary $\Leftrightarrow (\bar\partial_{G(A)})^{*H_0} = \partial_{G(A)}$

thus $\partial_{G(A)} = (G^{*H_0})^{-1} \circ \partial_A \circ G^{*H_0}$

$$\begin{array}{ccc} \mathcal{A}^{1,1}_{H_0}(E) & \longleftrightarrow & \mathcal{H}^{1,1}(E) \\ G \updownarrow & & \downarrow G \\ \mathcal{A}^{1,1}_{H_0}(E) & \longleftrightarrow & \mathcal{H}^{1,1}(E) \end{array}$$

$$\bar\partial_2 = G \circ \bar\partial_1 \circ G^{-1}$$

The space of equivalent classes of holomorphic structures of E is $\mathcal{A}^{1,1}_{H_0} / \mathcal{G}^{\mathbb{C}}$

D-U-Y Thm. $(E, \bar\partial_E)$ stable $\Rightarrow \exists!$ H-E metric H.

$h = H_0^{-1} H > 0 \quad \exists G \in \mathcal{G}^{\mathbb{C}}$ s.t. $G^{*H_0} \circ G = h$

$$F_{D_{G(A_0)}} = G \circ F_{D_H} \circ G^{-1}$$

$D_{A_0} = D_{(\bar\partial_E, H_0)}$

$D_H = D_{(\bar\partial_E, H)} \notin \mathcal{A}_{H_0}$

$$D_{G(A_0)} = G\circ \bar\partial_{A_0}\circ G^{-1} + (G^{*H_0})^{-1}\circ \partial_{A}\circ G^{*H_0}$$

$$F_{D_{G(A_0)}} \overset{(*)}{=} G\circ F_{D_H}\circ G^{-1} \qquad \Rightarrow \sqrt{-1}\Lambda_\omega F_{D_{G(A_0)}} = \lambda \mathrm{Id}$$

$$D_{G(A_0)} \in \mathcal{A}^{1,1}_{H_0}$$

$$(*)\ F_{D_{G(A_0)}} = \bar\partial_{G(A_0)}\partial_{G(A_0)} + \partial_{G(A_0)}\bar\partial_{G(A_0)}$$

$$= G\circ\bar\partial_{A_0}\circ G^{-1}\circ (G^*)^{-1}\circ \partial_{A_0}\circ G^* + (G^*)^{-1}\circ\partial_{A_0}\circ G^*\circ G\circ \bar\partial_{A_0}\circ G^{-1}$$

$$= G\circ\bar\partial_{A_0}\circ \underbrace{h^{-1}\circ \partial_{A_0}\circ h}_{\substack{\shortparallel \\ h^{-1}\circ(\partial_{A_0}h)+\partial_{A_0}}}\circ G^{-1} + G\circ h^{-1}\circ G_{A_0}\circ h_0\circ \bar\partial_{A_0}\circ G^{-1}$$

$$= G\circ\left(\bar\partial_{A_0}\circ\partial_{A_0} + \partial_{A_0}\circ\bar\partial_{A_0} + \bar\partial_{A_0}(h^{-1}\circ(\partial A_0 h))\right)\circ G^{-1}$$

$$= G\circ F_{D_H}\circ G^{-1} \qquad \begin{cases} D_H = D_{A_0} + h^{-1}(\partial A_0 h) \\ F_{D_H} = F_{H_0} + \bar\partial A_0(h^{-1}\partial A_0 h)\end{cases}$$

Thus if $D_{A_0}\in \mathcal{A}^{1,1}_{H_0}$ define a stable structure $(E,\bar\partial_{A_0})$

$\exists$ a H-E connection $D_{G(A_0)}$ in the $\mathcal{G}^{\mathbb{C}}$-orbit of A_0.

Conversely if $D_{A_0}\in\mathcal{A}^{1,1}_{H_0}$ is H-E, then $(E,\bar\partial_{A_0})$

is polystable.

In 9.3, we gave

$$\underbrace{\int_M |F_A|^2 \frac{\omega^n}{n!}}_{\text{Yang-Mills functional}} = \int_M |\sqrt{-1}\Lambda_\omega F_A|^2 \frac{\omega^n}{n!} + 4\pi^2 \int_M \left(2c_2(E) - c_1(E)\wedge c_1(E)\right) \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$= \int_M \left(|\sqrt{-1}\Lambda_\omega F_A - \lambda \mathrm{Id}|^2 + \lambda^2 \operatorname{rank}(E) \right) \frac{\omega^n}{n!} + \underbrace{\cdots}_{\text{free of the choice of } A}$$

H-E connection $\longrightarrow$ absolute minimum of Y-M functional

$$F_{A_t} = (D_{A_0} + \theta_t)^2 = F_{A_0} + D_{A_0}\theta_t + \theta_t \wedge \theta_t$$

$$\frac{d}{dt}\Big|_{t=0} \int_M |F_{A_t}|^2 \frac{\omega^n}{n!}$$

$$= 2\int_M \operatorname{Re} \left\langle D_{A_0} \frac{d\theta}{dt}, F_{A_0} \right\rangle \frac{\omega^n}{n!} = 2\operatorname{Re} \int_M \left\langle \frac{d\theta}{dt}, D_{A_0}^* F_{A_0} \right\rangle \frac{\omega^n}{n!}$$

$$\Rightarrow D_{A_0}^* F_{A_0} = 0$$

Kähler case:

$$\bar\partial_A^* = -\sqrt{-1}[\Lambda_\omega, \partial_A]$$

$$\partial_A^* = \sqrt{-1}[\Lambda_\omega, \bar\partial_A]$$

$$\sqrt{-1}\Lambda_\omega F_A = \lambda \mathrm{Id} \Rightarrow \bar\partial_A^* F_A = \partial_A^* F_A = 0$$

11.2 Donaldson's heat flow. H-Y-M flow

H-Y-M flow: $H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id})$

Y-M flow: $\frac{\partial A}{\partial t} = -D^* F_A$

$\frac{\partial H}{\partial t}(X,Y) = \langle H^{-1}\frac{\partial H}{\partial t}(X), Y\rangle$ $\quad H^{-1}\frac{\partial H}{\partial t} \in \Gamma(\mathrm{End}(E))$

$(E, \bar\partial_E)$

$h(t) = H_0^{-1}H(t) \in GL(E)$ $\quad \partial_H - \partial_{H_0} = h^{-1}\partial_{H_0} h$

$F_H - F_{H_0} = \bar\partial_E(h^{-1}\partial_{H_0} h)$

$$H^{-1}\frac{\partial H}{\partial t} = h^{-1}\frac{\partial h}{\partial t}$$

$$= -2\sqrt{-1}\Lambda_\omega(F_{H_0} + \bar\partial_E(h^{-1}\partial_{H_0} h)) + 2\lambda \mathrm{Id}$$

$$\Rightarrow \frac{\partial h}{\partial t} = -2\sqrt{-1}\Lambda_\omega \bar\partial_E \partial_{H_0} h + 2\sqrt{-1}\,\bar\partial_E h \cdot h^{-1} \cdot \partial_{H_0} h - 2\sqrt{-1}\, h \Lambda_\omega F_{H_0} + 2\lambda h$$

$$\begin{cases} h(t)|_{t=0} = \mathrm{Id}_E \end{cases} \qquad h^{*H_0} = h$$

Line Bundle case: $h = e^f$ $\quad \frac{\partial f}{\partial t} = \Delta f + 2\sqrt{-1}\Lambda_\omega(\bar\partial f \wedge \partial f) + (*)$

Rmk H-Y-M flow is a nonlinear strictly parabolic equation.

Y-M flow:

(E, H_0)

$$\begin{cases} \frac{\partial A}{\partial t} = -D_A^* F_A \\ A|_{t=0} = A_0 \in \mathcal{A}_{H_0}^{1,1} \end{cases}$$

$(E, \bar{\partial}_{A_0})$ is holomorphic.

D_{A_0} is the chern connection

If H-Y-M has a long-time solution, then $\exists!$ solution of Y-M in the complex gauge group of $A_0 \in \mathcal{A}_{H_0}^{1,1}$

$\sigma(t) \in \mathcal{G}^{\mathbb{C}}$ $\quad \sigma(t)^{*H_0} \circ \sigma(t) = h(t) = H_0^{-1} H(t)$

$H(t)$: solution of H-Y-M

$\tilde{A}(t) = \sigma(t)(A_0)$. then $F_{D_{\sigma(t)(A_0)}} = \sigma(t) \circ F_{D_{H(t)}} \circ \sigma^{-1}(t)$

$$\sigma^*(t)^{-1} \frac{\partial h}{\partial t} \sigma^{-1}(t) = (\sigma^*)^{-1} \frac{\partial \sigma^*}{\partial t} + \frac{\partial \sigma}{\partial t} \cdot \sigma^{-1}$$

$$\begin{aligned} &\| \\ &-2\sqrt{-1}\, \sigma \Lambda_\omega F_{D_H} \circ \sigma^{-1} + 2\lambda \mathrm{Id}_E \\ &\| \\ &-2\sqrt{-1}\, \Lambda_\omega F_{\tilde{A}(t)} + 2\lambda \mathrm{Id}_E \end{aligned}$$

$$\frac{\partial \tilde{A}(t)}{\partial t} = \frac{\partial}{\partial t}\left(\sigma \circ \bar{\partial}_{A_0} \circ \sigma^{-1} + (\sigma^*)^{-1} \circ \partial_{A_0} \circ \sigma^*\right)$$

$$= -\bar{\partial}_{\tilde{A}}\left(\frac{\partial \sigma}{\partial t} \circ \sigma^{-1}\right) + \partial_{\tilde{A}}\left((\sigma^*)^{-1} \frac{\partial \sigma^*}{\partial t}\right)$$

$$= \frac{1}{2}(\partial_{\tilde{A}} - \bar{\partial}_{\tilde{A}})\left(\frac{\partial \sigma}{\partial t} \circ \sigma^{-1} + (\sigma^*)^{-1} \frac{\partial \sigma^*}{\partial t}\right) - \frac{1}{2}(\partial_{\tilde{A}} + \bar{\partial}_{\tilde{A}})\left(\underbrace{\frac{\partial \sigma}{\partial t} \circ \sigma^{-1} - (\sigma^*)^{-1} \frac{\partial \sigma^*}{\partial t}}_{-2\alpha(t)}\right)$$

$$= -\sqrt{-1}(\partial_{\hat{A}} - \bar{\partial}_{\hat{A}}) \wedge_\omega F_{\hat{A}} + \tfrac{1}{2}(\partial_{\hat{A}} + \bar{\partial}_{\hat{A}})(\alpha(t))$$

$\alpha^*(t) = -\alpha(t)$

Let $S(t) \in \mathcal{G}$ be the unique solution to

$$\frac{dS}{dt} = S\alpha \qquad S(0) = I$$

Let $A(t) = S(t)(\hat{A}(t))$

$$\frac{\partial A(t)}{\partial t} = \frac{\partial}{\partial t}(S \circ D_{\hat{A}(t)} \circ S^{-1})$$

$$= -\sqrt{-1}(\partial_{A(t)} - \bar{\partial}_{A(t)}) \wedge_\omega F_{A(t)} = -D_A^* F_A$$

$$\frac{\partial \bar{\partial}_{\hat{A}}}{\partial t} = \sqrt{-1}\, \bar{\partial}_{\hat{A}} \wedge_\omega F_{\hat{A}}$$

$$\frac{\partial \partial_{\hat{A}}}{\partial t} = -\sqrt{-1}\, \partial_{\hat{A}} \wedge_\omega F_{\hat{A}}$$

Uniqueness: If $\hat{A}(t)$ is another solution

$$\frac{\partial \hat{A}(t)}{\partial t} = -D_{\hat{A}}^* F_{\hat{A}} = -\sqrt{-1}(\partial_{\hat{A}} - \bar{\partial}_{\hat{A}}) \wedge_\omega F_{\hat{A}}$$

Let $g(t) \in \mathcal{G}^{\mathbb{C}}$ $\frac{\partial g}{\partial t} g^{-1} = (-\sqrt{-1}\wedge_\omega F_{\hat{A}} + \lambda \mathrm{Id})$. $g(0) = \mathrm{Id}_E$

then $\frac{\partial}{\partial t}(g^{-1}(\hat{A}(t))) = \frac{\partial}{\partial t}\left(g^{-1} \circ \bar{\partial}_{\hat{A}} \circ g + g^* \circ \partial_{\hat{A}} \circ (g^*)^{-1}\right)$

$= 0 \quad \Rightarrow \hat{A}(t) = g(t)(A_0)$

$g^* H_0 g = \hat{h}(t) = H_0^{-1} \hat{H}(t) \qquad \hat{H}(t) = H_0 \hat{h}(t)$

$$F_{\hat{A}(t)} = g \circ F_{D_{\hat{A}(t)}} \circ g^{-1}$$

$$\Rightarrow \begin{cases} \hat{h}^{-1}\dfrac{\partial \hat{h}}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_{D_{\hat{H}(t)}} - \lambda \mathrm{Id}) \\ \hat{h}|_{t=0} = \mathrm{Id} \end{cases}$$

by uniqueness of H-Y-M. $\hat{H}(t) = H(t)$

$g^* g = \hat{h} = h$

$$\tilde{S} = g \circ \sigma^{-1} \Rightarrow \begin{cases} \tilde{S}^* S = \mathrm{Id} \\ \tilde{S}(0) = \mathrm{Id}_E \end{cases}$$

$$\frac{\partial}{\partial t}\hat{S} = \frac{\partial}{\partial t}\left(\frac{1}{2}\tilde{S} + \frac{1}{2}\tilde{S}^{*-1}\right)$$

$$= \frac{1}{2}\frac{\partial \tilde{S}}{\partial t} - \frac{1}{2}\tilde{S}^{*-1}\frac{\partial \tilde{S}^*}{\partial t}\tilde{S}^{*-1}$$

$$= \frac{1}{2}\hat{S}\left(-\frac{\partial \sigma}{\partial t}\cdot\sigma^{-1} + (\sigma^*)^{-1}\frac{\partial \sigma^*}{\partial t}\right)$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}\hat{S} = \hat{S}\alpha(t) \\ \tilde{S}(0) = \mathrm{Id}_E \end{cases}$$ thus $\tilde{S}(t) = S(t)$

then $\hat{A}(t) = \tilde{S}(t)\,\sigma(t)(A_0) = A(t)$ #.

Donaldson Stable $(E, \bar{\partial}_E) \Rightarrow H(t)$ is a solution of H-Y-M

$$H(t) \to H_\infty \quad t \to \infty$$

$$\downarrow$$

H-E metric

11.3 Basic Estimates

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id}) \\ H(t)|_{t=0} = H_0 \end{cases}$$

$h(t) = H_0^{-1}H(t)$.

Lemma. $\left(\dfrac{\partial}{\partial t} - \Delta\right) \mathrm{tr}(\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id}) = 0$

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id}|_H^2 = -4|\bar\partial_E(\sqrt{-1}\Lambda_\omega F_H)|_H^2 \le 0$$

Pf. $\Phi(H) = \sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id} = \sqrt{-1}\Lambda_\omega(F_{H_0} + \bar\partial_E(h^{-1}\partial_{H_0} h)) - \lambda \mathrm{Id}$

$$\frac{\partial}{\partial t}\Phi(H) = \sqrt{-1}\Lambda_\omega \bar\partial_E\left(-h^{-1}\frac{\partial h}{\partial t} h^{-1}\partial_{H_0} h + h^{-1}\partial_{H_0}\left(h \circ h^{-1} \circ \frac{\partial h}{\partial t}\right)\right)$$

$$= \sqrt{-1}\Lambda_\omega \bar\partial_E\left(-h^{-1}\frac{\partial h}{\partial t} h^{-1}\partial_{H_0} h + h^{-1}\partial_{H_0} h \circ h^{-1} \circ \frac{\partial h}{\partial t} + \partial_{H_0}\left(h^{-1}\frac{\partial h}{\partial t}\right)\right)$$

$$= \sqrt{-1}\Lambda_\omega \bar\partial_E \partial_H\left(h^{-1}\frac{\partial h}{\partial t}\right)$$

$$\frac{\partial}{\partial t}\mathrm{tr}\,\Phi(H) = \sqrt{-1}\Lambda_\omega \mathrm{tr}\left(\bar\partial_E \partial_H \left(h^{-1}\frac{\partial h}{\partial t}\right)\right)$$

$$= \sqrt{-1}\Lambda_\omega \bar\partial\partial\, \mathrm{tr}\left(\underbrace{h^{-1}\frac{\partial h}{\partial t}}_{-2\Phi(H)}\right)$$

$$= -2\sqrt{-1}\Lambda_\omega \bar\partial\partial\, \mathrm{tr}(\Phi(H)) = \Delta\, \mathrm{tr}(\Phi(H))$$

$$\Delta|\Phi|^2 = -2\sqrt{-1}\Lambda_\omega \bar\partial\partial\, \mathrm{tr}(\Phi H^{-1} \Phi^* H)$$

$$= \sum_{i=1}^n 2\partial_i \bar\partial_i \langle \Phi, \Phi\rangle_H$$

$$= 2\partial_i\left(\langle D^{0,1}_{\bar\partial_i}\Phi, \Phi\rangle + \langle \Phi, D^{1,0}_{\partial_i}\Phi\rangle\right)$$

$$= 2\left(\langle D^{0,1}_{\bar\partial_i}\Phi, D^{0,1}_{\bar\partial_i}\Phi\rangle + \underline{\langle D^{1,0}_{\partial_i}\Phi, D^{1,0}_{\partial_i}\Phi\rangle}\right)$$

$|\partial_H \Phi|^2 = |\bar\partial_E \Phi|^2$

$$+\underbrace{\langle D^{1,0}_{\partial_i} D^{0,1}_{\bar\partial_i}\Phi, \Phi\rangle + \langle \Phi, D^{0,1}_{\bar\partial_i} D^{1,0}_{\partial_i}\Phi\rangle}$$

$$2\langle [F_H(\partial_i, \bar\partial_i)], \Phi\rangle + 4\,\mathrm{Re}\langle \Phi, D^{0,1}_{\bar\partial_i} D^{1,0}_{\partial_i}\Phi\rangle.$$

$$\frac{\partial}{\partial t}|\Phi|^2_H = \frac{\partial}{\partial t}\operatorname{tr}\Phi H^{-1}\bar\Phi^T H = 2\,\mathrm{Re}\langle \frac{\partial}{\partial t}\Phi, \Phi\rangle + 2\langle [\Lambda_\omega F_H, \Phi], \Phi\rangle$$

#.

Lemma Stable bundle admits unique H-E metric

Pf If H, K are two H-E metric

$h = K^{-1}H \qquad F_H - F_K = \bar\partial_E(h^{-1}\partial_K h)$

$$0 = \operatorname{tr}(h\sqrt{-1}(\Lambda_\omega F_H - \Lambda_\omega F_K))$$

$$= -\frac{1}{2}\Delta \operatorname{tr} h + \underbrace{\operatorname{tr}(-\sqrt{-1}\Lambda_\omega \bar\partial h\, h^{-1}\partial_K h)}_{0 \leq} \Rightarrow \Delta \operatorname{tr} h \geq 0$$

Thus $\operatorname{tr} h = \text{const}$ (compact without boundary)

h is automorphism

$\bar\partial h = 0 \qquad$ stable $\Rightarrow$ simple $\Rightarrow h = \mu\,\mathrm{Id}$

#.

$H(t)$. $K(t)$ two families of metrics $\qquad h(t)=K^{-1}(t)H(t)$

① $$\left(\Delta-\frac{\partial}{\partial t}\right)\operatorname{tr}h = 2\operatorname{tr}(-\sqrt{-1}\Lambda_\omega\bar\partial h\, h^{-1}\partial_K h)-2\operatorname{tr}(\sqrt{-1}h(\Lambda_\omega F_H-\Lambda_\omega F_K))$$
$$-\operatorname{tr}(K^{-1}H\,\underbrace{H^{-1}\frac{\partial H}{\partial t}}-\underbrace{K^{-1}\frac{\partial K}{\partial t}}K^{-1}H)$$

② $$\left(\Delta-\frac{\partial}{\partial t}\right)\operatorname{tr}h^{-1} = 2\operatorname{tr}(-\sqrt{-1}\Lambda_\omega\bar\partial h^{-1}\, h\partial_H h)-2\operatorname{tr}(\sqrt{-1}h^{-1}(\Lambda_\omega F_K-\Lambda_\omega F_H))$$
$$-\operatorname{tr}(H^{-1}K\,\underbrace{K^{-1}\frac{\partial K}{\partial t}}-\underbrace{H^{-1}\frac{\partial H}{\partial t}}H^{-1}K)$$

③ $$2(\operatorname{tr}h)^{-1}\operatorname{tr}(-\sqrt{-1}\Lambda_\omega\bar\partial h\, h^{-1}\partial_K h)-(\operatorname{tr}h)^{-2}|d\operatorname{tr}h|^2\geq 0$$

④ $$2(\operatorname{tr}h^{-1})^{-1}\operatorname{tr}(-\sqrt{-1}\Lambda_\omega\bar\partial h^{-1}\, h\,\partial_H h^{-1})-(\operatorname{tr}h^{-1})^{-2}|d\operatorname{tr}h^{-1}|^2\geq 0$$

$$\Rightarrow 2(\operatorname{tr}h+\operatorname{tr}h^{-1})^{-1}\{-\sqrt{-1}\Lambda_\omega\bar\partial h\, h^{-1}\partial_K h-\sqrt{-1}\Lambda_\omega\bar\partial h^{-1}\, h\,\partial_H h^{-1}\}$$
$$\geq \frac{\operatorname{tr}h\,|d\operatorname{tr}h|^2+(\operatorname{tr}h^{-1})|d\operatorname{tr}h^{-1}|^2}{(\operatorname{tr}h+\operatorname{tr}h^{-1})\operatorname{tr}h\,\operatorname{tr}h^{-1}}$$
$$\geq \frac{1}{(\operatorname{tr}h+\operatorname{tr}h^{-1})^2}|d(\operatorname{tr}h+\operatorname{tr}h^{-1})|^2 \qquad ⑤$$

Pf of ③. normal coordinates $\{z^1,\cdots z^m\}$

$K(e_1,\cdots e_r)$ $\quad \langle e_i,e_j\rangle_{H_0}=\delta_{ij}$

Then for H. WLOG H is diagonal at P. i.e. h

$$\frac{1}{2}\frac{|d\,\mathrm{tr}h|^2}{(\mathrm{tr}\,h)^2}=\sum_{i=1}^{n}\left|\sum_{\alpha=1}^{r}\bar\partial_i h_\alpha^\alpha\right|^2\left(\sum_{\alpha=1}^{r}h_\alpha^\alpha\right)^{-2}$$

$$=\sum_{i=1}^{n}\left(\sum_{\alpha=1}^{r}h_\alpha^\alpha\right)^{-2}\left|\sum_{\alpha=1}^{r}(\bar\partial_i h_\alpha^\alpha)(h_\alpha^\alpha)^{-\frac{1}{2}}(h_\alpha^\alpha)^{\frac{1}{2}}\right|^2.$$

$$\overset{\text{Schwarz}}{\le}\sum_{i=1}^{n}\left(\sum_{\alpha=1}^{r}|\bar\partial_i h_\alpha^\alpha|^2(h_\alpha^\alpha)^{-1}\right)\left(\sum_{\beta=1}^{r}h_\alpha^\alpha\right)^{-1}.$$

$$\mathrm{tr}(-\sqrt{-1}\Lambda_\omega\bar\partial h\cdot h^{-1}\partial_k h)=\sum_{i=1}^{n}\bar\partial_i h_\beta^\alpha(h^{-1})_\alpha^\gamma\,\partial_i h_\gamma^\beta$$

$$=\sum_{i=1}^{n}\sum_{1\le\alpha,\beta\le r}\bar\partial_i h_\beta^\alpha(h_\alpha^\alpha)^{-1}\partial_i h_\alpha^\beta$$

$$=\sum_{i=1}^{n}\sum_{\alpha,\beta}|\bar\partial_i h_\beta^\alpha|^2(h_\alpha^\alpha)^{-1}\ge\sum_{i=1}^{n}\left|\sum_{\alpha=1}^{r}\bar\partial_i h_\alpha^\alpha\right|^2(h_\alpha^\alpha)^{-1}$$

$$\ge\frac{1}{2}\frac{|d\,\mathrm{tr}h|^2}{\mathrm{tr}\,h}\qquad\#.$$

Def Donaldson's distance:

$$\sigma(H,K)=\mathrm{tr}H^{-1}K+\mathrm{tr}K^{-1}H-2\,\mathrm{rank}\,E$$

then $\sigma(H,K)\ge 0.$ $\qquad\sigma(H,K)=0\Leftrightarrow H=K.$

Lemma. $H(t), K(t)$ the solutions of H-Y-M flow.

then $(\Delta-\frac{\partial}{\partial t})\,\sigma(H(t),K(t))\ge 0.$

If $H(0)=K(0)$, then $H(t)=K(t).$

pf. $h(t) = k^{-1}(t) h(t)$

$$(\Delta - \frac{\partial}{\partial t}) \operatorname{tr} h = 2\operatorname{tr}(\sqrt{-1}\Lambda_\omega \bar{\partial} h \cdot h^{-1} \cdot \partial_K h) \overset{①}{\geq} 0.$$

$$(\Delta - \frac{\partial}{\partial t}) \operatorname{tr} h^{-1} = 2\operatorname{tr}(\sqrt{-1}\Lambda_\omega \bar{\partial} h^{-1} h \cdot \partial_{H} h^{-1}) \overset{②}{\geq} 0$$

#

Lemma. $H(t)$ a solution of H-Y-M flow. $H(0) = H_0$

then (i) $(\Delta - \frac{\partial}{\partial t}) \log(\operatorname{tr} H_0^{-1} H(t) + \operatorname{tr} H^{-1} H_0) \geq -2|\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id|_{H_0}$

(ii) $\Delta \log(\operatorname{tr} H_0^{-1} H + \operatorname{tr} H^{-1} H_0) \geq -2|\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id|_{H_0}$

pf. $h(t) = H_0^{-1} H(t)$ ①② gives

$$(\Delta - \frac{\partial}{\partial t}) \operatorname{tr} h = 2\operatorname{tr} h(\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id) - 2\operatorname{tr} h(\sqrt{-1}\Lambda_\omega \cancel{F_H} - \lambda Id)$$
$$- \operatorname{tr}(\cancel{H_0^{-1} H(t) H^{-1}} \frac{\partial H}{\partial t}) + 2\operatorname{tr}(\sqrt{-1}\Lambda_\omega \bar{\partial} h\, h^{-1} \partial_K h)$$

$$(\Delta - \frac{\partial}{\partial t}) \operatorname{tr} h^{-1} = -2\operatorname{tr}(h^{-1}(\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id)) + 2\operatorname{tr}(\cancel{h^{-1}}(\sqrt{-1}\Lambda_\omega F_H - \lambda Id))$$
$$+ \operatorname{tr}(H^{-1} \cancel{\frac{\partial H}{\partial t}} h^{-1}) + 2\operatorname{tr}(\sqrt{-1}\Lambda_\omega \bar{\partial} h^{-1} \cdot h \cdot \partial_H h^{-1})$$

$$(\Delta - \frac{\partial}{\partial t}) \log(\operatorname{tr} h + \operatorname{tr} h^{-1}) = (\operatorname{tr} h + \operatorname{tr} h^{-1})^{-1} (\Delta - \frac{\partial}{\partial t})(\operatorname{tr} h + \operatorname{tr} h^{-1})$$
$$- \underline{(\operatorname{tr} h + \operatorname{tr} h^{-1})^{-2} |d(\operatorname{tr} h + \operatorname{tr} h^{-1})|^2}_{*}$$

$$= 2(\operatorname{tr} h + \operatorname{tr} h^{-1})^{-1} [\operatorname{tr}(h \sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id) - \operatorname{tr}(h^{-1}(\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda Id))]$$
$$+ \underline{2(\operatorname{tr} h + \operatorname{tr} h^{-1}) \{-\sqrt{-1}\Lambda_\omega \bar{\partial} h \cdot h^{-1} \partial_{H_0} h - \sqrt{-1}\Lambda_\omega \bar{\partial} h^{-1} \cdot h \cdot \partial_H h^{-1}\}} - * \overset{\text{by } ⑤}{\geq} 0$$

$\geq -2|\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda \mathrm{Id}|_{H_0}$ #

11.4 Existence of Long-time Solution of H-Y-M flow

Lemma. If $H(t)$ is the solution of H-Y-M flow defined for $0 \leq t < T < +\infty$ then $H(t)$ converges in C^0-topology to some continuous non-degenerate metric H_T.

pf. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall s', s'' \in (0, \delta)$ $\sigma(H(s'), H(s'')) < \varepsilon$.

$H(t) = H(t+s')$ $K(t) = H(t+s'')$

By the maximum principle of parabolic equation

$$\sup_M \sigma(H(t+s'), H(t+s'')) \leq \sup_M \sigma(H(s'), H(s'')) < \varepsilon.$$

then $\forall a, b \in (T-\delta, T)$ $\sup_M \sigma(H(a), H(b)) < \varepsilon$.

thus $t \to T$. $H(t)$ Cauchy $H(t) \to H_T$

$(\Delta - \frac{\partial}{\partial t}) |\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id}|^2 \geq 0$, thus

$$\sup_M |\sqrt{-1}\Lambda_\omega F_{H(t)} - \lambda \mathrm{Id}|_{H(t)} \leq \sup_M |\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda \mathrm{Id}|_{H_0} = C_0$$

$$\frac{\partial}{\partial t} \log \mathrm{tr} H_0^{-1} H = \left| \frac{1}{\mathrm{tr} H_0^{-1} H} \mathrm{tr}\, H_0^{-1} H \; H^{-1} \frac{\partial H}{\partial t} \right|$$

$$\leq 2 |\sqrt{-1}\Lambda_\omega F_{H(t)} - \lambda \mathrm{Id}|_{H(t)}$$

$$\log \operatorname{tr} H_0^{-1} H(s) - \log r \le \int_0^s 2 |\sqrt{-1}\Lambda_\omega F_{H(t)} - \lambda \mathrm{Id}|_{H(t)} \, dt$$
$$\le 2C_0 s < 2C_0 T$$
$$\Rightarrow \operatorname{tr} H_0^{-1} H(s) \le r e^{2C_0 T}$$

similarly $\operatorname{tr} H^{-1}(s) H_0 \le r e^{2C_0 T}$ $\Rightarrow \sigma(H_0, H(s)) \le C_1$

thus H_T is non-degenerate #.

$\operatorname{tr}(-\sqrt{-1}\bar\partial h h^{-1} \partial_{H_0} h) \sim |\partial_{H_0} h|^2_{H_0} = |\bar\partial h|^2_{H_0} \sim |h^{-1}\partial_{H_0} h|^2_{H_0} \triangleq |\eta|^2_{H_0}$

⑥ $\frac{1}{2}\Delta \operatorname{tr} h = \operatorname{tr}(-\sqrt{-1}\Lambda_\omega \bar\partial h \, h^{-1} \partial_{H_0} h) + \operatorname{tr} \sqrt{-1} h (\Lambda_\omega F_{H_0} - \Lambda_\omega F_{H_t})$

⑦ $$\left(\Delta - \frac{\partial}{\partial t}\right) \operatorname{tr} h = 2 \operatorname{tr}(-\sqrt{-1}\bar\partial h \, h^{-1} \partial_{H_0} h) + \operatorname{tr}(h (\sqrt{-1}\Lambda F_{H_0} - \lambda \mathrm{Id}))$$
$$\ge C_2 |\eta|^2_{H_0} - |\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda \mathrm{Id}| \operatorname{tr} h$$

⑧ $$\left(\Delta - \frac{\partial}{\partial t}\right) |\eta|^2 \ge |\nabla \eta|^2 - C_4 |\eta|^2 - C_5$$

C_2, C_4, C_5 depending only on $\operatorname{tr} h + \operatorname{tr} h^{-1}$

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|\eta|^2 + C_6 \operatorname{tr} h) \ge |\nabla \eta|^2 + |\eta|^2 - C_7$$

maximum principle $\Rightarrow \sup\limits_{M \times [0,T)} |\eta|^2 \le C_8$

Prop. H-Y-M flow has a unique solution $H(t)$ existing for $0 \le t < \infty$

11.5 Donaldson's Functional

$$\mathcal{M}(K,H)=\int_0^1\int_M tr[(\sqrt{-1}\Lambda_\omega F_H-\lambda Id)H^{-1}(s)\frac{\partial H}{\partial s}]\frac{\omega^n}{n!}ds$$

where $H(s)$ is a path connecting K and H $\quad H(0)=K$, $H(1)=H$

Lemma. $\mathcal{M}(K,H)$ is independent of the choice of H.

Lemma $H(\tau,s)$ $\quad H(\tau,0)=K \quad H(\tau,1)=H(\tau)$

$$\frac{d}{d\tau}\mathcal{M}(K,H(\tau))=\frac{d}{d\tau}\int_0^1\int_M tr(\sqrt{-1}\Lambda_\omega F_{H(\tau,s)}-\lambda Id)H^{-1}\frac{\partial H}{\partial s})\frac{\omega^n}{n!}ds$$

$$=\int_M(tr(\sqrt{-1}\Lambda_\omega F_H-\lambda Id)H^{-1}\frac{\partial H}{\partial \tau})\frac{\omega^n}{n!}\Big|_{s=0}^{1}$$

$$=\int_M \underline{tr(\sqrt{-1}\Lambda_\omega F_{H(\tau)}-\lambda Id)H^{-1}\frac{\partial H}{\partial \tau}}\frac{\omega^n}{n!}$$

$$=-2\int_M|\sqrt{-1}\Lambda_\omega F_{H(\tau)}-\lambda I|^2$$

thus H-Y-M flow is gradient flow of Donaldson's functional

Lemma. ① $\mathcal{M}(H_0,H')+\mathcal{M}(H',H'')=\mathcal{M}(H_0,H'')$

② $\mathcal{M}(H,aH)=0$

Lemma. $H(0)$ is a critical point of $\mu(K,H)$ iff $H(0)$ is a H-E metric and $\mu(K,H)$ attains the absolute minimum at $H(0)$

11.6 Donaldson-Uhlenbeck-Yau's Thm

$H_0 = Ke^f$. $\quad \mathrm{tr}(\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda \mathrm{Id})$

$$= \mathrm{tr}(\sqrt{-1}\Lambda_\omega (F_K + \bar\partial\partial f\ \mathrm{Id}) - \lambda \mathrm{Id})$$

solve $\frac{1}{2}\Delta f = \sqrt{-1}\Lambda_\omega \partial\bar\partial f = \frac{1}{\mathrm{rank}(E)} \mathrm{tr}(\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id})$

thus by conformal transform. we can assume $\mathrm{tr}(\sqrt{-1}\Lambda_\omega F_{H_0} - \lambda \mathrm{Id}) = 0$.

H-Y-M flow $\begin{cases} H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_H - \lambda \mathrm{Id}) \\ H(t)|_{t=0} = H_0. \end{cases}$

$$\Phi(H(t)) = \sqrt{-1}\Lambda_\omega F_{H(t)} - \lambda \mathrm{Id}.$$

Recall: $(\Delta - \frac{\partial}{\partial t})\mathrm{tr}(\Phi(H(t))) = 0$

then by initial condition $\mathrm{tr}(\Phi(H(t))) = \mathrm{tr}(\Phi(H_0)) = 0$

$\Rightarrow \operatorname{tr}\left(h^{-1}\frac{\partial h}{\partial t}\right) = \frac{\partial}{\partial t}\log\det h$ $\qquad \Rightarrow \det h(t) = \det h(0) = 1$

$\qquad \parallel$

$-2\operatorname{tr}(\Phi(H(t)) = 0$

Let $s(t) = \log h(t)$.

$\operatorname{tr} s = \log\det h(t) = 0$.

$H = H_0 e^s \quad \det(H_0^{-1}H) = 1 \quad \operatorname{tr} s = 0$.

$$M(H_0, H) = \int_0^1 \int_M \operatorname{tr}\left[\left(\sqrt{-1}\Lambda_\omega F_{H(t)} - \lambda \mathrm{Id}\right) s\right] \frac{\omega^n}{n!}\, dt$$

$$= \int_M \operatorname{tr}\left(s\sqrt{-1}\Lambda_\omega F_{H_0}\right) + \langle \Psi(s)(\bar{\partial}_E s), \bar{\partial}_E s \rangle_{H_0} \frac{\omega^n}{n!}$$

Chapter 12 Calabi–Yau Thm

12.1 Calabi–Yau's Thm

Thm (Calabi-Yau) (M, ω_0) Kähler. then for any real $(1,1)$ form $\psi \in 2\pi C_1(M)$ $\exists!$ Kähler metric $\omega \in [\omega_0]$ s.t. $\rho(\omega) = \psi$.

$$\rho(\omega_0) - \psi = \sqrt{-1}\,\partial\bar{\partial} F \qquad F \in C^\infty(M, \mathbb{R})$$

$\omega = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$. then $\rho(\omega) = \psi \Leftrightarrow \rho(\omega) - \rho(\omega_0) = \psi - \rho(\omega_0)$

$$\Leftrightarrow -\sqrt{-1}\,\partial\bar{\partial}\log\frac{\omega^n}{\omega_0^n} = -\sqrt{-1}\,\partial\bar{\partial} F$$

$$\Leftrightarrow \log\frac{\omega^n}{\omega_0^n} = F + C$$

$$\Leftrightarrow e^F = \frac{(\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n}{\omega_0^n}$$

complex Monge–Ampère

Locally $(z^1, \dots, z^n)$. $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$.

$$\det((g_0)_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det((g_0)_{i\bar{j}})$$

Let $(g_0)_{i\bar{j}} = (\psi_0)_{i\bar{j}}$.

$$\det((\psi_0 + \varphi)_{i\bar{j}}) = e^F \det((g_0)_{i\bar{j}})$$

12.2 Complex M-A Equation

$$(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)^n = e^F\omega_0^n \qquad F=0 \ni \varphi=c. \text{ solved}$$

$$\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)^n}{\omega_0^n} = e^{tF+C_t} \quad (\star) \qquad \int_M (e^{tF+Ct}-1)\omega_0^n = 1$$

Define $T = \{t\in[0,1] \mid (\star) \text{ can be solved}\}$

(1) $T\neq\varphi \to T=0$ ✓.

(2) T is open

(3) T is closed

$\Rightarrow T=[0,1]$

(2): $B_1 = \{\varphi\in C^{k,\alpha} \mid \omega_0+\sqrt{-1}\partial\bar\partial\varphi>0 \quad \int_M \varphi\frac{\omega_0^n}{n!}=0 \quad k\geq 2\}$

$B_2 = \{\varphi\in C^{k-2,\alpha} \mid \int_M e^\varphi \frac{\omega_0^n}{n!}=1\}$

$\Psi: B_1\to B_2. \qquad \varphi\mapsto \log\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi)^n}{\omega_0^n}$

differential $D\Psi_{\varphi_{t_0}}: T_{\varphi_{t_0}}B_1 \to T_{\Psi(\varphi_{t_0})}B_2.$

$u = \frac{\partial\varphi_t}{\partial t}\Big|_{t=t_0} \qquad T_{\varphi_{t_0}}B_1 = \{u\in C^{k,\alpha} \int_M u\frac{\omega_0^n}{n!}=0\}$