

几何测度论

Ref <Introduction to Geometric Measure Theory> Leon Simon

Chap1 测度论前置

1 基本记号

外测度 μ regular: $\forall A \subset X \quad \exists \mu$ -measurable $B \supset A$ s.t. $\mu(A) = \mu(B)$

Borel regular: Borel 集均可测

且 $\forall A \subset X \quad \exists$ Borel 集 $B \supset A$ s.t. $\mu(A) = \mu(B)$

X 上外测度 μ . $Y \subset X \Rightarrow X$ 上外测度 $\mu \llcorner Y$: $(\mu \llcorner Y)(Z) = \mu(Y \cap Z)$

Thm (Caratheodory 标准) (X, d) 上外测度 μ 满足

$\mu(A) + \mu(B) = \mu(A \cup B)$ ($\forall d(A, B) > 0$). 则 Borel 集均 μ -可测

pf. 只需证闭集 μ -可测. 即证对 C 闭, $\forall S \subset X \quad \mu(S) < \infty$ 有

$$\mu(S) \geq \mu(S \setminus C) + \mu(S \cap C)$$

$$C_j = \{x \in X \mid d(x, C) \leq \frac{1}{j}\} \Rightarrow d(S \setminus C_j, S \cap C) > 0$$

$$\Rightarrow \mu(S) \geq \mu(S \setminus C_j) + \mu(S \cap C)$$

$$\text{且证 } \mu(S \setminus C_j) \rightarrow \mu(S \setminus C) \text{ 即可}$$

X 上的 Borel regular 测度 μ 称为 open σ -有限. 若

$$X = \bigcup_j V_j \quad V_j \text{ 开} \quad \mu(V_j) < \infty.$$

例 可分度量空间上局部有限 $\Rightarrow$ open σ -finite

Thm. X 为度量空间 μ open σ -有限 Borel regular. 则

$$\mu(A) = \inf_{\substack{U \text{ 开} \\ U \supset A}} \mu(U) \quad (\forall A \subset X)$$

$$\mu(A) = \sup_{\substack{C \text{ 闭} \\ C \subset A}} \mu(C) \quad (\forall \text{ 可测 } A \subset X)$$

2. Hausdorff 测度

$$\mathcal{H}_\delta^m(A) = \inf_{\substack{\{C_j\} \text{ cover } A \\ \text{diam } C_j < \delta}} \sum_{j=1}^{\infty} \left(\frac{\text{diam } C_j}{2} \right)^m$$

$\downarrow$
 $\mathbb{R}^n$ 中单位球体积.

$$\mathcal{H}^m(A) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^m(A)$$

Fact. $\mathcal{H}^m(A)$ is Borel regular.

$$\text{Thm. } \underbrace{\mathcal{L}^n(A)}_{\text{Lebesgue 测度}} = \mathcal{H}^n(A) = \mathcal{H}_\delta^n(A) \quad (\forall A \subset \mathbb{R}^n, \delta > 0)$$

3 Densities

(Xd) μ 外测度 $\Theta^{*n}(\mu, A, x) = \overline{\lim}_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}$

Θ_*^n $\lim$ -

$A = X$ 时 $\Theta_{(x)}^{*n}(\mu, x)$

若 $\Theta^{*n}(\mu, A, x) = \Theta_*^n(\mu, A, x)$ 则 x 为 $\Theta^n(\mu, A, x)$

Thm (比较定理) μ Borel regular 外测度 $t \geq 0$ $A_1 \subset A_2 \subset X$

则 $\Theta^{*n}(\mu, A_2, x) \geq t \Rightarrow t \mathcal{H}^n(A_1) \leq \mu(A_2)$
($\forall x \in A_1$)

Pf. 首先承认如下两个覆盖有关的引理

引理① B 为 X 中闭球族且直径有界. 则 $\exists$ 两两不交子族 $B' \subset B$ s.t.

$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B' \in B'} \hat{B}$ 其中若 $B = B_\rho(x)$ 则 $\hat{B} = B_{3\rho}(x)$

进一步可使 $B \in \mathcal{B} \Rightarrow \exists B' \in B'$ $B \cap B' \neq \emptyset$ 且 $\text{diam}(B') \geq \frac{1}{2} \text{diam}(B)$ (*)

$\inf \{ \text{diam} B : x \in B \in \mathcal{B} \} = 0$ (则 $\hat{B}' \supset B$)
 $\uparrow$ ($\forall x \in A$)

② B 同上 A 被 B covered finely. 且 $B' \subset B$ 为满足(*)的不交子族. 则 $\forall \{B_i, B_N\} \subset B'$ $A \setminus \bigcup_{j=1}^N B_j \subset \bigcup_{B \in \mathcal{B} \setminus \{B_i, B_N\}} \hat{B}$

回来. 不妨 $\mu(A_2) < \infty$ $t > 0$ 则 $\forall \tau \in (0, t)$ $\Theta^{*n}(\mu, A_2, x) > \tau$ ($x \in A_1$)

$\forall \delta > 0$ $\mathcal{B} = \{ \text{闭球 } B_\rho(x) : x \in A_1, 0 < \rho < \frac{\delta}{2}, \mu(A_2 \cap B_\rho(x)) \geq \tau \omega_n \rho^n \}$

则 $B' \subset \mathcal{B}$ 同引理

又 $B_i, B_N \in B'$ 有

$\sum_{j=1}^N \mu(A_2 \cap B_j) \leq \mu(A_2) < \infty$ 故 B' 可数.

$\{B_{\rho_1}(x_1), B_{\rho_2}(x_2), \dots\}$

由引理 $\forall N \geq 1$. $A_1 \setminus \bigcup_{j=1}^N B_{\rho_j}(x_j) \subset \bigcup_{j=N+1}^{\infty} B_{\rho_j}(x_j)$

$$\Rightarrow \sum_{j=1}^{\infty} \omega_n \rho_j^n \leq \sum_{j=1}^{\infty} \mu(A_2 \cap B_{\rho_j}(x_j)) \leq \mu(A_2) < \infty \quad (*)$$

又 $A_1 \subset \bigcup_{j=1}^N B_{\rho_j}(x_j) \cup \bigcup_{j=N+1}^{\infty} B_{\rho_j}(x_j)$. 故

$$\mathcal{H}_{\delta}^n(A_1) \leq \sum_{j=1}^N \omega_n \rho_j^n + \sum_{j=N+1}^{\infty} \omega_n \rho_j^n \leq \mu(A_2)$$

$$\Rightarrow \text{令 } N \rightarrow \infty \text{ 由 } (*) \quad \sum \mathcal{H}_{\delta}^n(A_1) \leq \mu(A_2)$$

$$\text{令 } \delta \rightarrow 0 \quad \sum \mathcal{H}^n(A_1) \leq \mu(A_2)$$

Thm. (Upper Density) μ Borel regular $A \mu$ -可测 $\mu(A) < \infty$

$$\text{则 } \Theta^{*n}(\mu, A, x) = 0 \text{ for } \mu^n\text{-a.e. } x \in X \setminus A$$

Pf. $\subset \subset A$ 闭 $t > 0$ $S_t = \{x \in X \mid \Theta^{*n}(\mu, A, x) \geq t\}$

$$x \in S_t \text{ 有 } \Theta^{*n}(\mu, A \cap (X \setminus C), x) = \Theta^{*n}(\mu, A, x) \geq t$$

$$\xrightarrow{\text{由引理}} t \mathcal{H}^n(S_t) \leq \mu(A \setminus C)$$

$$\text{又 } \inf_{\substack{C \subset A \\ C \text{ 闭}}} \mu(C \setminus A) = 0 \Rightarrow \mathcal{H}^n(S_t) = 0 \quad \checkmark$$

Cor. $A \subset \mathbb{R}^n$ $\mathbb{L}^n$ -可测 $\mathbb{R}^n \setminus \Theta^n(\mathbb{L}^n, A, x) = 1$ $\mathbb{L}^n$ -a.e. $x \in \mathbb{R}^n \setminus A$
(Density Thm) 且 $\Theta^n(\mathbb{L}^n, A, x) = \begin{cases} 0 & \mathbb{L}^n\text{-a.e. } x \in X \setminus A \\ 1 & \mathbb{L}^n\text{-a.e. } x \in A \end{cases}$

Pf. 利用 $(\omega_n \rho^n)^{-1} \mathbb{L}^n(A \cap B_{\rho}(x)) + (\omega_n \rho^n)^{-1} \mathbb{L}^n(B_{\rho}(x) \setminus A) = 1 \quad \rho \rightarrow 0$.

Thm. $A \subset X$ 为 $\mathcal{H}^n$ -可测 $\mathbb{R}^n$

$$(1) \text{ 若 } \mathcal{H}^n(A) < \infty \text{ 则 } \Theta^{*n}(\mathcal{H}^n, A, x) \leq 1 \quad \mathcal{H}^n\text{-a.e. } x \in A$$

$$(2) \text{ 若 } \mathcal{H}_{\delta}^n(A) < \infty (\forall \delta > 0). \text{ 则 } \Theta^{*n}(\mathcal{H}_{\delta}^n, A, x) \geq 2^{-n} \text{ for } \mathcal{H}^n\text{-a.e. } x \in A$$

4. Differential Theorems

μ, μ_0 Borel regular
 $\downarrow$
 locally finite

$$\Theta^{\mu_0}(\mu, x) = \begin{cases} \lim_{p \rightarrow 0} \frac{\mu(B_p(x))}{\mu_0(B_p(x))} & x \in X \setminus (U_0 \cup V_0) \\ \infty & x \in U_0 \setminus V_0 \\ 0 & x \in V_0 \end{cases}$$

$$U_0 = \{x \in X : \mu_0(B_p(x)) = 0 \quad \forall p > 0\}$$

$$V_0 = \{x \in X : \mu(B_p(x)) = 0 \quad \forall p > 0\}$$

$$\mathbb{R}^1 \Theta^{\mu_0}(\mu, x) = \Theta^{\mu_0}(\mu, x) \quad (x \in \mathbb{R}^1, \mu_0 = \mathcal{L}^1)$$

(SVP)

Def. 外测度 μ_0 on X 满足 相对 Vitali 性质 若

$\forall A \subset X \quad \mu_0(A) < \infty$. B 为球心在 A 中的闭球族且 cover A finely

$\exists$ 互不相交的子族 $B' = \{B_{p_j}(x_j)\} \quad \text{s.t.} \quad \mu_0(A \setminus \bigcup_{j=1}^{\infty} B_{p_j}(x_j)) = 0$

Rmk. ① X 可分. $\mathbb{R}^1 \Theta^{\mu_0}$ Borel regular. $\mu_0(X) < \infty$. 且 $\exists C$ s.t.

$$\mu_0(B_p(x)) \leq C \mu_0(B_{p/2}(x)) \quad (\forall \text{ 闭球 } B_p(x))$$

则 μ_0 满足 SVP

② $\mathbb{R}^1$ 上有限 Borel regular μ_0 满足 SVP

与之前类似可证下面的命题

Thm. μ, μ_0 开 G-有限 Borel regular μ_0 满足 SVP. $A \subset X \quad t \geq 0$

$$\mathbb{R}^1 \Theta^{\mu_0}(\mu, x) \geq t \quad (\forall x \in A) \Rightarrow \mu(A) \geq t \mu_0(A)$$

Cor. μ, μ_0 同上. $\mathbb{R}^1 \Theta^{\mu_0}(\mu, x) < \infty$ for μ_0 -a.e. $x \in X$

General Upper Density Thm: μ, μ_0 Borel regular
 $\#G$ -a.e. sup

$$A \text{ } \mu\text{-TVR} \mid \underline{A}_\mu(A) < \infty. \quad \text{R) } \Theta^{*u_0}(\mu \ll A, x) = 0 \text{ for } \mu\text{-a.e. } x \in X \mid A$$

General Density Thm: $\mu \neq 0$ -有限 Borel regular. SVP

$$A \text{ is } \mu\text{-}\overline{\sigma}\text{-R). Then } \lim_{\rho \rightarrow 0} \frac{\mu(A \cap B_\rho(x))}{\mu(B_\rho(x))} = \begin{cases} 1, & \mu\text{-a.e. } x \in A \\ 0, & \mu\text{-a.e. } x \in X \setminus A \end{cases}$$

则可以用它们证明如下 Lebesgue 微分定理

Cor. X, μ 同 GDT. $f: X \rightarrow \mathbb{R}$ 局部 μ -可积. 则

$$\lim_{\rho \rightarrow 0} (\mu(B_\rho(x)))^{-1} \int_{B_\rho(x)} f d\mu = f(x) \quad \text{for } \mu\text{-a.e } x \in X$$

pf $f - f^+ - f^- \Rightarrow$ 不恒为 $f \geq 0$ 且不恒为 $\mu(X) < \infty$

$$L_{\text{usin}} \Rightarrow \exists i \nexists j A_j, \quad u(X \setminus \bigcup A_j) = 0 \quad f|_{A_j} \text{ cts}$$

又由GDT. $\lim_{p \rightarrow \infty} \frac{\mu(A_i \cap B_p(x))}{\mu(B_p(x))} = \chi_A, \mu\text{-a.e. on } X$

由 f cts. $\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(x) \cap A_j} f d\mu}{\mu(B_\rho(x))} = f(x)$ μ -a.e. on A_j

再对 $\nu = \int f d\mu$ 和 μ 用 GUDT $\Rightarrow \lim_{p \rightarrow 0} \frac{\int_{B_p(x)} V_j f d\mu}{\mu(B_p(x))} = 0$ μ -a.e. on A_j .

与上述度类似定义下度 $\Theta_{*}^{\mu_0}(\mu, x)$ 也有如下的比较定理

Thm μ_0, μ 开 σ -有限 Borel regular $t > 0$ $A \subset X$

$\Theta_{\mu_0}^*(\mu, x) \leq t$ ($\forall x \in A$) 则

(1) 若 μ SVP. 则 $\mu(A) \leq t\mu_0(A)$

(2) 若 μ_0 SVP 则 $\mu(A \setminus B) \leq t\mu_0(A)$ 其中 B Borel $\mu_0(B) = 0$
 $\mu \ll (X \setminus B) \ll \mu_0$

Thm (比较定理) μ, μ_0 同上 则

(1) 若 μ SVP. 则 $\exists \mu$ -零测 Borel 集 S s.t. $\Theta^{\mu_0}(\mu, x)$ 在 $X \setminus S$ 存在

(2) 若 μ_0 SVP. $\dots \mu_0 \dots \dots \Theta^{\mu_0}(\mu, x) \dots \dots$ 且有

事实上 $\Theta^{\mu_0}(\mu, x)$ 即为 Radon-Nikodym 定理中的 "derivative"

Thm (Radon-Nikodym) μ, μ_0 同上 μ_0 SVP.

(1) 若 $\mu \ll \mu_0$ (则 μ 也 SVP). 且

$$\mu(A) = \int_A \Theta^{\mu_0}(\mu, x) d\mu_0(x) \quad \forall \text{ Borel } A \subset X$$

(2) 若 $\mu \ll \mu_0$ 且 $\mu \not\ll \mu_0$ 则 $\exists$ Borel Z $\mu_0(Z) = 0$

$$\text{s.t. } \mu(A) = \int_A \Theta^{\mu_0}(\mu, x) d\mu_0(x) + (\mu \ll Z)(A) \quad \forall \text{ Borel } A \subset X$$

(3) 若 μ SVP. 则 (2) 中的 Z 为 $Z = \{x \in X, \Theta^{\mu_0}(\mu, x) = \infty\}$

5. Radon 测度与表示定理

Def. X Hausdorff 则外测度 μ 为 Radon 测度. 若满足

μ Borel regular 且紧有限 (R1)

$$\mu(A) = \inf_{\substack{u \supset A \\ u \text{ 开}}} \mu(u) \quad \forall A \subset X \quad (R2)$$

$$\mu(u) = \sup_{\substack{K \subset u \\ K \text{ 紧}}} \mu(K) \quad \forall \text{ 开 } u \subset X \quad (R3)$$

Prop. 若 μ Radon 则上的 (R3) 对一族的 A $\mu(A) < \infty$ 均成立

Thm. X LCH, μ Radon, $1 \leq p < \infty$. 则 $C_c(X)$ 在 $L^p(\mu)$ 中稠密

Pf. $f \in L^p(\mu)$. $\varepsilon > 0$. 取 simple function $\varphi = \sum_{j=1}^n a_j \chi_{A_j}$ (A_j 互交)

s.t. $\|f - \varphi\|_p < \varepsilon$ 且 $\|\varphi\|_p < \infty$. (则 $\mu(A_j) < \infty$)

对 $M > \max_{1 \leq j \leq n} |a_j|$ 取紧 $K_j \subset A_j$ s.t. $\mu(A_j \setminus K_j) < \frac{\varepsilon^p}{2^{p+1} M^p n}$

再取开 U_j s.t. $\mu(U_j \setminus K_j) < \frac{\varepsilon^p}{2^{p+1} M^p n}$

不妨 $U_1 \cap \dots \cap U_n \neq \emptyset$

由 LCH 空间的 Urysohn 引理 $\exists g_j \in C_c(X)$

s.t. $g_j = a_j$ on K_j $\{g_j \neq 0\}$ 包含在 U_j 的闭集中

且 $\sup |g_j| \leq |a_j|$.

则令 $g = \sum_{j=1}^n g_j$ 有 $g = \varphi$ on K_j

$$\sup |g| = \sup |\varphi| < M$$

1) $\varphi - g = 0$ on $\bigcup_j (U_j \setminus K_j) \cup (A_j \setminus K_j)$

$$\Rightarrow \int_X |\varphi - g|^p d\mu \leq \sum_j \int_{(U_j \setminus K_j) \cup (A_j \setminus K_j)} |\varphi - g|^p d\mu$$

$$\leq (2M)^p \sum_j \mu(U_j \setminus K_j) + \mu(A_j \setminus K_j) \leq \varepsilon^p$$

$$\Rightarrow \|f - g\|_p < 2\varepsilon \quad \checkmark$$

有 Riesz Hilbert

Thm (Riesz 表示) X LCH. $L: C_c(X, \underline{H}) \rightarrow \mathbb{R}$ 线性且

$$\sup_{\substack{f \in C_c(X, H) \\ |f| \leq 1 \\ \text{supp } f \subseteq K}} L(f) < \infty \quad \forall K \subset X' \text{ 紧}$$

1) $\exists$ Radon measure μ 和 Borel 映射 $\nu: X \rightarrow H$ $|\nu| = 1$ μ -a.e. on X

$$\text{s.t. } L(f) = \int_X \langle f, \nu \rangle d\mu \quad \forall f \in C_c(X, H)$$

Thm (紧性定理) X LCH 6-紧 $\{\mu_k\}$ Radon 测度

且 $\sup_k \mu_k(K) < \infty$ ($\forall K \subset X'$ 紧). 则 $\exists$ 子列 $\{\mu_{k_j}\}$

s.t. $\mu_{k_j} \rightarrow \mu$ (i.e. $\forall f \in C_c(X)$ $\mu_{k_j}(f) \rightarrow \mu(f)$)

$\{f: X \rightarrow \mathbb{R} \mid f \text{ 有界, 紧支}\}$

Pf. $K_1 \subset K_2 \subset \dots$ 紧 s.t. $X = \bigcup_j K_j$ $F_{k_j}: C(K_j) \rightarrow \mathbb{R}$
 $f \mapsto \int_{K_j} f d\mu_{k_j}$

由 Banach-Alaoglu $\exists F_{j, k_j}$ 有界 $F_{j, k_j}: C(K_j) \rightarrow \mathbb{R}$ s.t. $F_{j, k_j}(f) \rightarrow F_j(f)$
 $(\forall f \in C(K_j))$

再用对偶性证明 Riesz 表示定理.

Chap 2 分析前置

1. Lipschitz 常数

Thm ACX $f: A \rightarrow \mathbb{R}$ Lipschitz. 则 $\exists \tilde{f}: X \rightarrow \mathbb{R}$ s.t. $\begin{cases} \text{Lip } \tilde{f} = \text{Lip } f \\ f = \tilde{f}|_A \end{cases} (*)$
 还可使得 $\sup_X |\tilde{f}| = \sup_A |f|$

p.f. 令 $L = \text{Lip } f$. $\tilde{f}(x) = \inf_{z \in A} (f(z) + Ld(x, z))$

可证 $\tilde{f}$ 满足 $(*)$. 再验证即可

Thm (Rademacher) f Lipschitz on $\mathbb{R}^n$ 则 f $\mathbb{R}^n$ -a.e. 可微

p.f. $v \in S^{n-1}$ 定义 $D_v f(x) = \frac{d}{dt} f(x+tv)|_{t=0}$ (无论是否存在)

由 $|\frac{f(y)-f(x)}{|y-x|}| \leq \text{Lip } f$ 故 $D_v f(x)$ 存在 $\Leftrightarrow \limsup_{t \rightarrow 0} \frac{f(x+tv) - f(x)}{t} = \liminf_{t \rightarrow 0} \dots$

则 $A_v = \{x \in \mathbb{R}^n : D_v f(x) \text{ 不存在}\}$ Borel 测度 $\Rightarrow \mathbb{R}^n$ 可测

$\left(\lim_{s \rightarrow \infty} \sup_{0 < |t| \leq \frac{1}{s} } \frac{f(x+tv) - f(x)}{t} \right)$
 LSC $\rightarrow$ Borel

$\varphi(t) = f(x+tv)$ AC for fixed $x, v \Rightarrow$ 对几乎所有 t 可微

则 $\forall L \geq 0$. $A_v \cap L$ 包含在 $\mathcal{H}'$ -0 测度集中 $\stackrel{\text{Fubini}}{\Rightarrow} A_v$ 为 $\mathbb{R}^n$ -可测 $(*)$

$\forall \zeta \in C_c^\infty(\mathbb{R}^n)$. $h > 0$

$$\int_{\mathbb{R}^n} \frac{f(x+hv) - f(x)}{h} \zeta(x) d\mathbb{I}^n(x) = - \int_{\mathbb{R}^n} \frac{\zeta(x) \cdot \zeta(x-hv)}{h} f(x) d\mathbb{I}^n(x)$$

$$\stackrel{(*)}{\Rightarrow} \int D_v f \zeta = - \int f D_v \zeta = - \int f v \cdot \nabla \zeta$$

$$\stackrel{\text{DCT}}{=} - \sum_{j=1}^n v_j \int f D_j \zeta = \sum_{j=1}^n v_j \int \zeta D_j f = \int \zeta v \cdot \nabla f$$

ζ 任意 $\Rightarrow \forall v \in S^{n-1}$. $\nabla f(x)$ 存在且 $D_v f(x) = v \cdot \nabla f(x)$ for $\mathbb{I}^n$ -a.e. x
 故 $\|\nabla f(x)\| \leq L$

取 S^{n-1} 的可数稠密子集 $\{v_k\}$ $A_k \triangleq \{x: \nabla f(x), D_{v_k} f(x) \text{ 存在且 } D_{v_k} f(x) = \nabla f(x) \cdot v_k\}$

又 $A = \bigcup_{k=1}^{\infty} A_k$ 有 $1^n(\mathbb{R}^n \setminus A) = 0$ 且 $D_{v_k} f(x) = v_k \cdot \nabla f(x)$ ($\forall x \in A, k \in \mathbb{Z}^+$)

则 $\forall x \in A, v \in S^{n-1}, h > 0$ 定义 $Q(x, v, h) = \frac{f(x+hv) - f(x)}{h} - v \cdot \nabla f(x)$

$\forall \varepsilon > 0, \exists P$ s.t. $S^{n-1} \subset \bigcup_{i=1}^P B_\varepsilon(v_i)$

又 $\lim_{h \rightarrow 0} Q(x, v_i, h) = 0 \quad \exists \delta_i > 0$ s.t. $|Q(x, v_i, h)| < \varepsilon$ ($\forall x, |h| < \delta_i$)

则 $\forall v \in S^{n-1}$ 取 i s.t. $|v - v_i| < \varepsilon$ 有

$$\begin{aligned} |Q(x, v, h)| &\leq |Q(x, v, h) - Q(x, v_i, h)| + |Q(x, v_i, h)| \\ &\leq \frac{1}{|h|} |f(x+hv) - f(x+hv_i)| + |v - v_i| |\nabla f(x)| + |Q(x, v_i, h)| \\ &< (2L + 1)\varepsilon \quad (0 < |h| < \delta = \min_{1 \leq i \leq P} \delta_i) \end{aligned}$$

$\Rightarrow f$ 在 x 处可微

Thm (C' 逼近) $f: \mathbb{R}^n \rightarrow \mathbb{R}$ Lipschitz. 则 $\forall \varepsilon > 0, \exists g \in C'(\mathbb{R}^n)$
s.t. $1^n(\{f(x) \neq g(x)\} \cup \{\nabla f(x) \neq \nabla g(x)\}) < \varepsilon$

pf. 可以参考如下

Whitney Extension Thm: $C \subset \mathbb{R}^n$ 凸, $h: C \rightarrow \mathbb{R}, v: C \rightarrow \mathbb{R}^n$ 均连续

若 $\forall K \subset C$ 有 $\lim_{\substack{y \rightarrow x \\ y \in K}} R(x, y) = 0$ uniformly for $x \in K$

$$\text{其中 } R(x, y) = \frac{h(y) - h(x) - v(x) \cdot (y - x)}{|y - x|}$$

则 $\exists g \in C'(\mathbb{R}^n)$ s.t. $g|_C = h, \nabla g = v$ on C

$$\mathbb{R}^n \text{ 对 } R(x,y) = \frac{|f(y) - f(x) - \nabla f(x) \cdot (y-x)|}{|y-x|}$$

由 Rademacher's: $\eta_k(x) = \sup_{0 < |y-x| < k} |R(x,y)| \rightarrow 0$ for $\mathbb{L}^n$ -a.e. x

$$\text{Eg. } \exists \mathbb{L}^n\text{-measurable } E_j \subset B_j(0) \setminus B_{j-1}(0) \quad \mathbb{L}^n((B_j(0) \setminus B_{j-1}(0)) \cap E_j) < \frac{\varepsilon}{2^{j+1}}$$

$$\text{s.t. } \eta_k \Rightarrow 0 \quad \text{on } E_j$$

$$\text{又由 Lebesgue's } \exists \frac{1}{2} C_j \subset E_j \quad \mathbb{L}^n(E_j \setminus C_j) < \frac{\varepsilon}{2^{j+1}} \quad \nabla f|_{C_j} \text{ cts}$$

$$\text{令 } C = \bigcup_{j=1}^{\infty} C_j \cap \mathbb{R}^n \Rightarrow \nabla f|_C \text{ cts.} \quad \mathbb{L}^n(\mathbb{R}^n \setminus C) < \varepsilon$$

$$\eta_k \rightarrow 0 \text{ 内点处 on } C$$

由 Whitney 定理立得

6-10

Thm $\underline{X}$ 度量空间 $A \subset X$ $\mathcal{H}^m$ -可测 且 $\mathcal{H}^m(A) < \infty$

$$f: A \rightarrow Y \text{ Lip.} \quad \text{记 } N(f,y) = \mathcal{H}^0(f^{-1}y) \quad (\text{即 } f^{-1}(y) \text{ 点数})$$

$$\text{则 (i) } f(A) \text{ } \mathcal{H}^m\text{-可测 且 } \mathcal{H}^m(f(A)) \leq (\text{Lip } f)^m \mathcal{H}^m(A)$$

$$(ii) N(f,y) \text{ 为关于 } y \text{ 的 } \mathcal{H}^m\text{-可测函数. 且}$$

$$\int_Y N(f,y) d\mathcal{H}^m \leq (\text{Lip } f)^m \mathcal{H}^m(A)$$

$$\text{Pf. (i) 由 } \mathbb{R}^n\text{-性质 6-10, } \exists k_1, k_2, \dots \subset \mathbb{R}^n \quad K_j \subset A \text{ s.t. } \mathcal{H}^m(A \setminus \bigcup_j K_j) = 0$$

$$\Rightarrow \mathcal{H}^m(f(A \setminus \bigcup_j K_j)) = 0 \Rightarrow A \text{ } \mathcal{H}^m\text{-可测}$$

$$\text{又 } f(K_j) \text{ cpt} \Rightarrow \text{Borel} \Rightarrow \mathcal{H}^m\text{-可测}$$

$$A \subset \bigcup_j C_j, \text{ diam } C_j < \delta, \quad \mathbb{R}^n \text{ 中 } f(A) \subset \bigcup_j f(C_j) \quad \text{diam } f(C_j)$$

$$\leq (\text{Lip } f) \delta \\ < (1 + \text{Lip } f) \delta$$

$$\text{则 } \mathcal{H}_{(1+\text{Lip } f)\delta}^m(f(A)) \leq (\text{Lip } f)^m \sum_j \omega_m \left(\frac{\text{diam } C_j}{2} \right)^m$$

$$\delta \rightarrow 0 \text{ 即得}$$

(ii) $A = \bigcup_{j=1}^{\infty} A_{ij}$. $\text{diam}(A_{ij}) < \frac{1}{j}$. A_{i+j} 包含于某个 A_{ik}

$$\stackrel{\text{HST}}{\Rightarrow} \int_Y N(f, y) d\mathcal{H}^m(y) = \lim_{i \rightarrow \infty} \int_Y \sum_j \chi_{f(A_{ij})} d\mathcal{H}^m$$

$$= \lim_{i \rightarrow \infty} \sum_j \underbrace{\mathcal{H}^m(f(A_{ij}))}$$

$$\leq (\text{Lip } f)^m \sum_j \mathcal{H}^m(A_{ij}) = (\text{Lip } f)^m \mathcal{H}^m(A)$$

Thm. X $\mathbb{R}^n$ -测度 $m \in \{1, 2, \dots\}$ $k > 0$ (正整数) $A \subset X$ $\mathcal{H}^{m+k}$ -可测 $\mathcal{H}^{m+k}(A) < \infty$

$f: A \rightarrow \mathbb{R}^n$ Lip. 则 $\mathcal{H}^k(f^{-1}y)$ 为 $\mathbb{R}^m$ -可测函数

$$\text{且 } \int_{\mathbb{R}^m} \mathcal{H}^k(f^{-1}y) d\mathbb{R}^m(y) \leq \frac{\omega_m \omega_k}{\omega_{m+k}} (\text{Lip } f)^m \mathcal{H}^{m+k}(A)$$

特别地 $\forall R > 0$. $\mathcal{H}^k(f^{-1}y) \leq R$ 除了 $\mathbb{R}^m$ -可测集 E 中

$$\text{其中 } \mathbb{R}^m(E) \leq \frac{\omega_m \omega_k}{\omega_{m+k}} (\text{Lip } f)^m \frac{\mathcal{H}^{m+k}(A)}{R}$$

Def. $\Omega \subset \mathbb{R}^n$ 有界开集为 Lipschitz 域. 若 $\exists 0 < \delta \leq T$ s.t.

$\forall y \in \partial\Omega$. $\exists v \in S^{n-1}$ 和 Lip 函数 $u: B_{\delta}(0) \cap v^\perp \rightarrow (-T, T)$ s.t.

$$U_y \cap \Omega = \{y + x + tv : x \in B_{\delta}(0) \cap v^\perp, t < u(x)\}$$

$$U_y \cap \partial\Omega = \text{---} = \}$$

$$\text{其中 } U_y = \text{---} \quad -T < t < T$$

Lemma. $\Omega \subset \mathbb{R}^n$ 有界开集 则 Ω 为 Lipschitz 域

2. BV函数

$U \subset \mathbb{R}^n$ 开 $u \in L^1_{loc}(U)$ 对 $u \in BV_{loc}(U)$ 若

$$\forall W \subset U \exists c(W) < \infty \text{ s.t. } \int_W u \operatorname{div} g \, d\mathbb{L}^n \leq c(W) \sup |g|$$

$$\text{其中 } g = (g^1 \dots g^n) \quad g^j \in C_c^\infty(W)$$

由Riesz表示 $\exists$ Radon测度 μ 和 Borel 向量场 $v = (v^1 \dots v^n)$ $|v| = 1$ a.e.

$$\text{s.t. } \int_U u \operatorname{div} g \, d\mathbb{L}^n = \int_U g \cdot v \, d\mu \quad (D_j u \mapsto v_j \, d\mu)$$

记全变差测度为 μ

$$\text{若 } u \in W^{1,1}_{loc}(U) \Rightarrow d\mu = |Du| \, d\mathbb{L}^n \quad v_j = \begin{cases} \frac{D_j u}{|Du|}, & |Du| \neq 0 \\ 0, & |Du| = 0 \end{cases}$$

$u \in BV_{loc}(U)$

$$\begin{aligned} |Du|(W) &= \sup_{\substack{|g| \leq 1 \\ g \text{ Lip}}} \int_W u \operatorname{div} g \, d\mathbb{L}^n \\ &= \int_W |Du| \end{aligned}$$

BV函数也可用“磨光” $\varphi_\sigma(x) = \sigma^{-n} \varphi(\frac{x}{\sigma})$ $\varphi: \begin{cases} C_c^\infty(\mathbb{R}^n) \\ \varphi(x) = \varphi(1-x) \\ \int \varphi = 1 \\ \operatorname{supp} \varphi \subset B_1(0) \end{cases}$

对 $u \in L^1_{loc}(U)$ 令 $\tilde{u} = \begin{cases} u & \text{on } U_\sigma \\ 0 & \text{on } U_\sigma^c \end{cases}$

$$U_\sigma = \{x \in U: d(x, \partial U) > \sigma\}$$

$$u^{(\sigma)} = \varphi_\sigma * \tilde{u}$$

Lemma. $u \in BV_{loc}(U)$ 则 $u^{(\sigma)} \rightarrow u$ in $L^1_{loc}(U)$

$$|Du^{(\sigma)}| \rightarrow |Du| \quad (\text{Radon测度收敛})$$

Thm (BV 泛数收敛性定理) $\{u_k\} \subseteq BV_{loc}(U)$ 且满足

$$\sup_{k \geq 1} (\|u_k\|_{L^1(W)} + \int_W |Du_k|) < \infty \quad (\forall W \subset\subset U)$$

则 $\exists$ 列 $u_{k'} \rightarrow u$ in $L^1_{loc}(U)$ 且 $\int_W |Du| \leq \liminf \int_W |Du_{k'}| \quad \forall W \subset\subset U$
 $BV_{loc}(u)$

(证略)

Poincaré 引理: U 有界开子集. $\xi \in U$ s.t. $B_{\delta R}(\xi) \subset U \subset B_R(\xi)$

$u \in BV(U)$. 则 $\forall \theta \in (0,1) \exists \beta \in \mathbb{R}$ with $\min\{L^n\{u(x) \geq \beta\}, L^n\{u(x) \leq \beta\}\} \geq \theta L^n(U)$ (*)

$$\text{有 } \int_U |u - \beta| dL^n \leq CR \int_U |Du| \quad C = C(\theta, \delta, n)$$

pf. 不妨 $R=1, \xi=0$. 取 $W \subset U$ s.t. $\int_W |u - \beta| dL^n \geq \frac{1}{2} \int_U |u - \beta| dL^n$

对 $\frac{\theta}{2}$ (*) 成立. 注意对 $W, \frac{\theta}{2}$ 条件 (*) 成立.

取方便 u_0 由 Poincaré. $\int_W |u_0 - \beta^{(0)}| dL^n \leq C \int_W |Du_0| dL^n$

取子列 不妨 $\beta^{(0)} \rightarrow \beta$

令 $\epsilon \rightarrow 0$. 由 $|Du^{(0)}| \rightarrow |Du| \quad u^{(0)} \xrightarrow{L^1_{loc}} u$ 立得.

Lemma. $U \subset \mathbb{R}^n$ 如上 $u \in BV(\mathbb{R}^n)$ 且 $\text{supp } u \subseteq \bar{U}$ 则

$$\int_{\mathbb{R}^n} |Du| \leq C (\int_U |Du| + R^{-1} \int_U |u| dL^n)$$

3. 面积公式

$$\lambda: \mathbb{R}^1 \rightarrow \mathbb{R}^{n+k} \quad \lambda(\mathbb{R}^1) \subset F \quad \rightarrow n\text{-dim subspace}$$

$$\text{取正交变换 } \varphi \text{ s.t. } \varphi(F) = \mathbb{R}^n \times \{0\} \quad p: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ 投影}$$

$$\mathbb{R}^1 \ni p \circ \varphi \circ \lambda: \mathbb{R}^1 \rightarrow \mathbb{R}^n \quad \mathcal{L}^n((p \circ \varphi \circ \lambda)(A)) = \underbrace{|\det(p \circ \varphi \circ \lambda)|}_{=1} \mathcal{L}^1(A)$$

$$\Rightarrow \mathcal{H}^n(\lambda(A)) = \sqrt{\det(\lambda^* \circ \lambda)} \mathcal{H}^1(A)$$

再对 $f: A \rightarrow \mathbb{R}^m$ 非 Lip 可通过上面情形逼近得到

$$\mathcal{H}^n(f(A)) = \int_A J_f d\mathcal{H}^n \quad J_f = \sqrt{\det(df)^* \circ df}$$

$$f \text{ 可测时也有 } \int_{\mathbb{R}^m} \mathcal{H}^n(f^{-1}(y) \cap A) d\mathcal{H}^n(y) = \int_A J_f d\mathcal{H}^n$$

$$\begin{matrix} \Rightarrow \\ \lambda \geq 0 \\ \mathcal{H}^n\text{-meas.} \end{matrix} \quad \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} h(x) d\mathcal{H}^n(y) = \int_A h J_f d\mathcal{H}^n$$

$$\text{例 1. } \gamma: [a, b] \rightarrow \mathbb{R}^n \text{ inj EP curve } \Rightarrow \mathcal{H}^1(\gamma(A)) = \int_A |\dot{\gamma}| dt' \quad (\text{EP usual length})$$

$$\text{② } M \xrightarrow[n\text{-dim}]{\text{embed } C^1} \mathbb{R}^{n+k} \quad \mathbb{R}^1 \ni \mathcal{H}^1 \llcorner M \text{ 在 } M \text{ 上 } n\text{-dim 体积 (Riem 长度) 积分}$$

$$\text{③ } \Omega \subset \mathbb{R}^n \quad M = \text{Graph}(u) \subseteq \mathbb{R}^{n+1} \quad u \in C^1(\Omega)$$

$$\mathbb{R}^1 \ni \mathcal{H}^n(M) = \int_{\Omega} \sqrt{1 + |Du|^2} dx$$

4. 子流形第一变分

$$M \xrightarrow[n\text{-dim}]{C^2 \text{ 子流形}} \mathbb{R}^{n+k}$$

$$B_y: T_y M \times T_y M \rightarrow (T_y M)^\perp$$

$$(T, \eta) \mapsto - \sum_{\alpha=1}^k (\eta \cdot D_T v^\alpha) v^\alpha|_y$$

其中 $v^1, \dots, v^k$ 满足 $v^\alpha(z) v^\beta(z) = \delta_{\alpha\beta}$ for z near y
 $v^\alpha(z) \in (T_z M)^\perp$

$$H(y) = \sum_{i=1}^n B_y(T_i, T_i) \in (T_y M)^\perp \quad T_i \text{ 为 } T_y M \text{ 的 标准基}$$

$$= - \sum_{\alpha=1}^k (\operatorname{div}_M v^\alpha) v^\alpha \quad \text{平均曲率向量}$$

Fact. $\varphi: U \rightarrow \mathbb{R}^{n+k} \quad C^1 \quad \varphi(U) \subset M \quad \varphi(0) = y \quad \varphi_s(0,0) = T$
 $\varphi_t(0,0) = \eta$
 观察域
 $\text{则 } B_y(T, \eta) = - \left(\frac{\partial^2 \varphi}{\partial s \partial t}(0,0) \right)^\perp$

现对 $\varphi: (-\varepsilon, \varepsilon) \times M \rightarrow \mathbb{R}^{n+k}$

$$\begin{cases} \varphi(0, x) = x & x \in M \\ \varphi(t, x) = x & \forall t, x \in M \setminus K \end{cases} \quad K \text{ 紧}$$

且 $\frac{\partial \varphi(t, x)}{\partial t}$ 为 C^1 的

则 $X(y) = \frac{\partial \varphi(t, y)}{\partial t} \Big|_{t=0}$ 记 $\varphi_t(y) = \varphi(t, y)$

则 $\varphi_t(y) = y + t(X(y) + E(t, y))$

$$\xrightarrow{\hspace{1cm}} \sup_{y \in K} (|E(t, y)| + \sum_{\substack{T \in T_y M \\ |T|=1}} |D_T E(t, y)|) \xrightarrow{t \rightarrow 0} 0$$

故知 $\frac{d}{dt} \mathcal{H}^n(\varphi_t(M \cap K)) = - \int_M \operatorname{div}_M X$

$$X = X^T + X^\perp \quad X^\perp = \sum_{\alpha=1}^k (v^\alpha \cdot X) v^\alpha \Rightarrow \operatorname{div}_M X^\perp = \sum_{\alpha=1}^k (v^\alpha \cdot X) \operatorname{div}_M v^\alpha = -X \cdot \underline{H}$$

$$\text{又 } \int_M \operatorname{div}_M X^T = 0 \Rightarrow \int_M \operatorname{div}_M X d\mathcal{H}^n = - \int_M X \cdot \underline{H} d\mathcal{H}^n$$

$$\stackrel{||}{=} \frac{d}{dt} \mathcal{H}^n(\varphi_t(M \cap K)) \Big|_{t=0}$$

first variation.

(n+1)-dim Riem mfd

Def. $M \subset N$ 称为静止的 (or 极小 submfd) 若 $\frac{d}{dt} \mathcal{H}^n(\varphi_t(M \cap K)) \Big|_{t=0} = 0 \quad \forall K, \varphi$

Lemma. N (n+k)-dim C^2 embed submfd $\hookrightarrow \mathbb{R}^{n+L}$

C^2 embed U -dim M

$$\text{By } M \subset N \text{ 静止} \Leftrightarrow \int_M \operatorname{div}_M X = - \int_M \underline{H}_M^N X \quad (\forall X \in C^1, \text{cpt supp in } M)$$

$$\text{其中 } \underline{H}_M^N(y) = \sum_{i=1}^n B_y^N(T_i, T_i)$$

Pf. $v^1, \dots, v^L$ 为与 M 垂直的向量场并正交族 其中 $v^1, \dots, v^k$ 与 N 相切 $v^{k+1}, \dots, v^L \perp N$

$X = X^T + X^\perp$ (关于 N) 对 T_i 为 $T_y M$ 基底

$$\begin{aligned} \operatorname{div}_M X &= \operatorname{div}_M X^T + \sum_{j=k+1}^L (v^j \cdot X) \operatorname{div}_M v^j \\ &= \operatorname{div}_M X^T + \sum_{j=k+1}^L (v^j \cdot X) \sum_{i=1}^n T_i D_{T_i} v^j \\ &= \operatorname{div}_M X^T - \sum_{i=1}^n X \cdot B_y^N(T_i, T_i) \end{aligned}$$

$$\text{则由 } M \subset N \text{ 静止} \Leftrightarrow \int_M \operatorname{div}_M X^T = 0 \quad \checkmark$$

5. 第二变分

n 同前 要求 $\frac{\partial^2 \varphi(t, y)}{\partial t^2} \in C'$ $Z(y) = \frac{\partial^2 \varphi(t, y)}{\partial t^2} |_{t=0}$

$$\Rightarrow \varphi_t(x) = x + tX_x + \frac{1}{2}t^2(Z_x + E(t, x))$$

$$\mathcal{H}^n(\varphi_t(M \cap K)) = \int_M J_{\varphi_t} d\mathcal{H}^n$$

$$D_{T_i} \varphi_t(x) = T_i + t D_{T_i} X + \frac{t^2}{2} (D_{T_i} Z + E_i)$$

$$\Rightarrow (J_{\varphi_t}(x))^2 = \delta_{ij} + t(T_i D_{T_j} X + T_j D_{T_i} X) + t^2 \left(\frac{1}{2}(T_i D_{T_j} Z + T_j D_{T_i} Z) + (D_{T_i} X)(D_{T_j} X) + E_{ij} \right)$$

$$\text{由 } \det(I+A) = 1 + \text{tr}(A) + \frac{1}{2}(\text{tr} A)^2 - \frac{1}{2}\text{tr}(A^2) + O(|A|^3)$$

$$\begin{aligned} \det(J_{\varphi_t}(x))^2 &= 1 + 2t \text{div}_M X + t^2 \left(\text{div}_M Z + \sum_{i,j=1}^n |D_{T_i} X|^2 \right. \\ &\quad \left. + 2(\text{div}_M X)^2 - \frac{1}{2} \sum_{i,j=1}^n (T_i D_{T_j} X + T_j D_{T_i} X)^2 + F(t, x) \right) \\ &= 1 + 2t \text{div}_M X + t^2 \left(\text{div}_M Z + \sum_{i,j=1}^n |(D_{T_i} X)^\perp|^2 \right. \\ &\quad \left. + 2(\text{div}_M X)^2 - \sum_{i,j=1}^n (T_i D_{T_j} X)(T_j D_{T_i} X) \right) + t^2 F \\ \Rightarrow J_{\varphi_t} &= 1 + t \text{div}_M X + \frac{t^2}{2} \left(\dots + F \right) \end{aligned}$$

$$\Rightarrow \frac{d^2}{dt^2} \mathcal{H}^n(\varphi_t(M \cap K)) |_{t=0} = \int_M \text{div}_M Z + (\text{div}_M X)^2 + \sum_{i,j=1}^n |(D_{T_i} X)|^2 - \sum_{i,j=1}^n (T_i D_{T_j} X)(T_j D_{T_i} X) d\mathcal{H}^n$$

Cor. $M \subset \mathbb{R}^n$ 光滑, $\mathbb{R}^n$ 中 X 有 compact support in M $X_y \in (T_y M)^\perp$ ($\forall y \in M$) $\mathbb{R}^n$

$$\frac{d^2}{dt^2} \mathcal{H}^n(M_t) |_{t=0} = \int_M \left(\sum_{i,j=1}^n |D_{T_i} X|^2 - \sum_{i,j=1}^n (X \cdot B(T_i, T_j))^2 \right) d\mathcal{H}^n$$

6. C^1 -Sard 定理

Thm (面积公式) $M: \mathbb{R}^m \rightarrow \mathbb{R}^n$ m -dim C^1 -embed mfd. $A \subset M$ Borel

$f: A \rightarrow \mathbb{R}^n$ Lipschitz $\mathbb{R}^n$

$$\int_A J_f^* d\mathcal{H}^m = \int_{\mathbb{R}^n} \mathcal{H}^k(A \cap f^{-1}(y)) d\mathcal{L}^m(y)$$

$$\text{其中 } J_f^*(x) = \sqrt{|\det(df_x) \circ (df_x)^*|}$$

它可以用来证明下面的 C^1 -Sard 定理

Thm (C^1 -Sard) M 如上. $f: M \rightarrow \mathbb{R}^n$ C^1 $\mathbb{R}^n$ 对 $\mathcal{L}^m$ -a.e. $y \in f(M)$

$f^{-1}(y)$ 分解为 k -dim C^1 -embed submfd 与 $\mathcal{H}^k$ -零测集

$$\text{pf. } C = \{x \in M: J_f^*(x) = 0\} = \{x \in M: \text{rank}(df_x) < m\}$$

$$\text{则由面积公式 } \mathcal{H}^k(C \cap f^{-1}(y)) = 0$$

又由隐函数定理. $f^{-1}(y) \setminus C$ 要么为空. 要么为 k -dim submfd

故 $f^{-1}(y) = (f^{-1}(y) \setminus C) \cup (f^{-1}(y) \cap C)$ 为想要分解.

Rmk 若 $f \in C^{k+1}$ 则 Sard's Thm $\rightarrow f^{-1}(y) \cap C = \emptyset$ for $\mathcal{L}^m$ -a.e. y .

故 $f^{-1}(y) = \emptyset$ 或 k -dim C^{k+1} -embed submfd for $\mathcal{L}^m$ -a.e. y

$$\text{Cor. } g \geq 0 \text{ } \mathcal{H}^m\text{-meas.} \quad \text{则 } \int_M g J_f^* d\mathcal{H}^m = \int_{\mathbb{R}^n} \int_{f^{-1}(y)} g d\mathcal{H}^k d\mathcal{L}^m(y)$$

Chap 3 Countably n-Rectifiable Sets

1. 3C号

Def. $M \subset \mathbb{R}^{n+k}$ is said to be countably n-rectifiable if $M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n)$ where $\mathcal{H}^n(M_0)=0$ $F_j: \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ Lip

Lem. M is C_n -R $\Leftrightarrow M \subset \bigcup_{j=1}^{\infty} N_j$ N_j n-dim embedded C^1 manifold $\mathcal{H}^n(M_0)=0$

Pf $\Leftarrow$ for N_j . locally $\psi: V \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ C^1 Lipschitz $\checkmark$

$\Rightarrow C^1$ 逼近定理: $\exists C^1$ G_j s.t. $\mathcal{H}^n(\{x: F_j(x) \neq G_j(x)\}) < \frac{1}{j}$

$Z_j \triangleq \mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} \{x: F_j(x) = G_{ij}(x)\}$ $\mathcal{H}^n(Z_j)=0$

$F_j(\mathbb{R}^n) \subset F_j(Z_j) \cup \bigcup_{i=1}^{\infty} G_{ij}(\mathbb{R}^n)$ $\xrightarrow{F_j \text{ Lip}} \mathcal{H}^n(F_j(Z_j))=0$

R) $M \subset \underbrace{M_0 \cup N_0}_{\mathcal{H}^n=0} \cup \bigcup_{j=1}^{\infty} G_{ij}(\mathbb{R}^n)$ 其中 $N_0 = \bigcup_{j=1}^{\infty} F_j(Z_j)$

再令 $C_{ij} = \{x \in \mathbb{R}^n: JG_{ij}(x) = 0\}$ 也 $\mathcal{H}^n$ 0-VR

$G_{ij}(C_{ij}^c)$ 由该点出发 then $F_j \circ \psi$ 为 n-dim C^1 submfd $\checkmark$

Def. M $\mathcal{H}^n$ -可测 且 $\mathcal{H}^n(M \cap K) < \infty$ ($\forall K$ 紧)

R) n-dim $P \subset \mathbb{R}^{n+k}$ 满足 $\lim_{\lambda \rightarrow 0} \int \gamma_{x,\lambda}(m) f(y) d\mathcal{H}^n(y) = \int_P f(y) d\mathcal{H}^n(y)$

其中 $\gamma_{x,\lambda}(y) = \lambda^{-n} \gamma(y-x)$ $(\forall f \in C_c^0(\mathbb{R}^{n+k}))$

则称 P 为近似切平面 记为 $T_x M$

Thm. M $\mathcal{H}^n$ -rectifiable and $\forall K \subset \mathbb{R}^n$, $\mathcal{H}^n(M \cap K) < \infty$

$$M \text{ Cn-R} \Leftrightarrow T_x M \text{ exists for } \mathcal{H}^n\text{-a.e. } x \in M$$

pf $\Rightarrow M = \bigcup_{j=0}^{\infty} M_j$. $\mathcal{H}^n(M_0) = 0$. $\underbrace{M_j \subset N_j}_{\mathcal{H}^n\text{-rectifiable}} \text{ embedded C'}$

$$R > 0 \quad f \in C_c^0(\mathbb{R}^{n+k}) \text{ with } f = 0 \text{ in } B_{R/2}^c$$

$$x \in M = (M \setminus N_j) \cup (N_j \setminus (N_j \setminus M_j))$$

$$\underbrace{\int_{\gamma_{x,\lambda}(M)} f d\mathcal{H}^n}_{\textcircled{1}} = \underbrace{\int_{\gamma_{x,\lambda}(N_j)} f d\mathcal{H}^n}_{\textcircled{2}} - \underbrace{\int_{\gamma_{x,\lambda}(N_j \setminus M_j)} f d\mathcal{H}^n}_{\textcircled{3}} + \underbrace{\int_{\gamma_{x,\lambda}(M \setminus N_j)} f d\mathcal{H}^n}_{\textcircled{4}}$$

$\downarrow (\lambda \rightarrow 0)$
 $\int_{T_x N_j} f d\mathcal{H}^n$

$$|\textcircled{4}| \leq \sup |f| \mathcal{H}^n(B_{R/2} \cap \gamma_{x,\lambda}(M \setminus N_j)) = \sup |f| \lambda^n \mathcal{H}^n(B_{R/2} \cap M \setminus N_j) \rightarrow 0$$

类似 $\textcircled{3} \rightarrow 0 \Rightarrow T_x M \text{ exists if } T_x N_j \text{ for } \mathcal{H}^n\text{-a.e. } x \in M_j \rightarrow 0$

证毕

可以将上面的积分先对 λ 求导。

Def. M $\mathcal{H}^n$ -rectifiable $\theta > 0$ $\mathcal{H}^n$ -rectifiable $\int_M \theta d\mathcal{H}^n < \infty$ ($\forall K$ cpt)

对 $x \in \mathbb{R}^{n+k}$, n -dim P_x 称为 θ -平面 π_x w.r.t. θ 若

$$\lim_{\lambda \downarrow 0} \int_{\gamma_{x,\lambda}(M)} f(y) \theta(x + \lambda y) d\mathcal{H}^n(y) = \theta(x) \int_{P_x} f(y) d\mathcal{H}^n(y) \quad (\forall f \in C_c^0(\mathbb{R}^{n+k}))$$

Thm. M, θ 同上. 则 M Cn-R $\Leftrightarrow M$ 有近似切空间 $T_x M$ w.r.t. θ

for $\mathcal{H}^n$ -a.e. $x \in M$

2. Area/Co-Area

局部有限

设 M $\mathcal{H}^n$ -可测 $C \subset \mathbb{R}^n$ 则 $M = \bigcup_{i=1}^{\infty} M_i$ 同上一节

f 为 u 上局部 Lip. $u \subseteq \mathbb{R}^{n+k}$ 为包含 M 开集. 则可定义 (与分解无关!)

$$\nabla^M f(y) = \nabla^{M_i} f(y) \quad (y \in M_i)$$

$$\mapsto d^M f_x: T_x M \rightarrow \mathbb{R}^n$$

$$T \mapsto D_T f(x) = \langle T, \nabla^M f(x) \rangle$$

若 f 为 $\mathbb{R}^{n+k}$ -值函数 有 Jacobian

$$J_f^M(x) = \sqrt{\det J(x)} = \sqrt{\det((d^M f_x)^* \circ d^M f_x)}$$

同样有面积公式

$$\int_A J_f^M d\mathcal{H}^n = \int_{\mathbb{R}^{n+k}} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y)$$

以及 对 $h \geq 0$ $\mathcal{H}^n$ -可测

$$\begin{aligned} \int_M h J_f^M d\mathcal{H}^n &= \int_{\mathbb{R}^{n+k}} \int_{f^{-1}(y)} h d\mathcal{H}^0 d\mathcal{H}^n(y) \\ &\stackrel{f|_M \text{ 可测}}{=} \int_{f(M)} h \circ f^{-1} d\mathcal{H}^n \end{aligned}$$

$f: u \rightarrow \mathbb{R}^m$ $n = m+k$

再对 $J_f^{M*}(x) = \sqrt{\det(d^M f_x) \circ (d^M f_x)^*}$ 有 Coarea 公式

$$\int_A J_f^{M*} d\mathcal{H}^n = \int_{\mathbb{R}^m} \mathcal{H}^k(A \cap f^{-1}(y)) d\mathcal{L}^m(y)$$

以及 对 $g \geq 0$ $\mathcal{H}^n$ -可测

$$\int_A g J_f^{M*} d\mathcal{H}^n = \int_{\mathbb{R}^m} \int_{f^{-1}(y) \cap M} g d\mathcal{H}^k d\mathcal{L}^m(y)$$

3. Purely Unrectifiable Sets, Structure Thm

Def. $S \subset \mathbb{R}^{n+k}$ 称为 purely n -unrectifiable. 若它不含 $\mathcal{H}^n$ -正则的 C_n -R 集

Lemma A 为 $\mathcal{H}^n$ -finite 的 $\mathbb{R}^{n+k}$ 中子集. $\mathcal{R} \setminus A = \underbrace{R}_{C_n-R} \cup \underbrace{P}_{P_n-U}$

Pf. 若 A $\mathcal{H}^n$ -正则, 则 $A = \bigcup_{j=1}^{\infty} A_j$, A_j 正则, $\mathcal{H}^n(A_j) < \infty$

则取 Borel $C_j \subset A_j$ s.t. $\mu(A_j \setminus C_j) = 0$

令 $\alpha_j = \sup \{ \mathcal{H}^n(S) : S \subset C_j, S \text{ } C_n\text{-R Borel} \}$

$\Rightarrow \exists R_{ij} \subset C_j$ C_n -R Borel $\mathcal{H}^n(R_{ij}) > \alpha_j - \frac{1}{i}$

$R_j = \bigcup_i R_{ij}$ 则显然由 α_j 的定义 $C_j \setminus R_j$ P_n -U

则 $R = \bigcup_j R_j$ $P = A \setminus R \in P_n$.

若 A $\mathcal{H}^n$ -不可正则, $A = \bigcup_{j=1}^{\infty} A_j$ $\mathcal{H}^n(A_j) < \infty$.

则取 B_j 包含 A_j Borel 且 $\mathcal{H}^n(B_j) = \mathcal{H}^n(A_j)$

令 $B = \bigcup_j B_j \stackrel{\text{前半}}{=} R \cup P. \Rightarrow A = (A \cap R) \cup (A \cap P)$ ✓

Lemma. 记 P_L 为 $\mathbb{R}^{n+k} \rightarrow L(n\text{-dim 子空间})$ 的正交投影

对 $1 \leq j_1 < j_2 < \dots < j_n \leq n+k$. $P_{j_1, \dots, j_n}$ 为 $\mathbb{R}^{n+k} \rightarrow \text{span}\{e_{j_1}, \dots, e_{j_n}\}$ 投影

若 $S \subset \mathbb{R}^{n+k}$ 满足 $\mathcal{H}^n(P_{j_1, \dots, j_n}(S)) = 0 \quad \forall 1 \leq j_1 < \dots < j_n \leq n+k$

则 S 为 P_n -U

Pf. 若不然, 则由上 Lemma $\exists n\text{-dim } C'\text{-embed submfld } U$

s.t. $\mathcal{H}^n(S \cap U) > 0 \Rightarrow \exists x \in S \cap U. \quad \mathcal{H}^n(S \cap U \cap B_p(x)) > 0$

($\exists \rho > 0$)

$\exists j_1, \dots, j_n$ s.t. $P_{j_1, \dots, j_n} / (N \cap \tilde{B}_\rho(x))$ 为 C^1 -diffeo

$\mathbb{R}^n \setminus H^n(P_{j_1, \dots, j_n}(S \cap N \cap \tilde{B}_\rho(x))) > 0$. 矛盾!

Thm (Structure Thm) $Q \subset \mathbb{R}^{n+k}$ P_n -U $Q = \bigcup_{j=1}^\infty Q_j$ $H^n(Q_j) < \infty$

$\mathbb{R}^n \setminus H^n(p(Q)) = 0$ for $\int$ G -a.e $p \in O(n+k, n)$
Haar 测度

证明:

Cor. (Rectifiability Thm) $A \subset \mathbb{R}^{n+k}$ 可写成 $A = \bigcup_{j=1}^\infty A_j$ $H^n(A_j) < \infty$

若 $\forall A$ 的 H^n -正则子集 B 有 $H^n(p(B)) > 0$ for $p \in M \subset O(n+k, n)$

其中 $\sigma(M) > 0$. 则 A C_n -R

4. Locally finite perimeter

Def. $U \subset \mathbb{R}^{n+1}$ 开, $E \subset \mathbb{R}^{n+1}$ L^{n+1} -可测. E 有 locally finite perimeter in U 若 $\chi_E \in BV_{loc}(U)$

$$|v| = 1 \text{ } \mu_E\text{-a.e.}$$

(R) $\exists \mu_E = |D\chi_E|$ Radon, $v = (v_1, \dots, v_{n+1})$ s.t. $\int_{E \cap U} \operatorname{div} g \, dL^{n+1} = - \int_U g \cdot v \, d\mu_E$
(*)

Reduced boundary: $\partial^* E = \{x \in U: \underbrace{\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(x)} v \, d\mu_E}{\mu_E(B_\rho(x))}}_{= v_E(x)} \text{ 存在且长度为 } 1\}$

Thm (De Giorgi) E 有 locally finite perimeter in U . (R) $\mu_E = \mathcal{H}^n \llcorner \partial^* E$
 $\partial^* E \subset \mathbb{C}\text{-R}$, 且 $\forall x \in \partial^* E$, $T_x \partial^* E$ 存在.

$$T_x \partial^* E = \{y \in \mathbb{R}^{n+1}: y \cdot v_E(x) = 0\}$$

此外 v_E 为“内法向量”: $\exists x, \lambda \{ \lambda(y-x): y \in E \} \rightarrow \{y \in \mathbb{R}^{n+1}: y \cdot v_E(x) > 0\}$
在 $L^{loc}(\mathbb{R}^{n+1})$ 中收敛 ($\forall x \in \partial^* E$)

pf $y \in \partial^* E$, 若 $y = 0$, $v(0) = (0, \dots, 0, 1) \Rightarrow \lim_{\rho \downarrow 0} \frac{\int_{B_\rho(0)} |v| \, d\mu_E}{\mu_E(B_\rho(0))} = \begin{cases} 1, & i = n+1 \\ 0, & \text{其它} \end{cases}$

$\forall \zeta \in C_0'(U)$ $\operatorname{supp} \zeta \subseteq B_\rho(0) \subseteq U$. (R) $\int_U v_{n+1} \zeta \, d\mu_E = - \int_U \chi_E D_{n+1} \zeta \, dL^{n+1}$
 $\leq \int_E |D\zeta| \, dL^{n+1}$

对 $\zeta_1 \in \chi_{B_\rho(0)}$ s.t. $\lim_{\rho \downarrow 0} \int_E |D\zeta_1| = \frac{d}{d\rho} L^{n+1}(E \cap B_\rho(0))$ (a)

$$\Rightarrow \int_{B_\rho(0)} v_{n+1} \, d\mu_E \leq \frac{d}{d\rho} L^{n+1}(E \cap B_\rho(0))$$

(?)

例) 对 $\rho, \rho_0 = d(0, \partial U)$, 有 $\mu_E(B_\rho(0)) \leq 2 \frac{d}{d\rho} \int_{E \cap B_\rho(0)} 1^{n+1}$
 $= 2H^n(E \cap \partial B_\rho(0)) \leq 2(n+1) \omega_{n+1} \rho^n$

例) 由 Lebesgue 定理 of BV 函数 $\exists \rho_k > 0$ s.t. $\chi_{\eta_{0,\rho_k}(E)} \rightarrow \chi_F$ in L^1_{loc} .

$\Rightarrow \lim_{k \rightarrow \infty} \int_{\eta_{0,\rho_k}(E)} D_i \zeta d\ell^{n+1} = \int_F D_i \zeta d\ell^{n+1}$ F : locally finite peri.

$\zeta_k(x) = \zeta(\rho_k^{-1}x)$ //

$-\rho_k^{-n} \int_U \zeta_k \cdot v_i d\mu_E \rightarrow 0 \quad (i=1, \dots, n)$

例) $F \subset \mathbb{R}^n \times H \quad H \subset \mathbb{R} \quad \ell^1$ -测度

(*) 对 $g = \zeta_k \ell_{n+1} \Rightarrow 0 \leq \rho_k^{-n} \int_U \zeta_k v^{n+1} d\mu_E = \int_{\eta_{0,\rho_k}(E)} D_{n+1} \zeta$
 $\rightarrow \int_F D_{n+1} \zeta = \int_{\mathbb{R}^n} \int_H \frac{\partial \zeta}{\partial x^{n+1}} dx^{n+1} dx'$

例) $\chi_U \uparrow \Rightarrow F = \{x \in \mathbb{R}^{n+1}, x^{n+1} < \lambda\}$

对 $\zeta \geq 0$ $\text{supp } \zeta \subset U$, $0 < d(\text{supp } \zeta, \partial U)$. 由 Sobolev 不等式

$(\int_U |\zeta \varphi_\epsilon * \chi_E|^{\frac{n+1}{n}} d\ell^{n+1})^{\frac{n}{n+1}} \leq C \int_U |D(\zeta \varphi_\epsilon * \chi_E)| d\ell^{n+1}$
 $\leq C(\int_U \zeta |D(\varphi_\epsilon * \chi_E)| d\ell^{n+1} + \int_U \varphi_\epsilon * \chi_E |D\zeta| d\ell^{n+1})$

$\Rightarrow (\int_E \zeta^{\frac{n+1}{n}} d\ell^{n+1})^{\frac{n}{n+1}} \leq C(\int_U \zeta d\mu_E + \int_E |\zeta| d\ell^{n+1})$

(0) $\Rightarrow \int_{E \cap B_\rho(0)} 1^{n+1} d\ell^{n+1} \leq C \frac{d}{d\rho} \int_{E \cap B_\rho(0)} 1^{n+1}$

$\rightarrow 1 \leq C \frac{d}{d\rho} \int_{E \cap B_\rho(0)} 1^{n+1} d\ell^{n+1} \quad \forall \rho \in (0, \rho_1)$

例) $C\rho^{n+1} \leq \int_{E \cap B_\rho(0)} 1^{n+1} d\ell^{n+1}$

$\Rightarrow$ 对 $\lambda = 0$

用 $U \setminus E$ 替换 $E \Rightarrow C\rho^{n+1} \leq \int_{(B_\rho(0) \setminus E)} 1^{n+1} d\ell^{n+1}$

$$\text{By } \mu_{0,p}(E) \rightarrow \mu_{\{x^{n+1} < 0\}} \quad (p \rightarrow 0)$$

$$\downarrow \quad \mathcal{H}^n \llcorner \{x^{n+1} = 0\}$$

$$\mu_{1,p}(E) \rightarrow \mathcal{H}^n \llcorner \{x \in \mathbb{R}^{n+1} : (x-y) \cdot \nu_E(y) = 0\} \quad (y \in \partial^* E)$$

$$\text{特别地 } (\mu_n \rho^n)^{-1} \mu_E(B_\rho(y)) \xrightarrow[\text{(*)}]{\text{a.s.}} \mathcal{H}^n \llcorner \partial^* E \in \mathcal{M}_E \text{ in } \mathcal{U}$$

$$\text{由 } \bar{y} \in \partial^* E \quad C_n - R(n^{\frac{1}{2}})$$

$$\bigcup_{j=0}^{\infty} M_j$$

$$\mathcal{H}^n(M_0) = 0.$$

$$M_j \subset N_j \quad n\text{-dim embed } C^1 \text{ submanifold } (j \geq 1)$$

$$\text{By upper density } \stackrel{(\text{**})}{\Rightarrow} \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\mathcal{H}^n(B_\rho(x) \cap N_j)} = 1 \quad \text{for } \mathcal{H}^n\text{-a.e } x \in M_j$$

$$\text{Radon-Nikodym} \\ \Rightarrow$$

$$\mu = \mathcal{H}^n \llcorner \partial^* E \quad \checkmark$$

Chap 4. Rectifiable n-Varifolds

1.12号

$U \subset \mathbb{R}^m$ 开. $M \subset U$ C^1 -可微. H^n -可测. θ 为正 H^n -可测. 且 $\int_M \theta dH^n < \infty$ (VK opt)

则定义 **rectifiable n-varifold** $\nu(M, \theta) = \{(\tilde{M}, \tilde{\theta}) : \tilde{M} \subset U, C^1\text{-可微}, H^n(\tilde{M} \Delta M) = 0, \tilde{\theta} = \theta \text{ a.e. on } M\}$
 θ : multiplicity function

若 θ 取整数值 则称为 integer multiplicity rectifiable varifold

希望研究: stationary varifold

仅有近似切空间 $\rightarrow$ 不能引 C^1 的 f ! 变分法需要修正

$V = \nu(M, \theta)$. $\mu_V \equiv H^n \llcorner \theta$ Radon $\Rightarrow \mu_V(A) = \int_M \theta dH^n$.
 mass: $M(V) = \mu_V(U) = \int_M \theta dH^n$

切空间 $T_x V \equiv T_x M$ (与 M 选取无关!)

$\text{supp } V \equiv \text{supp } \mu_V$

$A \subset \mathbb{R}^m$ 可测 $V \llcorner A \equiv \nu(M \cap A, \theta|_A)$

$V_k \rightarrow V$ 是 H^n Radon 测度收敛 $\mu_{V_k} \rightarrow \mu_V$

$f: \text{supp } V \cap U \rightarrow W$ Lip. proper, inj $\rightarrow f\# V = \nu(f(M), \theta \circ f^{-1})$

总结: $f\# V = \nu(f(M), \tilde{\theta})$ $\tilde{\theta}(y) = \sum_{x \in f^{-1}(y) \cap M} \theta(x)$

有 $M(f\# V) = \int_M J_f^M \theta dH^n$

2. 第二变分

$$v = \underline{v}(M, \theta) \text{ in } U \subseteq \mathbb{R}^{n+k} \quad \varphi: (-\varepsilon, \varepsilon) \times U \rightarrow \mathbb{R}^{n+k} \subset \mathbb{C}'$$

$$\text{且 } \frac{\partial \varphi}{\partial t} \in \mathbb{C}'$$

$$\begin{cases} \varphi(0, x) = x & \forall x \in U \\ \varphi(t, x) = x & \forall t, x \in U \setminus \bigcup_{q \in \text{cpt}} \{x\} \end{cases}$$

$$X = \frac{\partial \varphi}{\partial t} \Big|_{t=0} \in \mathbb{C}' \text{ 且 } \text{supp } X \subset U$$

$$M(\varphi_{t\#}(V \llcorner K)) = \int_{M \cap K} J_{\varphi_t}^M \theta d\mathcal{H}^n$$

$$\text{与之前类似地} \quad \frac{d}{dt} M(\varphi_{t\#}(V \llcorner K)) = \int_M \text{div}_M X d\mu_v$$

$$\text{其中 } \text{div}_M X = \sum_{j=1}^{n+k} \nabla_j^M X^j = \sum_{j=1}^{n+k} e_j \cdot \nabla^M X^j$$

Def. V stationary if $\frac{d}{dt} M(\varphi_{t\#}(V \llcorner K)) \Big|_{t=0} = 0 \quad (\forall \varphi_t)$

$$\Leftrightarrow \int \text{div}_M X d\mu_v = 0 \text{ for } \forall \mathbb{C}' X \text{ cpt supp in } U$$

$$N \subset \mathbb{R}^{n+k} \text{ (n+k dim)} \quad M \subset N \text{ } \mathcal{H}^n\text{-可测} \text{ 且 } \theta > 0 \text{ on } M \text{ s.t. } \int_{M \cap K} \theta d\mathcal{H}^n < \infty \quad (\forall K \text{ cpt})$$

则称 V 为 N 中 rectifiable varifold

$$\text{可类似讨论 } \varphi: (-\varepsilon, \varepsilon) \times N \rightarrow N, \quad \frac{X}{\varepsilon} \text{ 为 } N \text{ 中 } \text{cpt} \text{ 子集}$$

$$V \text{ 在 } N \text{ 中 stationary} \Leftrightarrow \int_M \text{div}_M X d\mu_v = 0$$

$$\forall X \in \mathbb{C}' \quad \text{Supp } X \subseteq \text{cpt subset of } N, \quad X_\alpha \in T_x N \quad (\forall x \in N)$$

$$\text{又经 } N \text{ 上 } \text{div}_M X = \text{div}_M X^T - \underline{H}_M^N X$$

$$\text{故 } V \subset N \text{ stationary} \Leftrightarrow \int_M \text{div}_M X d\mu_v = - \int_M X \cdot \underline{H}_M^N d\mu$$

3. 利用性公式

$U \subset \mathbb{R}^{n+k}$ 开 $V = \nu(M, \theta)$ stationary in U

$$\Leftrightarrow \int_M \operatorname{div}_M X dM_\nu = 0$$

先取 $X_x = \gamma(r)(x - \xi)$ $\xi \in U$ 固定 $r = |x - \xi|$

$\gamma: \mathbb{R} \rightarrow [0, 1]$ C^1

$$\begin{cases} \gamma(t) \leq 0 & \forall t \\ \gamma(t) = 1 & t \leq \rho \\ \gamma(t) = 0 & t \geq R \end{cases}$$

$$R \gg \rho > 0 \\ B_R(\xi) \subseteq U$$

$\forall f \in C^1(U)$ x s.t. $T_x M$ 存在 $(e^j): \mathbb{R}^{n+k} \rightarrow T_x M$ 的基

$$\mathbb{R}^j \nabla^M f(x) = \sum_{j=1}^{n+k} e^j \cdot D_e f(x) e_j$$

$$\Rightarrow \operatorname{div}_M X = \sum_{j=1}^{n+k} e_j \cdot \nabla^M X^j$$

$$= \gamma(r) \underbrace{\sum_{j=1}^{n+k} e^j}_{\text{基}} + r \gamma'(r) \underbrace{\sum_{j=1}^{n+k} e^j \frac{(x^j - \xi^j)}{r}}_{\|P_{T_x V}(\frac{x - \xi}{r})\|^2}$$

$$\text{故 } n \int \gamma(r) d\mu + \int r \gamma'(r) |\nabla^M r|^2 d\mu = 0$$

$$\left(\text{可取 } \frac{h(x)}{r} \gamma(r) \Rightarrow n \int \gamma(r) h d\mu + \int r \gamma'(r) |\nabla^M r|^2 h d\mu = - \int \gamma(r) (x - \xi) \cdot \nabla^M h d\mu \right)$$

$$\text{再对 } \varphi: \mathbb{R} \rightarrow [0, 1] \quad \begin{cases} \varphi(t) = 1, t \leq 1 \\ \varphi(t) = 0, t \geq 1 + \varepsilon \\ \varphi(t) \leq 0 \end{cases} \quad \gamma(r) = \varphi\left(\frac{r}{\rho}\right) \quad (1 + \varepsilon)\rho < R$$

$$\Rightarrow n \int_M \varphi\left(\frac{r}{\rho}\right) d\mu - \rho \frac{d}{d\rho} \int_M |\nabla^M r|^2 \varphi\left(\frac{r}{\rho}\right) d\mu = 0$$

$$\Rightarrow n I(\rho) - \rho I'(\rho) = -\rho \int_M \frac{\partial}{\partial \rho} [\varphi\left(\frac{r}{\rho}\right)] |\nabla^M r|^2 d\mu \quad \text{其中 } I(\rho) = \int_M \varphi\left(\frac{r}{\rho}\right) d\mu$$

$$\Rightarrow \frac{d}{d\rho} (\rho^{-n} I(\rho)) = \int_M \rho^{-n} \frac{\partial}{\partial \rho} [\varphi\left(\frac{r}{\rho}\right)] |\nabla^M r|^2 d\mu$$

$$2(1+\varepsilon)^{-n} r^{-n} \frac{\partial}{\partial \rho} \varphi \leq \rho^{-n} \frac{\partial}{\partial \rho} \varphi \leq r^{-n} \frac{\partial}{\partial \rho} \varphi$$

$$\Rightarrow (1+\varepsilon)^{-n} J(\rho) \leq \frac{d}{d\rho} (\rho^{-n} I(\rho)) \leq J(\rho). \text{ 其中 } J(\rho) = \int_M r^{-n} |D^{\perp} r|^2 \varphi d\mu$$

$$\text{在 } [0, \rho] \text{ 上积分 } \int_M (1+\varepsilon)^{-n} r^{-n} (\varphi(\frac{\rho}{r}) - \varphi(\frac{\sigma}{r})) |D^{\perp} r|^2 d\mu \leq \rho^{-n} I(\rho) - \sigma^{-n} I(\sigma) \\ \leq \int_M r^{-n} (\varphi(\frac{\rho}{r}) - \varphi(\frac{\sigma}{r})) |D^{\perp} r|^2 d\mu$$

$$\varepsilon \geq 0, \mathbb{R} \ni \varphi \geq \chi_{(-\infty, 1]}$$

$$\Rightarrow \frac{\mu_\nu(B_\rho(\xi))}{\omega_n \rho^n} \cdot \frac{\mu_\nu(B_\sigma(\xi))}{\omega_n \sigma^{-n}} = \omega_n^{-1} \int_{B_\rho(\xi) \setminus B_\sigma(\xi)} \frac{|D^{\perp} r|^2}{r^n} d\mu$$

fundamental
monotonicity identity

$$\text{特别地 } (\omega_n \rho^n)^{-1} \mu_\nu(B_\rho(\xi)) \uparrow \Rightarrow \Theta^n(\mu_\nu, \xi) = \lim_{\rho \rightarrow 0} \frac{\mu_\nu(B_\rho(\xi))}{\omega_n \rho^n} \text{ 存在}$$

$$\sigma \geq 0 \Rightarrow (\omega_n \rho^n)^{-1} \mu_\nu(B_\rho(\xi)) \geq \Theta^n(\mu_\nu, \xi)$$

$$\text{Cor (usc)} \quad \Theta^n(\mu_\nu, \xi) \geq \overline{\lim_{x \rightarrow \xi}} \Theta^n(\mu_\nu, x)$$

$$\text{pf. } \rho > 0, \xi \in (0, 1), \xi_j \rightarrow \xi, \mathbb{R}^1 \text{ 大时 } B_{(1-\varepsilon)\rho}(\xi_j) \subset B_\rho(\xi)$$

$$\Rightarrow (1-\varepsilon)^n \Theta^n(\mu_\nu, \xi_j) \leq (\omega_n \rho^n)^{-1} \mu_\nu(B_{(1-\varepsilon)\rho}(\xi_j)) \leq (\omega_n \rho^n)^{-1} \mu_\nu(B_\rho(\xi))$$

$$\varepsilon \rightarrow 0 \Rightarrow \limsup_{j \rightarrow \infty} \Theta^n(\mu_\nu, \xi_j) \leq (\omega_n \rho^n)^{-1} \mu_\nu(B_\rho(\xi))$$

$\rho \rightarrow 0 \checkmark$

$$\text{对 } V = \nu(M, \Theta) \text{ 可取典范代表 } (M_\nu, \Theta_\nu)$$

$$M_\nu = \{x \in M, \Theta^n(\mu_\nu, x) > 0\}$$

$$\Theta_\nu(x) = \Theta^n(\mu_\nu, x) \quad (\text{upper semi cts})$$

$$\text{故 } \Theta_\nu^{-1}([0, \infty)) \text{ 在 } M \text{ 中稠密} \Rightarrow M_\nu \text{ 稠密}$$

Def. $V = \nu(M, \theta) \subset U \subset \mathbb{R}^{n+k}$ 的 ν 有正体积元 μ_ν 且 $\mu_\nu \in U$. if

$$\int_M \operatorname{div}_M X d\mu_\nu = - \int_M X \cdot \underline{H} d\mu_\nu \quad \forall X \in C_c^1(U, \mathbb{R}^{n+k})$$

$\mathbb{R}^1 \subset U$ stationary $\Leftrightarrow GMC = 0$
(General mean curvature)

Rmk. $V \subset N \subset \overset{C^2 \text{ embed}}{\mathbb{R}^{n+k}}$ stationary $\Leftrightarrow GMC = \underline{H}_M^N$ in U
($n+k$)-dim

希望将 $\underline{H} = 0$ 的情形推广到 $\underline{H}$ 有界

$$\begin{aligned} \text{回+2取 } X = h\gamma(r)(x-\xi) &\Rightarrow \int (h\gamma(r) + r\gamma'(r)|\nabla^M \gamma|^2) h d\mu_\nu = - \int (x-\xi) \cdot (h\underline{H} + \nabla^M h) \gamma(r) d\mu \\ \gamma(r) = \varphi(\frac{r}{\rho}) &\Rightarrow n \int h \varphi d\mu - \rho \frac{d}{d\rho} \int h |\nabla^M \gamma|^2 \varphi d\mu = E_h(\rho) \\ &= \rho^{n-1} \int (h\underline{H} + \nabla^M h)(x-\xi) \varphi d\mu \end{aligned}$$

设 $B_R(\xi) \subset U$ 且 $R \sup_{B_R(\xi)} |\underline{H}| \leq \Lambda$ on M

$$\begin{aligned} \text{同前有 } (1+\varepsilon)^{-n} J(\rho) &\leq \frac{d}{d\rho} (\rho^{-n} I(\rho)) + \underbrace{E_1(\rho)}_{\substack{E(\rho)\rho^{-n}I(\rho) \\ \text{"0"}}} \leq J'(\rho) \\ \text{有 } -(1+\varepsilon) R^{-1} \wedge \rho^{-n} I(\rho) &\leq E_1(\rho) \leq (1+\varepsilon) R^{-1} \wedge \rho^{-n} I(\rho) \end{aligned}$$

$$\wedge F(\rho) = \int_0^\rho E(t) dt \in \left[-(1+\varepsilon) \frac{\Lambda \rho}{R}, (1+\varepsilon) \frac{\Lambda \rho}{R} \right]$$

$$\Rightarrow (1+\varepsilon)^{-n} e^{-(1+\varepsilon)\wedge} J'(\rho) \leq \frac{d}{d\rho} (e^{F(\rho)} \rho^{-n} I(\rho)) \leq e^{(1+\varepsilon)\wedge} J'(\rho)$$

取 $\varepsilon \rightarrow 0$

$$\Rightarrow e^{\frac{F(\rho)}{\omega_n \rho^n}} = e^{\frac{F(\rho)}{\omega_n \rho^n}} = \omega_n^{-1} G(\rho, \rho) \int_{B_{\rho(\xi)} \setminus B_\delta(\xi)} r^{-n} |\nabla^M \gamma|^2 d\mu$$

$$\text{其中 } F(\rho) \in \left[-\frac{\Lambda \rho}{R}, \frac{\Lambda \rho}{R} \right] \quad G(\rho, \rho) \in [e^{-\wedge}, e^{\wedge}]$$

4. L^p -平均的单调性公式

上面及 $H \in L^\infty$. 现在设 $H \in L^{p>n}$. 设 $(R^{2n} \int_{B_R(\xi)} |H|^p d\mu_\nu)^{\frac{1}{p}} \leq \Lambda$

$$\begin{aligned} \text{R.1 } |p^{-n} \int_{\mathbb{R}^n} p^{-n}(x-\xi) H \varphi d\mu| &\stackrel{\text{Hölder}}{\leq} (1+\varepsilon) p^{-n} \|H\|_{L^p(\mu_\nu, L_{B_R(\xi)})} (I(p))^{1-\frac{1}{p}} \\ &= (1+\varepsilon) R^{-1} \left(\frac{p}{R}\right)^{-\frac{n}{p}} \left(R^{2n} \int_{B_R(\xi)} |H|^p d\mu_\nu\right)^{\frac{1}{p}} \\ &\stackrel{d^{-\frac{1}{p}} \leq \frac{1}{p} + \varepsilon}{\leq} (1+\varepsilon) \wedge R^{-1} \left(\frac{p}{R}\right)^{-\frac{n}{p}} \left(\frac{1}{p} + p^{-n} I(p)\right) \end{aligned}$$

$$\Rightarrow (1+\varepsilon)^{-n} J'(p) \leq \frac{d}{dp} (p^{-n} I(p)) + F_0(p) \left(\frac{1}{p} + p^{-n} I(p)\right) \leq J'(p)$$

$$F(p) = \int_0^p F_0(t) dt \quad \text{有 } |F(p)| \leq (1+\varepsilon) \wedge K \left(\frac{p}{R}\right)^{1-\frac{n}{p}} \leq (1+\varepsilon) \wedge \underbrace{K}_{\frac{p}{p-n}}$$

$$\Rightarrow (1+\varepsilon)^{-n} e^{-(1+\varepsilon) \wedge K} J'(p) \leq \frac{d}{dp} (e^{F(p)} p^{-n} I(p) + \frac{1}{p} e^{F(p)}) \leq e^{(1+\varepsilon) \wedge K} J'(p)$$

$$\begin{aligned} &\left(e^{F(p)} \frac{\mu_\nu(B_p(\xi))}{\omega_n p^n} + \frac{1}{p} (e^{F(p)} - 1) \right) - \left(e^{F(\sigma)} \frac{\mu_\nu(B_\sigma(\xi))}{\omega_n \sigma^n} + \frac{1}{\sigma} (e^{F(\sigma)} - 1) \right) = \\ &\quad \text{其中 } G(\sigma, p) \in [e^{-K \wedge}, e^{K \wedge}] \quad G(\sigma, p) \int_{B_p(\xi) \setminus B_\sigma(\xi)} r^{-n} |D^{\frac{1}{2}} r|^2 d\mu \\ &\quad |F(p)| \leq \wedge K \left(\frac{p}{R}\right)^{1-\frac{n}{p}} \end{aligned}$$

同前类似有 $\Theta^n(\mu_\nu, \xi) = \lim_{\sigma \rightarrow 0} (\omega_n \sigma^n)^{-1} \mu_\nu(B_\sigma(\xi))$ 存在. A

$$\left(e^{F(p)} \frac{\mu_\nu(B_p(\xi))}{\omega_n p^n} + \frac{1}{p} (e^{F(p)} - 1) \right) - \Theta^n(\mu_\nu, \xi) = G(p) \underbrace{\int_{B_p(\xi)} r^{-n} |D^{\frac{1}{2}} r|^2 d\mu}_{< \infty} \quad (*)$$

且 $\Theta^n(\mu_\nu, \xi)$ 上半连续

可取类代表 (μ_ν, Ω_ν) 同前 $\nu = \nu(\mu_\nu, \Omega_\nu)$

5. Local Conical Approximation

Thm. $\Gamma > 0, p > n, \lambda \in (0, \frac{1}{4}), \delta \in [0, \frac{1}{16}]$ $V = v(\mu, \theta)$ 满足

$$0 \in \text{supp } \mu, \Theta^n(\mu, 0) \leq \Gamma$$

$$\omega_n^{-1} \mu(B_{1/2}(0)) \leq \Theta^n(\mu, 0) + \delta \quad \left(\int_{B_{1/2}(0)} |H|^p d\mu \right)^{\frac{1}{p}} \leq \delta$$

则 $\forall \xi \in \check{B}_{1/2}(0) \setminus B_{2\lambda}(0), \tau \in [1, 1]$ $\lambda \leq \rho \leq \min\{|\xi| - \lambda, (1 + \delta^{\frac{1}{p}})^{-1}(1 - |\xi|)\}$

$$\text{有 } \frac{\mu(B_{1-\delta^{\frac{1}{p}}\rho}(\xi))}{\omega_n \rho^n} - C(\delta, \lambda) \leq \frac{\mu(B_{\tau\rho}(\tau\xi))}{\omega_n (\tau\rho)^n} \leq \frac{\mu(B_{(1+\delta^{\frac{1}{p}})\rho}(\xi))}{\omega_n \rho^n} + C(\delta, \lambda)$$

$$C(\delta, \lambda) = \lambda^n \delta + \delta^{\frac{1}{p}} |\omega_n \lambda| \quad C = C(n, p, \Gamma)$$

Remark. $\nabla^M H = 0$ 且 $\omega_n^{-1} \mu(\check{B}_{1/2}(0)) = \Theta^n(\mu, 0)$ 则 $\delta = 0$ $\lambda \neq 0$ 时

$$\therefore \rho \downarrow 0 \text{ 有 } \Theta^n(\mu, \tau\xi) = \Theta^n(\mu, \xi) \quad (\forall \tau \in [0, 1])$$

$\Rightarrow (\eta_{0, \tau} \# V) \llcorner B_{1/2}(0) = V \llcorner B_{1/2}(0)$ $\mathbb{R}^n V \llcorner B_{1/2}(0)$ is a cone.

6. Poincaré and Sobolev 不等式

设 $V = v(\mu, \theta)$ 在 U 中有 LMC. 设 $0 \geq 1$ μ -a.e. in U

则由上半连续. 若 $H \in L_{\text{loc}}^{p > n}(\mu)$ 则 $\Theta^n(\mu, x) \geq 1$ in $\text{supp } \mu \cap U$

首先对 $\tilde{I}(\rho) = \int \varphi(\frac{\cdot}{\rho}) h d\mu$ 有

$$\frac{\partial}{\partial \rho} (\rho^{-n} \tilde{I}(\rho)) = \rho^{-n} \frac{\partial}{\partial \rho} \int |\nabla r|^{-1} h \varphi d\mu + \rho^{-n-1} \int (x - \xi) \cdot [\nabla^M h + \frac{Hh}{\varphi}] \varphi d\mu$$

$$\Rightarrow \frac{\partial}{\partial \rho} (\rho^{-n} \tilde{I}(\rho)) \geq \rho^{-n-1} \int (x - \xi) \cdot (\nabla^M h + Hh) \varphi(\frac{\cdot}{\rho}) d\mu \in \mathbb{R}$$

Robt. (a) 若 $|\underline{h}| \leq \Gamma$. $\Rightarrow R \geq \rho^{-n-1} \int r |\nabla^M h| \varphi(\frac{\cdot}{\rho}) d\mu - (\Gamma \rho) \rho^{-n-1} \varphi(\rho)$

$$\stackrel{\varphi \geq 0}{\Rightarrow} \frac{1}{\omega_n \sigma^n} \int_{B_{\sigma}(\xi)} h d\mu \leq e^{\Lambda \rho} \left(\frac{1}{\omega_n \rho^n} \int_{B_{\rho}(\xi)} h d\mu + \frac{1}{n \omega_n} \int_{B_{\rho}(\xi)} \frac{|\nabla^M h|}{|x-\xi|^{n-1}} d\mu \right) \quad (1)$$

(b) $\underline{h} \geq \frac{1}{2} |\varphi| \Rightarrow R \geq \rho^{-n-1} \int r (|\nabla^M h| + \frac{1}{2} |\underline{h}|) \varphi(\frac{\cdot}{\rho}) d\mu$

$$\stackrel{\varphi \geq 0}{\Rightarrow} \frac{1}{\omega_n \sigma^n} \int_{B_{\sigma}(\xi)} h d\mu \leq \frac{1}{\omega_n \rho^n} \int_{B_{\rho}(\xi)} h d\mu + \frac{1}{n \omega_n} \int_{B_{\rho}(\xi)} \frac{(|\nabla^M h| + \frac{1}{2} |\underline{h}|)}{|x-\xi|^{n-1}} d\mu \quad (2)$$

Thm (Poincaré-type 不等式) $\overset{20}{h} \in C^1(U)$, $B_{2\rho}(\xi) \subset U$, $|\underline{h}| \leq \Lambda$, $0 > \alpha$ μ -a.e.

且 $\mu(\{x \in B_{\rho}(\xi) : h(x) > 0\}) \leq (1-\alpha) \omega_n \rho^n$, $e^{\Lambda \rho} \leq 1+\alpha$, $\alpha \in (0,1)$

再设 $\mu(B_{2\rho}(\xi)) \leq \Gamma \rho^n$, $\exists \beta = \beta(n, \alpha, \Gamma) \in (0, \frac{1}{2})$, $C = C(n, \alpha, \Gamma) > 0$

$$\text{s.t.} \quad \int_{B_{\rho}(\xi)} h d\mu \leq C \rho \int_{B_{\rho}(\xi)} |\nabla^M h| d\mu$$

pf. 将 $\eta \in B_{\rho}(\xi) \cap \text{supp } \mu \setminus \Lambda(1)$ 有

$$h(\eta) \leq e^{\Lambda \rho} \left(\frac{1}{\omega_n ((1-\beta)\rho)^n} \int_{B_{\rho}(\xi)} h d\mu + \frac{1}{n \omega_n} \int_{B_{\rho}(\xi)} \frac{|\nabla^M h|}{|x-\eta|^{n-1}} d\mu \right)$$

取 $\gamma \uparrow C'$ 且 $\gamma(t) = 0$ ($t \leq 0$) 取 $t \geq 0$ 用 $\gamma(h-t)$ 代 λh .

$$\stackrel{\gamma \geq 0}{\Rightarrow} \gamma(h(\eta)-t) \leq \frac{1+\alpha}{n \omega_n} \int_{B_{\rho}(\xi)} \frac{\gamma'(h-t) |\nabla^M h|}{|x-\eta|^{n-1}} d\mu + (1-\alpha^2) (1-\beta)^{-n}$$

取 β s.t. $(1-\beta)^{-n} (1-\alpha^2) \leq 1 - \frac{\alpha^2}{2}$. 再取 $\gamma(t) = 1$ ($t \geq 1+\varepsilon$)

$$\Rightarrow 1 \leq C \int_{B_{\rho}(\xi)} \frac{\gamma'(h-t) |\nabla^M h|}{|x-\eta|^{n-1}} d\mu, \quad \eta \in B_{\rho}(\xi) \cap A_{t+\varepsilon}$$

其中 $A_T = \{x \in \text{supp } \mu : h(x) > T\}$

$$\begin{aligned} \Rightarrow \mu(A_{t+\varepsilon} \cap B_{\rho}(\xi)) &\leq C \int_{B_{\rho}(\xi)} \gamma'(h(x)-t) |\nabla^M h(x)| \left(\int_{B_{\rho}(\xi)} \frac{1}{|x-\eta|^{n-1}} d\mu(\eta) \right) d\mu(x) \\ &\leq C \Gamma \rho \int_{B_{\rho}(\xi)} \gamma'(h-t) |\nabla^M h| d\mu \end{aligned}$$

对 $t \in (0, \infty)$ 我们要有 $\int_{A_\varepsilon \cap B_{\rho(\varepsilon)}} (h - \varepsilon) \leq C \int_{B_{\rho(\varepsilon)}} |\nabla^m h| d\mu$
 $\varepsilon \rightarrow 0 \checkmark$

Thm (Sobolev-type 不等式) $h \in C_0^\infty(\Omega)$ $0 \leq h$ $n \geq 1$ $n \in \mathbb{R}$

$$\left(\int h^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq C \int (|\nabla^m h| + h |H|) d\mu \quad C = C(n)$$

Pf. Lemma f, g 为 $(0, \infty)$ 有界函数 $1 \leq \sigma^{-n} f(\sigma) \leq \rho^{-n} f(\rho) + \int_0^\rho \tau^{-n} g(\tau) d\tau$
 $10 < \sigma < \rho < \infty$
 $\mathbb{R}^n \ni 0 < \rho < \rho_0 = 2(f(\infty))^{-\frac{1}{n}}$ s.t. $f(\sigma\rho) \leq \frac{\sigma}{2} g(\rho) \cdot \rho$

$$\mathbb{R}^n \ni \sigma f(\rho) = \omega_n^{-1} \int_{B_{\rho(\sigma)}} h d\mu \quad g(\rho) = \omega_n^{-1} \int_{B_{\rho(\varepsilon)}} (|\nabla^m h| + h |H|) d\mu$$

可用引理 (因为 (2)) $\exists \rho \exists \xi \in \{x \in \text{supp } \mu : h(x) \geq 1\}$ $\rho < 2(\omega_n^{-1} \int_M h d\mu)^{\frac{1}{n}}$
s.t. $\int_{B_{\sigma\rho(\varepsilon)}} h d\mu \leq \frac{\sigma}{2} (\omega_n^{-1} \int_M h d\mu)^{\frac{1}{n}} \int_{B_{\rho(\varepsilon)}} |\nabla^m h| + h |H| d\mu$

取 ρ 使 $B_{\sigma\rho(\varepsilon)}$ 覆盖 $\{x \in M : h(x) \geq 1\} \Rightarrow$

$$\int_{\{x \in \text{supp } \mu : h(x) \geq 1\}} h d\mu \leq \frac{\sigma}{2} (\omega_n^{-1} \int_M h d\mu)^{\frac{1}{n}} \int_M (|\nabla^m h| + h |H|) d\mu$$

$\uparrow$
 $\gamma \in C^1$ $\gamma(t) = 1 (t > \varepsilon)$ $\gamma(t) = 0 (t \leq 0)$ $\gamma'(t) = \gamma(h-t)$

$$\mu(M_{t+\varepsilon}) \leq \frac{\sigma}{2} \omega_n (\mu(M_{t+\varepsilon}))^{\frac{1}{n}} \int_M \gamma' (|\nabla^m h| + \gamma |H|) d\mu$$

$$\Rightarrow (t+\varepsilon)^{\frac{1}{n-1}} \mu(M_{t+\varepsilon}) \leq \frac{\sigma}{2} \omega_n^{-\frac{1}{n}} \left(\int_M (h+\varepsilon)^{\frac{n}{n-1}} d\mu \right)^{\frac{1}{n}} \left(-\frac{d}{dt} \int_M \gamma |\nabla^m h| d\mu + \int_{M_t} |H| d\mu \right)$$

对 $t \in (0, \infty)$ 我们

$$\Rightarrow \int_{M_\varepsilon} (h^{\frac{n}{n-1}} - \varepsilon^{\frac{n}{n-1}}) d\mu \leq \frac{\sigma}{2} \omega_n^{-\frac{1}{n}} \left(\int_M (h+\varepsilon)^{\frac{n}{n-1}} d\mu \right)^{\frac{1}{n}} \int_M (|\nabla^m h| + h |H|) d\mu$$

$\varepsilon \rightarrow 0$ 得证

Chap 5 Allard Regularity Thm

1. 同胚逼近

$M \xrightarrow{n\text{-dim}} \mathbb{R}^{n+k}$
 $C^2\text{-embed}$

$$\forall \xi \in M. \exists W \subset \mathbb{R}^{n+k} \quad B_\rho(\gamma) \subset W$$

$$C^2 u: B_\rho(\gamma) \rightarrow \mathbb{R}^k \quad \xi + u(\gamma) = \xi \quad Du(\gamma) = 0$$

$$\text{且 } M \cap W = \{(x, u(x)) : x \in B_\rho(\gamma)\}$$

$$A(u) = \int_{B_\rho(\gamma)} J u \, d\mathcal{H}^n \quad J u = |\det J| \quad J(x) = (D_i(x, u(x)) : D_j(x, u(x))) \\ = (\delta_{ij} + u_i u_j)$$

$$\text{for } |Du| < \varepsilon_0 \text{ small} \quad A(u) = \int_{B_\rho(\gamma)} (1 + \frac{1}{2} |Du|^2 + F(Du)) \, d\mathcal{L}^n$$

$$F(p) \quad (|p| < \varepsilon_0) : |F(p)| \leq C|p|^4 \quad |D_{p_{ij}} F(p)| \leq C|p|^3$$

$$\rightarrow \zeta = 0 \text{ on } \partial B_\rho(\gamma)$$

$$\text{若 } M \text{ stationary} \Rightarrow \frac{d}{dt} A(u + t\zeta) = 0 \quad \zeta \in C_0^1(B_\rho(\gamma), \mathbb{R}^k)$$

$$\sum_{i=1}^n \int_{B_\rho(\gamma)} D_i u \cdot D_i \zeta \, d\mathcal{L}^n = \int_{B_\rho(\gamma)} \sum_{i=1}^n \sum_{j=1}^k A_{ij}(Du) D_i \zeta_j \\ (A_{ij}(p) = D_{p_{ij}} F(p))$$

$$\text{再取 } v \text{ s.t. } \begin{cases} v = u \text{ on } \partial B_\rho(\gamma) \\ \Delta v = 0 \text{ on } B_\rho(\gamma) \end{cases} \xrightarrow{\text{Dirichlet}} \int_{B_\rho(\gamma)} \sum_{i=1}^n D_i v \cdot D_i \zeta \, d\mathcal{L}^n = 0 \quad \textcircled{2}$$

$$\textcircled{1} \textcircled{2} \text{ 作差 } \zeta = u - v \Rightarrow \int_{B_\rho(\gamma)} |D(u-v)|^2 \, d\mathcal{L}^n = \int_{B_\rho(\gamma)} \sum_{i,j} A_{ij}(Du) D_i(u_j - v_j) \\ \stackrel{\text{Cauchy}}{\leq} \frac{1}{2} \int_{B_\rho(\gamma)} \sum A_{ij}^2 + \frac{1}{2} \int_{B_\rho(\gamma)} |D(u-v)|^2 \\ \Rightarrow \int_{B_\rho(\gamma)} |D(u-v)|^2 \, d\mathcal{L}^n \leq C \int_{B_\rho(\gamma)} |Du|^6$$

v 是 u 的一个好的逼近.

2 Lipschitz 逼近

$U \subseteq \mathbb{R}^{n+k}$ $V = \nu(M, \theta)$ rectifiable n -varifold $\in U$ 中 $\exists GMC \perp$

tilt excess $E(\xi, \sigma, T) = \sigma^{-n} \int_{B_\sigma(\xi)} |P_{T_{xM}} - P_T|^2 d\mu_\nu(x)$

$\begin{matrix} \mathbb{R}^{n+k} \\ U \text{ n-dim} \\ (e^{ij}) \quad (\xi^{ij}) \\ \uparrow \quad \uparrow \end{matrix}$

(3) $T = \mathbb{R}^n \times \{0\}$ $P_{T_{xM}} \quad P_T \Rightarrow |P_{T_{xM}} - P_T|^2 = \sum_{j=1}^k (e^{ij} - \xi^{ij})^2$

$= 2(1 - \sum_{j=1}^n e^{jj}) = 2 \sum_{j=n+1}^{n+k} e^{jj}$

$R_j E(\xi, \sigma, T) = 2\sigma^{-n} \int_{B_\sigma(\xi)} \sum_{j=1}^k |\nabla^M x^{n+j}|^2 d\mu = 2 \sum_{j=1}^k |\nabla^M x^{n+j}|^2$

Lem. $B_{\rho/2}(y) \subset U$ $y \in (0,1)$ $T \subseteq \mathbb{R}^{n+k}$ $\mathbb{R}^n$

$$\rho^{-n} \int_{B_{\rho/2}(y)} |P_{T_{xM}} - P_T|^2 d\mu_\nu(x) \leq C \rho^{-n} \int_{B_{\rho/2}(y)} \frac{d(x, \xi + T)^2}{\rho^2} d\mu + C \rho^{2-n} \int_{B_{\rho/2}(y)} |H|^2 d\mu$$

取 $\delta \in (0, \frac{1}{4})$ $\begin{cases} 1 \leq \theta \text{ n-a.e. } 0 \in \text{supp } \nu, B_{\rho/2}(0) \subset U \\ \omega_n \rho^{-n} \mu(B_{\rho/2}(0)) \leq 1 + \delta, (\rho^{p-n} \int_{B_{\rho/2}(0)} |H|^p d\mu)^{\frac{1}{p}} \leq \delta \end{cases} (*)$

取 θ 代表元. 由恒等式. 对 $y \in \text{supp } \nu \cap B_{2\delta\rho}(0)$ $\sigma \in (0, (1-2\delta)\rho]$

有 $1 - C\delta \leq (\omega_n \sigma^n)^{-1} \mu(B_\sigma(y)) \leq 1 + C\delta$

$\Rightarrow \delta \rightarrow 0$ 时. $\frac{1}{2} \leq 1 - C\delta \leq \frac{\mu(B_\sigma(y))}{\omega_n \sigma^n} \leq 1 + C\delta \leq 2$

$\sigma \rightarrow 0 \Rightarrow \theta(y) \leq 1 + C\delta \quad (y \in \text{supp } \mu \cap B_{2\delta\rho}(0))$

再由恒等式有 $\int_{B_{1/2\delta\rho}(y)} |x-y|^{-n} \left| \left(\frac{x-y}{|x-y|} \right)^{\perp} \right|^2 d\mu \leq C\delta$

Lemma (Affine Approximation) $\delta \in (0, \frac{1}{6})$ 且 $(*)$ 成立 $\mathbb{R}^1$

$\forall y \in \text{supp } \mu \cap B_{2\delta}(0)$ $\delta \in (0, 4\delta p)$. 有

$$\sup_{x \in \text{supp } \mu \cap B_\delta(y)} d(x, y + T(y, \delta)) \leq C \delta^{\frac{1}{2n+2}} \delta \quad C = C(n, k, p)$$

$y \in B_{2\delta}(0)$ $\delta \leq 4\delta$

Cor. 对 $T = T(y, \delta)$. 有 $\delta^{-n} \int_{B_{\delta p}(y)} |P_{T\mu} - P_T|^2 d\mu \leq C \delta^{\frac{1}{n+1}}$

Pf 由 $\mu(B_\delta(y)) \leq 2\omega_n \delta^n$ 及 Hölder 不等式. Lem.

$$\text{LHS} \leq C \sup_{x \dots} \frac{d(x, y + T)^2}{\delta^2} + C \left(\delta^{p-n} \int_{B_\delta(y)} |H|^p d\mu \right)^{\frac{2}{p}} \leq C \delta^{\frac{1}{n+1}}$$

Thm (Lipschitz Approximation) $L \in (0, 1]$ $\exists \beta = \beta(n, k, p) \in (0, \frac{1}{6}]$.

s.t. 对任意 $0 < \delta \leq (\beta L)^{\frac{1}{2n+2}}$ $p=1$ 若 $(*)$ 成立 且 $T(0, 4\delta) \in \mathbb{R}^1 \times \{0\}$.

则 $\exists$ Lipschitz $f: B_\delta(0) \rightarrow \mathbb{R}^k$ $\text{Lip } f \leq L$ $\sup |f| \leq C \delta^{\frac{1}{2n+2}}$

$$\begin{aligned} & \delta^{-n} \left(\mu(B_\delta(0) \cap (\text{supp } \mu \setminus \text{graph } f)) + \mathcal{H}^n(B_\delta(0) \cap (\text{graph } f \setminus \text{supp } \mu)) \right) \\ & \leq C L^2 \delta^{-n} \int_{B_{2\delta}(0)} |P_{T\mu} - P_{\mathbb{R}^1 \times \{0\}}|^2 d\mu \leq C L^2 \delta^{\frac{1}{n+1}} \quad C = C(n, k, p) \end{aligned}$$

Cor. $\exists \beta = \beta(n, k, p) \in (0, \frac{1}{4}]$ s.t. 在 Thm 4.14 下. 若 $\forall y \in \text{supp } \mu \cap B_{2\delta}(0)$

$$\sup_{\delta \in (0, \delta)} \delta^{-n} \int_{B_\delta(y)} |P_{T\mu} - P_{T_0}|^2 d\mu \leq \beta^2 L^2$$

$\mathbb{R}^1 \ni f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ Lip 且 $\text{Lip } f \leq L$ $\sup |f| \leq C \delta^{\frac{1}{2n+2}}$.

s.t. $\text{supp } \mu \cap B_\delta(0) = \text{graph } f \cap B_\delta(0)$

3. 调和逼近 2

Lemma (Harmonic Approximation) $\forall \varepsilon > 0 \quad \exists \delta = \delta(n, \varepsilon) > 0$ s.t

$$\text{若 } f \in W^{1,2}(B) \quad B = B_{\delta}(0), \text{ 且 } \int_B |\nabla f|^2 \leq 1 \quad \left| \int_B \nabla f \cdot \nabla \zeta \, d\mathbb{L}^n \right| \leq \delta \sup |\nabla \zeta|$$

($\forall \zeta \in C_c^\infty(B)$)

$$\text{则 } \exists \text{ 调和 } u \text{ s.t. } \int_B |\nabla u|^2 \leq 1 \quad \int_B |u - f|^2 \leq \varepsilon$$

p.f. 若不然, $\exists \varepsilon > 0 \quad f_k \in W^{1,2}(B)$

$$\begin{cases} \left| \int_B \nabla f_k \cdot \nabla \zeta \, d\mathbb{L}^n \right| \leq \frac{1}{k} \sup |\nabla \zeta| \\ \int_B |\nabla f_k|^2 \leq 1 \\ \int_B |f_k - u|^2 > \varepsilon \quad (*) \end{cases}$$

$$\text{令 } \lambda_k = \omega_n^{-1} \int_B f_k \, d\mathbb{L}^n$$

$$\text{Poincaré} \Rightarrow \int_B |f_k - \lambda_k|^2 \leq C \int_B |\nabla f_k|^2 \leq C$$

$$\text{Rellich} \Rightarrow \exists \text{ s.t. } f_k - \lambda_k \xrightarrow{L^2(B)} w \quad \nabla f_k \xrightarrow{L^2} \nabla w \quad \text{其中 } w \in W^{1,2}(B) \quad \int_B |\nabla w|^2 \leq 1$$

$$\text{则 } \int_B \nabla w \cdot \nabla \zeta \, d\mathbb{L}^n = 0 \Rightarrow w \text{ 为调和. } \int_B |f_k - w - \lambda_k|^2 \rightarrow 0$$

$$\text{又 } w - \lambda_k \text{ 为调和. } S(x) \text{ 为调和. } \checkmark$$

$$\text{因此对 } u \text{ 在 } B \text{ 中调和. 则 } \sup_{B_{\frac{\gamma}{2}}(0)} \zeta^2 |D^2 u| \leq (C \zeta^{-\frac{n}{2}} \|u\|_{L^2(B)})$$

$$\Rightarrow \sup_{B_{\frac{\gamma}{2}}(0)} \zeta^{2-n} |D^2 u| \leq (C \zeta^{-\frac{n}{2}} \|\nabla u\|_{L^2(B)})$$

$$\text{对线性近似 } (1x) = u(0) + x \cdot \nabla u(0)$$

$$\Rightarrow \begin{cases} |e(0)| = |u(0)| \leq C \zeta^{-\frac{n}{2}} \|u\|_{L^2(B)} & |\nabla e| \leq C \zeta^{-\frac{n}{2}} \|\nabla u\|_{L^2(B)} \end{cases}$$

$$\left| \sup_{B_{\gamma\zeta}(0)} |u - e| \leq (\gamma\zeta)^2 \sup_{B_{\gamma\zeta}} |D^2 u| \leq C \gamma^2 \zeta^{1-\frac{n}{2}} \|\nabla u\|_{L^2(B)} \right.$$

$$(\gamma \in (0, \frac{1}{4}))$$

$$\text{Rnk. 对 } u \text{ 逼近以有理. 则 } |u(0)| \leq C (\|u - f\|_{L^2(B)} + \|f\|_{L^2(B)})$$

$$\leq C (\sqrt{\varepsilon} + \|f\|_{L^2(B)})$$

$$|\nabla u(0)| \leq C$$

4. 主正则性定理

$$E_*(\xi, G, T) \triangleq \max \{ E(\xi, G, T), \delta^{-1} (G^{p-n} \int_{B_\xi(\xi)} |H|^p d\mu)^{\frac{2}{p}} \}$$

定理之前附加的(*)条件 $\begin{cases} \theta \geq \mu - ac & 0 \in \text{supp } V \quad B_{\rho}(0) \subset U \\ \omega_n^{-1} \rho^{-n} \mu(B_{\rho}(0)) \leq 1 + \delta & (G^{p-n} \int_{B_{\rho}(0)} |H|^p d\mu)^{\frac{1}{p}} \leq \delta \end{cases}$

Lemma (Tilt-Excess Decay Lemma) $\exists \eta = \eta(n, k, p) \quad \delta_0 = \delta_0(n, k, p)$

s.t. $\delta \in (0, \delta_0]$ 且条件(*)成立. 且 $G \in (0, \delta \rho]$ $\xi \in \text{supp } \mu \cap B_{\delta \rho}(0)$

$$T \subseteq \mathbb{R}^{n+k} \quad \text{则} \quad E_*(\xi, \eta G, S) \leq \eta^{2(1-\frac{n}{p})} E_*(\xi, G, T) \text{ for some } S$$

我们的主定理为

Thm (Allad Regularity) $p > n, \exists \eta = \eta(n, k, p) \quad \delta_0 = \delta_0(n, k, p) \in (0, \frac{1}{16}]$

s.t. 若 $\delta \in (0, \delta_0]$, (*)成立. 则 $\exists Q$ 为 $\mathbb{R}^{n+k}$ 的仿射子集

$$u = (u^1, \dots, u^k) \in C^{1, 1-\frac{n}{p}}(B_{\gamma \rho}^n(0); \mathbb{R}^k), \quad Du(0) = 0, \quad \text{supp } V \cap B_{\gamma \rho}(0) = Q(\text{graph } u) \cap B_{\gamma \rho}(0)$$

$$\text{且 } \rho^{-1} \sup |u| + \sup |Du| + \rho^{1-\frac{n}{p}} \sup_{\substack{x, y \in B_{\gamma \rho}(0) \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x - y|^{1-\frac{n}{p}}} \leq C \delta^{\frac{1}{2n+2}}$$

$$C = C(n, k, p) > 0$$

pf. $\xi \in \text{supp } V \cap B_{\frac{\delta \rho}{2}}(0) \quad S_0 \subseteq \mathbb{R}^{n+k}$ 任意.

由 Tilt-excess decay $\exists \delta_0, \eta$ s.t. $\forall \delta \leq \delta_0$. 则

$$E_*(\xi, \eta G, S_1) \leq \eta^{2(1-\frac{n}{p})} E_*(\xi, G, S_0) \text{ for some } S_1$$

则对 $G_0 = \frac{\delta \rho}{2} \quad \exists S_1, S_2, \dots$ s.t. $E_*(\xi, \eta^j G_0, S_j) \leq \eta^{2j(1-\frac{n}{p})} E_*(\xi, G_0, S_0)$

令 $T_0 = T(0, 2G_0)$. 由 Affine Approximation 给出 $E(0, G_0, T_0) \leq C\delta^{\frac{1}{n+1}}$

$$\text{取 } S_0 = T_0 \Rightarrow E_{\#}(\xi, \eta^j G_0, S_j) \leq \eta^{2j(1-\frac{n}{p})} E_0 \quad E_0 = E_{\#}(0, G_0, T_0)$$

$$T.E \text{ Decay: } (\eta^j G)^{-n} \int_{B_{\eta^j G}(\xi)} |P_{T_{j,m}} - P_{S_j}|^2 d\mu \leq E_{\#}(\xi, \eta^{j-1} G_0, S_{j-1})$$

$$\text{又是由: } (\eta^j G)^{-n} \int_{B_{\eta^j G}(\xi)} |P_{T_{j,m}} - P_{S_{j-1}}|^2 d\mu \leq \eta^{-jn} E(\xi, \eta^{j-1} G_0, S_{j-1})$$

$$\begin{aligned} \text{相加} \\ \Rightarrow |P_{S_{j-1}} - P_{S_j}|^2 &\leq C E_{\#}(\xi, \eta^{j-1} G_0, S_{j-1}) \\ &\leq C \eta^{2j(1-\frac{n}{p})} E_{\#}(\xi, G_0, S_0) \end{aligned}$$

$$\Rightarrow |P_{S_k} - P_{S_j}|^2 \leq C \eta^{2j(1-\frac{n}{p})} E_0$$

$$\text{取 } S(\xi) = \lim_{j \rightarrow \infty} S_j \quad \text{有 } |P_{S_k} - P_{S_j}|^2 \leq C \eta^{2j(1-\frac{n}{p})} E_0$$

$$|P_{S(\xi)} - P_{T_0}|^2 \leq C E_0$$

$$\begin{aligned} \forall G \in (0, \frac{G_0}{2}]. \text{ 取 } j \text{ s.t. } \eta^j \frac{G_0}{2} < G \leq \eta^{j-1} \frac{G_0}{2} \\ \text{由 } E_{\#}(\xi, G, S(\xi)) &\leq C \left(\frac{G}{G_0}\right)^{2(1-\frac{n}{p})} E_0 \quad (*) \\ &\leq C E_{\#}(\xi, G_0, T_0) \leq C E_0 \leq C \delta^{\frac{1}{2n+2}} \end{aligned}$$

取 $T_0 = \mathbb{R}^n \times \{0\}$. 对 $L_0 \in (0, \frac{1}{4}]$, $\delta \leq \delta_0 L_0^{2n+2}$ for some δ_0

有 Lipschitz 映射 $f: B_{\frac{G_0}{2}}(0) \rightarrow \mathbb{R}^k$ $\text{Lip } f \leq L_0$

$$\text{s.t. } \text{supp } V \cap B_{\frac{G_0}{4}}(0) = \underbrace{\text{graph } f}_{\Gamma} \cap B_{\frac{G_0}{4}}(0)$$

对 $\xi = (\xi', f(\xi')) \in \Gamma$

$$(*) \Rightarrow \lim_{G \rightarrow 0} G^{-n} \int_{B_{G, \xi} \cap \Gamma} |P_{T_{\xi, F}} - P_{S(\xi)}|^2 d\mathcal{H}^n = 0 \quad \mathcal{H}^{n-1} \llcorner \xi$$

$$\text{也取 } S(\xi) = T_{\xi} F$$

$$\mathbb{R}^n(x) \Rightarrow \sigma^{-n} \int_{B_{\sigma}(\xi) \cap F} |P_{T_x F} - P_{T_\xi F}|^2 dH^n \leq C \left(\frac{\sigma}{\sigma_0}\right)^{2(1-\frac{n}{p})} \varepsilon_0$$

$$P_\xi = P_{T_\xi F} \quad \text{Gram-Schmidt} \Rightarrow \sum_{j=1}^n (e_j, D_j f(\xi')) \quad 1 \leq j \leq n \quad F$$

$$P_\xi = \begin{pmatrix} I_{n \times n} & Df(\xi') \\ (Df(\xi'))^T & 0_{k \times k} \end{pmatrix} + F(Df(\xi'))$$

$$F(0)=0. \quad D_p F(0)=0$$

$$\text{又 } L_0 \text{ 足够小时 } |Df(x) - Df(\xi')|^2 \leq |P_{T_x F} - P_{T_{\xi'} F}|^2 \leq 3 |Df(x) - Df(\xi')|^2$$

$$\text{有 } \sigma^{-n} \int_{B_\sigma^{\hat{n}}(\xi')} |Df(x) - Df(\xi)| d\mathcal{L}^n(x) \leq C \left(\frac{\sigma}{\sigma_0}\right)^{2(1-\frac{n}{p})} \varepsilon_0$$

$$\mathbb{R}^n \text{ 对 } \mu\text{-a.e. } x_1, x_2 \in \text{supp } \nu \cap B_{\sigma_0}(0). \quad \text{令 } \sigma = |x_1 - x_2| \quad \xi = x_1 \text{ 和 } x_2$$

$$\text{对 } x \in B_\sigma^{\hat{n}}(x_1) \cap B_\sigma^{\hat{n}}(x_2)$$

$$\text{由 } |Df(x_1) - Df(x_2)|^2 \leq 2 |Df(x_1) - Df(x)|^2 + 2 |Df(x) - Df(x_2)|^2$$

$$\Rightarrow |Df(x_1) - Df(x_2)|^2 \leq C \left(\frac{|x_1 - x_2|}{\sigma_0}\right)^{1-\frac{n}{p}} \varepsilon_0^{\frac{1}{2}}$$

$$\mathbb{R}^n |f| \in C^{1, 1-\frac{n}{p}}. \quad \text{取 } \delta \text{ 足够小, 以满足及上面各条件. 令 } u=f \quad \gamma=\delta \in \mathbb{R}^q$$

$$\text{Cor } \forall \gamma \in (0,1). \quad \exists \delta = \delta(n, k, p, \gamma) \in (0,1) \quad \text{s.t. } \frac{\delta}{2} \leq \delta/4 \text{ (*) 成立. } p > n.$$

$$\delta \leq \delta_0 \quad \mathbb{R}^n \exists q \text{ 为 } \mathbb{R}^{n+k} \text{ 的线性映射且 } q \text{ 是同构.} \quad u = (u^1, \dots, u^k) \in C^{1, 1-\frac{n}{p}}(B_{\gamma\rho}(0), \mathbb{R}^k)$$

$$\text{s.t. } \text{supp } \nu \cap B_{\gamma\rho}(0) = q(\text{graph } u) \cap B_{\gamma\rho}(0)$$

$$\text{且 } p^{-1} \sup |u| + \sup |Du| + p^{1-\frac{n}{p}} \sup_{\substack{x, y \in B_{\gamma\rho}(0) \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x - y|^{1-\frac{n}{p}}} \leq C \delta^{\frac{1}{2n+2}}$$

$$C = C(n, k, p, \gamma)$$