

Complex Geometry

chapter 1 Rudiments of Several Complex Variables

1.1 Holomorphic Function

I. One Variable

$$\Omega \subset \mathbb{C} \text{ open, } f \in C^1(\Omega, \mathbb{C}) \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$f \text{ is holomorphic} \Leftrightarrow \bar{\partial} f = 0$$
$$\begin{aligned} f = u + iv \\ \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \end{aligned}$$

$$dz = dx + i dy \quad d\bar{z} = dx - i dy$$
$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Cauchy formula: $\bar{\Omega} \subset \mathbb{C}$ compact, boundary $\partial\Omega \subset \mathbb{C}$.

$f \in C^1(\bar{\Omega}, \mathbb{C})$, then

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \int_{\bar{\Omega}} \frac{1}{\pi(z-w)} \frac{\partial f}{\partial \bar{z}} dx \wedge dy$$

$$\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$$

II Several Variables

$$\text{holomorphic} \Leftrightarrow \text{complex analytic} \Leftrightarrow \frac{\partial f}{\partial \bar{z}_\nu} = 0 \quad (\forall \nu)$$

Polydisk: $D(a, R) = D(a_1, R_1) \times \dots \times D(a_n, R_n)$

"boundary" $S(a, R) = S(a_1, R_1) \times \dots \times S(a_n, R_n)$

Cauchy formula. If $\bar{\partial}f = 0$ then

$$f(w) = \frac{1}{(2\pi i)^n} \int_{S(a, R)} \frac{f(z_1, \dots, z_n)}{(z_1 - w_1) \dots (z_n - w_n)} dz_1 \dots dz_n$$

1.2 Domain of Holomorphy

Envelope of holomorphy of Ω : $D_\Omega = \bigcap_{f \in \mathcal{O}(\Omega)} D_f$

Thm (Hartogs) Ω open in $\mathbb{C}^n$ ($n \geq 2$). $K \subset \Omega$ compact and $\Omega \setminus K$ is connected. Then every holomorphic function $f \in \mathcal{O}(\Omega \setminus K)$ extends to $\tilde{f} \in \mathcal{O}(\Omega)$

Thm $\Omega \subset \mathbb{C}^n$ open. if Ω is convex, then it's a domain of holomorphy

Def $\Omega \subset \mathbb{C}^n$, for a compact set K in Ω

holomorphic hull $\hat{K} = \{x \in \Omega \mid |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(\Omega)\}$

If $\hat{K}$ is compact when K is compact, then Ω is holomorphically convex.

Thm (Cartan-Thullen) Ω a domain. then $\text{TF} \Leftrightarrow$

- (i) Ω is a domain of holomorphy
- (ii) Ω is a holomorphically convex domain
- (iii) $\exists f \in \mathcal{O}(\Omega)$ st. $D_f = \Omega$

1.3 Pseudoconvex Domain

Def. $f: \Omega \rightarrow [-\infty, +\infty]$ is plurisubharmonic if

- (i) f is upper continuous
- (ii) every complex line $L \subset \mathbb{C}^n$, $f|_{L \cap \Omega}$ is subharmonic

Thm. $f \in C^2(\Omega)$ is plurisubharmonic iff

$$\sum_{i,j=1}^n \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} w_i \bar{w}_j \geq 0 \quad \forall z \in \Omega, w \in \mathbb{C}^n$$

Def Ω a domain. $y \in \partial\Omega$, if $\exists$ a nbhd U of y and $\varphi \in C^1(U)$, st

$$(i) \Omega \cap U = \{x \in U \mid \varphi(x) < 0\}$$

$$(ii) d\varphi|_y \neq 0$$

$$(iii) \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \Big|_y \xi_i \bar{\xi}_j \geq 0 \text{ for } \forall \sum_{i=1}^n \frac{\partial \varphi}{\partial z_i} \Big|_y \xi_i = 0$$

then Ω is pseudconvex at y

Rmk. This is independent of φ

Real and Complex Hessian:

$$\mathbb{C}^n \subseteq \mathbb{R}^{2n} \quad z_i = x_i + \sqrt{-1} y_i \quad y_i = x_{n+i}$$

$$\text{Real Hessian} \quad \left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right)_{2n \times 2n} = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} & \frac{\partial^2 \varphi}{\partial x_i \partial x_{n+j}} \\ \frac{\partial^2 \varphi}{\partial x_{n+i} \partial x_j} & \frac{\partial^2 \varphi}{\partial x_{n+i} \partial x_{n+j}} \end{pmatrix}$$

$$\text{Complex Hessian} \quad \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)_{n \times n}$$

$$J: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$$

$$J_* \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_{n+i}} = \frac{\partial}{\partial y_i} \quad J_* \left(\frac{\partial}{\partial y_i} \right) = -\frac{\partial}{\partial x_i}$$

$$J_* \left(\frac{\partial}{\partial z_i} \right) = \sqrt{-1} \frac{\partial}{\partial z_i} \quad J_* \left(\frac{\partial}{\partial \bar{z}_i} \right) = -\sqrt{-1} \frac{\partial}{\partial \bar{z}_i}$$

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

$$\text{Herm}(n) = \{ H \mid \overline{H}^T = -H \}$$

$$\tilde{\mathcal{S}} = \{ K \in \text{Sym}(\mathbb{R}^{2n}) \mid [K, J] = KJ - JK = 0 \}$$

$$i: \text{Herm}(n) \rightarrow \tilde{\mathcal{S}}$$

$$H = A + J^{-1}B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \text{is an isomorphism}$$

$$\text{Prop 1} \quad \left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right) \geq 0 \Rightarrow \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) \geq 0$$

$$\textcircled{2} \quad \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) = 0 \Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right) \in \tilde{\mathcal{S}}$$

Prop Ω a domain in $\mathbb{C}^n$. if Ω is strongly

pseudoconvex at $y \in \partial\Omega$, then we can choose a defining function s.t. its complex Hessian is positive at y

Thm Ω is strongly pseudoconvex at $y \in \partial\Omega$

iff $\exists$ a holomorphic coordinate s.t. Ω is

strictly convex at $y \in \partial\Omega$

Chapter 2 Complex Manifolds

2.1 Holomorphic Map

$f: \mathbb{C}^n \rightarrow \mathbb{C}^m$ is holomorphic if $\forall f_\lambda$ is holomorphic.

$$(z_1, \dots, z_n) \in \mathbb{C}^n$$

$$z_i = x_i + \sqrt{-1} y_i$$

$$(w_1, \dots, w_m) \in \mathbb{C}^m$$

$$w_\lambda = u_\lambda + \sqrt{-1} v_\lambda$$

$$df = \frac{\partial u_\lambda}{\partial x_i} dx_i \otimes \frac{\partial}{\partial u_\lambda} + \frac{\partial u_\lambda}{\partial y_i} dy_i \otimes \frac{\partial}{\partial u_\lambda} + \frac{\partial v_\lambda}{\partial x_i} dx_i \otimes \frac{\partial}{\partial v_\lambda} + \frac{\partial v_\lambda}{\partial y_i} dy_i \otimes \frac{\partial}{\partial v_\lambda}$$

$$\partial f = \frac{\partial f_\lambda}{\partial z_i} dz_i \otimes \frac{\partial}{\partial w_\lambda} \quad \bar{\partial} f = \frac{\partial f_\lambda}{\partial \bar{z}_i} d\bar{z}_i \otimes \frac{\partial}{\partial w_\lambda}$$

$$\partial \bar{f} = \frac{\partial \bar{f}_\lambda}{\partial z_i} dz_i \otimes \frac{\partial}{\partial \bar{w}_\lambda} \quad \bar{\partial} \bar{f} = \frac{\partial \bar{f}_\lambda}{\partial \bar{z}_i} d\bar{z}_i \otimes \frac{\partial}{\partial \bar{w}_\lambda}$$

$$df = \partial f + \bar{\partial} \bar{f} + \bar{\partial} f + \partial \bar{f}$$

$$f \text{ is holomorphic} \Leftrightarrow \bar{\partial} f = 0 \Leftrightarrow \partial \bar{f} = 0$$

$$\begin{pmatrix} I & 0 \\ -i & I \end{pmatrix} \begin{pmatrix} \frac{\partial u_\lambda}{\partial x_i} & \frac{\partial v_\lambda}{\partial x_i} \\ \frac{\partial u_\lambda}{\partial y_i} & \frac{\partial v_\lambda}{\partial y_i} \end{pmatrix} \begin{pmatrix} I & 0 \\ i & I \end{pmatrix} \stackrel{\text{check}}{=} \begin{pmatrix} \frac{\partial f_\lambda}{\partial \bar{z}_i} + \frac{\partial f_\lambda}{\partial z_i} & \frac{\partial v_\lambda}{\partial x_i} \\ -2\sqrt{-1} \frac{\partial f_\lambda}{\partial \bar{z}_i} & \frac{\partial \bar{f}_\lambda}{\partial \bar{z}_i} \frac{\partial f_\lambda}{\partial \bar{z}_i} \end{pmatrix}$$

$$f \text{ holomorphic, then } \det \begin{pmatrix} \frac{\partial u_\lambda}{\partial x_i} & \frac{\partial v_\lambda}{\partial x_i} \\ \frac{\partial u_\lambda}{\partial y_i} & \frac{\partial v_\lambda}{\partial y_i} \end{pmatrix} = |\det \left(\frac{\partial f_\lambda}{\partial z_i} \right)|^2 \geq 0$$

2.2 Complex Manifold. Pseudogroup Structure

Def Complex Manifold

e.g. $\mathbb{C}P^{n-1} = \mathbb{C}^n \setminus \{0\} / \sim$

Def. D a domain of $\mathbb{R}^n$ or $\mathbb{C}^n$. A pseudogroup of transformations in D is a set Γ of local transformations

of D s.t. (i) $f \in \Gamma \Rightarrow f^{-1} \in \Gamma$

(ii) $f \in \Gamma, g \in \Gamma \Rightarrow g \circ f \in \Gamma$

(iii) $f \in \Gamma \Rightarrow f|_W \in \Gamma \quad \forall W \text{ open}$

(iv) $\text{Id} \in \Gamma$

(v) f : any local diffeomorphism of D . $D = \bigcup U_j$

$f|_{U_j} \in \Gamma(\forall j)$. then $f \in \Gamma$

Def. Γ as above. X paracompact Hausdorff. By a

system of local Γ -coordinates we mean a set $\{z_j\}_{j \in I}$ of local homeomorphisms z_j of X into D s.t. $z_j \circ z_k^{-1} \in \Gamma$

$\{W_\lambda\}$ $\{z_j\}$ are equivalent if $w_\lambda \circ z_j^{-1} \in \Gamma$

A Γ -structure on X is an equivalence class of systems

of local Γ -coordinates. X with a Γ -structure is a Γ -manifold

Submanifold:

$f: M \rightarrow N$ smooth. if f_* is injective, then f is an immersion

If f is injective, then M is an immersed submanifold.

$f(M)$ has the relative topology.

If $f: M \rightarrow f(M)$ is a homeomorphism, then it's an embedding

Prop. M a connected compact complex manifold

and f be a holomorphic function on M , then f is constant.

Cor There are no compact complex submanifold of $\mathbb{C}^n$

2.3 Almost Complex Manifold

Def. A $(1,1)$ -tensor $J \in \Gamma(T^*M \otimes TM)$ on a differential manifold M satisfying $J^2 = -Id$ is called an almost complex structure.

Rmk. Complex Manifold $\Rightarrow$ Almost Complex Manifold

$$F: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \quad (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (-y_1, \dots, -y_n, x_1, \dots, x_n)$$

$$(F)_* = J_{\mathbb{R}^{2n}} \quad J_{\mathbb{R}^{2n}} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \\ \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \\ \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}$$

$$f: \mathbb{C}^n \rightarrow \mathbb{C}^m \quad f \text{ holomorphic} \Leftrightarrow \bar{\partial} f = 0$$

$$\Leftrightarrow f_* \circ J_{\mathbb{R}^{2n}} = J_{\mathbb{R}^{2m}} \circ f_*$$

J above is local, needs to check it can be extended globally.

$$\{\varphi_j, U_j\}_{j \in I} \quad \varphi_j: U_j \rightarrow \mathbb{C}^n = \mathbb{R}^{2n}$$

$$J_j = (\varphi_j)_*^{-1} \circ J_{\mathbb{R}^{2n}} \circ (\varphi_j)_* \quad J_j^2 = -\text{Id}$$

$$U_j \cap U_k \neq \emptyset \quad J_k = (\varphi_j)_*^{-1} \circ (\varphi_k \circ \varphi_j^{-1})_*^{-1} \circ J_{\mathbb{R}^{2n}} \circ \underbrace{(\varphi_k \circ \varphi_j^{-1})_*}_{\text{holomorphic}} \circ (\varphi_j)_*$$

$$= J_j \quad \checkmark$$

Def (Nijenhuis Tensor) A (2,1)-tensor

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

Thm (Newlander - Nirenberg) (M.T) almost complex
 then J comes from a complex structure iff $N^J = 0$

Lemma. A real finite-dimensional vector space V which
 admits endomorphism $J: V \rightarrow V$ s.t. $J^2 = -\text{Id}$. is
 necessarily even dimensional.

Cor. Almost complex manifold must be even dimensional.

$f: M \rightarrow N$ holomorphic $\Leftrightarrow \psi \circ f \circ \varphi^{-1}: \mathbb{C}^m \rightarrow \mathbb{C}^n$ holomorphic

$$\Leftrightarrow (\psi \circ f \circ \varphi^{-1})_* \circ J_{\mathbb{R}^{2m}} = J_{\mathbb{R}^{2n}} \circ (\psi \circ f \circ \varphi^{-1})_*$$

$$\Leftrightarrow f_* \circ \varphi_*^{-1} \circ J_{\mathbb{R}^{2m}} \circ \varphi_* = \psi_*^{-1} \circ J_{\mathbb{R}^{2n}} \circ \psi_* \circ f_*$$

$$\Leftrightarrow f_* \circ J_M = J_N \circ f_*$$

Lemma Almost complex manifold must be oriented

Pf. $\forall p \in M \quad T_p M = \text{span} \{ x_1, Jx_1, \dots, x_n, Jx_n \}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ Jx_1 \\ \vdots \\ Jx_n \end{pmatrix} = \begin{pmatrix} A & B \\ D & C \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ Jy_1 \\ \vdots \\ Jy_n \end{pmatrix} \quad \begin{aligned} X &= AY + BJY \\ JX &= D'Y + CJY \\ &= AJY - BY \end{aligned}$$

Thus $D = -B \cdot A = C$

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A + iB)|^2 > 0 \quad (*)$$

Take $(x'_p, \dots, x_p^n, y'_p, \dots, y_p^n)$ s.t. $\frac{\partial}{\partial x'_p} = x_{i,p}$ $\frac{\partial}{\partial y'_p} = Jx_{i,p}$
 $(x'_q, \dots, x_q^n, y'_q, \dots, y_q^n)$ $\frac{\partial}{\partial x'_q} = x_{i,q}$ $\frac{\partial}{\partial y'_q} = Jx_{i,q}$

Then $\forall$ basis $(x', \dots, x^n, y', \dots, y^n)$ at p
 $(u', \dots, u^n, v', \dots, v^n)$ at q

$$(x', \dots, x^n, y', \dots, y^n) \xrightarrow[\text{change}]{\text{coordinate}} (x'_p, \dots, y_p^n) \xrightarrow{(*)} (x'_q, \dots, y_q^n) \xrightarrow[\text{change}]{\text{coordinate}} (u', \dots, u^n, v', \dots, v^n)$$

then the Jacobian should be positive.

#

2.4 The Complexified Tangent Bundle

(M, J) almost complex

$$TM^{\mathbb{C}} \cong TM \otimes_{\mathbb{R}} \mathbb{C} \quad \text{extend } J \text{ to } TM^{\mathbb{C}}$$

$$J(aX) = aJ(X) \quad (a \in \mathbb{C})$$

$T^{1,0}M$ (resp. $T^{0,1}M$) $\cong$ the eigenbundle of $TM^{\mathbb{C}}$
w.r.t. the eigenvalue J (resp. $-J$) of J

$$Z = X + JY \in T^{1,0}M \Rightarrow JX + J^2Y = JZ = JX - Y$$

$$\Rightarrow Y = -JX \quad X = JY$$

$$Z = X - J^2X$$

$$\text{Thus } T^{1,0}M = \{X - J^2X \mid X \in TM\}$$

$$T^{0,1}M = \{X + J^2X \mid X \in TM\}$$

$$\text{and } TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$$

On T^*M (still use J)

$$(J\theta)(X) \cong \theta(JX).$$

$$\theta \in T^*M \quad X \in TM$$

$$\Lambda_{\mathbb{C}}^*M \cong T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$$

$$\Lambda^{1,0}M \triangleq \{ \theta \in \Lambda_C^1 M \mid \theta(X) = 0 \quad \forall X \in T^{0,1}M \}$$

$$= \{ \theta \in \Lambda_C^1 M \mid J\theta = \bar{J}\theta \} = \{ \theta - \bar{J}J\theta \mid \theta \in \Lambda^1 M \}$$

$$\Lambda^{0,1}M = \{ \theta \in \Lambda_C^1 M \mid J\theta = -\bar{J}\theta \} = \{ \theta + \bar{J}J\theta \mid \theta \in \Lambda^1 M \}$$

$$\Lambda^{1,0}M = \overline{\Lambda^{0,1}M}$$

More generally, $\Lambda_C^k M = \{ \theta_1 \wedge \dots \wedge \theta_k \mid \theta_i \in \Lambda_C^1 M \}$

$$\Lambda^{p,q}M = \underbrace{\Lambda^{1,0}M \wedge \dots \wedge \Lambda^{p,0}M}_p \wedge \underbrace{\Lambda^{0,1}M \wedge \dots \wedge \Lambda^{0,q}M}_q$$

Lemma ① ω is $(k,0)$ iff $X_J \omega = 0 \quad (\forall X \in T^{0,1}M)$

② ω is $(1,1)$ iff $\omega(X,Y) = \omega(JX, JY) \quad \forall X, Y \in TM$

Pf. ②. $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$

$$\omega \in \Lambda^{1,1}M \Leftrightarrow \omega^{2,0} = 0 = \omega^{0,2}$$

$$[\omega(X,Y) = \omega(X^{1,0}, Y^{1,0}) + \omega(X^{0,1}, Y^{0,1}) + \omega(X^{1,0}, Y^{0,1}) + \omega(X^{0,1}, Y^{1,0})]$$

$$\Leftrightarrow \omega(X - \bar{J}JX, Y - \bar{J}JY) = \omega(X + \bar{J}JX, Y + \bar{J}JY) = 0$$

$$\Leftrightarrow \omega(X,Y) - \omega(JX, JY) - \bar{J}(\omega(JX,Y) + \omega(X, JY)) = 0$$

$$\Leftrightarrow \omega(X,Y) = \omega(JX, JY)$$

We assumed ω is real above. By taking conjugation it still holds when ω is complex. #

Prop (M, J) almost complex, TFAE:

(1) J is a complex structure.

(2) $N^J = 0$

(3) $T^{0,1}M$ is integrable

(4) $d\Gamma(\Lambda^{1,0}M) \subset \Gamma(\Lambda^{2,0}M \oplus \Lambda^{1,1}M)$

(5) $d\Gamma(\Lambda^{p,q}M) \subset \Gamma(\Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M)$

Pf. (1) $\Leftrightarrow$ (2) Newlander-Nirenberg

(2) $\Leftrightarrow$ (3) $Z_1 = X_1 + \sqrt{-1}JX_1, Z_2 = X_2 + \sqrt{-1}JX_2 \in \Gamma(T^{0,1}M)$

Check $[Z_1, Z_2] - \sqrt{-1}J[Z_1, Z_2] = N^J(X_1, X_2) - \sqrt{-1}JN^J(X_1, X_2)$

$T^{0,1}M$ integrable $\Leftrightarrow [Z_1, Z_2] \in T^{0,1}M$

$\Leftrightarrow [Z_1, Z_2] = \sqrt{-1}J[Z_1, Z_2]$

$\Leftrightarrow N^J(X_1, X_2) = 0$

(3) $\Rightarrow$ (4) $\theta \in \Gamma(\Lambda^{1,0}M)$

$d\theta \in \Gamma(\Lambda^{2,0}M \oplus \Lambda^{1,1}M) \Leftrightarrow (d\theta)^{0,2} = 0$

$\Leftrightarrow d\theta(X, Y) = 0 \quad \forall X, Y \in T^{0,1}M$

$[d\theta(X, Y) = \underbrace{X(\theta(Y))}_0 - \underbrace{Y(\theta(X))}_0 - \theta([X, Y])]$

$\Leftrightarrow \theta([X, Y]) = 0 \Leftrightarrow [X, Y] \in T^{0,1}M \xrightarrow{\text{Frobenius}} T^{0,1}M \text{ integrable}$

(4) $\Leftrightarrow$ (5) Just some trivial calculation

#

2.5 Holomorphic Objects on Complex Manifold

$(M, \mathcal{S})$ complex manifold. J

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$$

$$J(dx^i) = -dy^i \quad J(dy^i) = dx^i$$

By the last prop. $d = \partial + \bar{\partial}$

$$d^2 = 0 \Leftrightarrow \partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} = -\bar{\partial}\partial$$

Def. A vector field $Z \in \Gamma(T'^0 M)$ is called holomorphic if $Z(f)$ is holomorphic for every locally defined holomorphic function f

A (p,0) form θ is holomorphic if $\bar{\partial}\theta = 0$

Def. A real vector field X is called real holomorphic if $X - J \circ JX$ is holomorphic

Lemma X is a real vector field. TFAE:

(1) X is real holomorphic

(2) $L_X J = 0$

(3) The flow of X consists of holomorphic transformations

pf. $\{\varphi_t\}$ is generated by X

$$Y \in \Gamma(TM) \quad (L_X Y)_p = \lim_{t \rightarrow 0} \frac{1}{t} [Y_{p^{-1}(\varphi_t)} - (Y_{\varphi_t^{-1}(p)})] \\ = [X, Y]$$

$$\theta \in \Gamma(T^*M) \quad (L_X \theta)_p = \lim_{t \rightarrow 0} \frac{1}{t} [\theta_{p^{-1}(\varphi_t)} - (\varphi_t^{-1})^* \theta_{\varphi_t^{-1}(p)}]$$

(2) $\Leftrightarrow$ (3)

$$L_X (Y \otimes \theta) = \lim_{t \rightarrow 0} \frac{1}{t} [Y_p \otimes \theta_{p^{-1}(\varphi_t)} - (Y_{\varphi_t^{-1}(p)}) \otimes (\varphi_t^{-1})^* (\theta_{\varphi_t^{-1}(p)})]$$

$$\Rightarrow (L_X J)_p = \lim_{t \rightarrow 0} \frac{1}{t} [J_{p^{-1}(\varphi_t)} - (Y_{\varphi_t^{-1}(p)}) \circ (\varphi_t^{-1})^*]$$

$$(\varphi_s)_* \circ (L_X J) \circ (\varphi_s^{-1})^* = L_X ((\varphi_s)_* \circ J \circ (\varphi_s^{-1})^*) \\ = -\frac{d}{dt} [(\varphi_t)_* \circ J \circ (\varphi_t^{-1})^*]_p \Big|_{t=s}$$

$$L_X J = 0 \Leftrightarrow (\varphi_t)_* \circ J \circ (\varphi_t^{-1})^* = J \Leftrightarrow \varphi_t \text{ is holomorphic} \quad \#.$$

Lemma (Poincaré) $\mathcal{O}_M$ is a smooth manifold.

θ is closed r -form. Then $\forall p \in M \exists U \ni p$ and $(r-1)$ form ψ on U s.t. $\theta = d\psi$

② M is a complex manifold, θ is a $\bar{\partial}$ -closed

(p, q) form on M , then $\forall p \in M \exists U \ni p$ and $(p, q-1)$ form

ψ on U s.t. $\theta = \bar{\partial}\psi$

Lemma ($\partial\bar{\partial}$ Lemma): M is a complex manifold.

$\omega \in \Omega^{1,1}(M) \cap \Omega^2(M, \mathbb{R})$ Then ω is closed

iff $\forall p \in M, \exists U \ni p$ s.t. $\omega|_U = \sqrt{-1} \partial\bar{\partial}u$ for some real function u on U

Pf. $\Leftarrow \omega|_U = \sqrt{-1} \partial\bar{\partial}u$, then $d\omega = \sqrt{-1}(\partial + \bar{\partial})(\partial\bar{\partial}u)$
 $= \sqrt{-1}(\partial^2\bar{\partial}u - \partial\bar{\partial}^2u) = 0$

$\Rightarrow \exists$ locally real 1-form θ s.t. $\omega = d\theta$

$\theta = \theta^{1,0} + \theta^{0,1}$, $\theta^{1,0} = \overline{\theta^{0,1}}$

$\omega = d\theta = \partial\theta^{1,0} + (\bar{\partial}\theta^{1,0} + \partial\theta^{0,1}) + \bar{\partial}\theta^{0,1}$

$\omega \in (1,1)$, thus $\partial\theta^{1,0} = \bar{\partial}\theta^{0,1} = 0$

$\omega = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0}$ $\exists f$, $\theta^{0,1} = \bar{\partial}f \Rightarrow \theta^{1,0} = \partial\bar{f}$

$\omega = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0} = \partial\bar{\partial}(f - \bar{f}) = \sqrt{-1} \partial\bar{\partial}(2\operatorname{Im}f) \neq$

2.6 Complex and holomorphic vector Bundle

(M, E, π) is a holomorphic bundle

$$\wedge^{p,q} E = \wedge^{p,q} M \otimes E \quad (E \text{ valued } (p,q) \text{ form})$$

$$\bar{\partial}_E: T(\wedge^{p,q} E) \rightarrow T(\wedge^{p,q+1} E)$$

$$e_i = \psi_u^{-1}(p, (0, \dots, 1, \dots, 0)) \quad p \in U$$

$$\theta = \theta^i \otimes e_i \quad (\theta^i \in T(\wedge^{p,q} M)) \text{ then } \bar{\partial}_E \theta = \bar{\partial} \theta^i \otimes e_i$$

$$\text{Leibniz rule: } \bar{\partial}_E(\omega \wedge \sigma) = \bar{\partial}_E \omega \wedge \sigma + (-1)^{p+q} \omega \wedge \bar{\partial}_E \sigma$$

(*)
 $\forall \omega \in \wedge^{p,q}(E), \sigma \in \wedge^{r,s}(E)$

Def (Pseudo-holomorphic structure)

(E, M, π) complex bundle, M complex manifold

An operator $\bar{\partial}_E: \wedge^{p,q}(E) \rightarrow \wedge^{p,q+1}(E)$ satisfying

(*) is called a pseudo-holomorphic structure.

If moreover, $\bar{\partial}_E^2 = 0$, then $\bar{\partial}_E$ is called a holomorphic structure

A section σ in a pseudo-holomorphic bundle $(E, \bar{\partial}_E)$ is called holomorphic if $\bar{\partial}_E \sigma = 0$

Lemma A pseudo-holomorphic bundle $(E, \bar{\partial}_E)$ of rank r is holomorphic $\Leftrightarrow \forall P \in M, \exists U \ni P \quad \exists$ holomorphic basis $\{\sigma_i(x)\}_{i=1}^k$

Thm. A complex bundle (E, M, π) is holomorphic iff it has a holomorphic structure $\bar{\partial}_E$.

p.f. $\Leftarrow \{\sigma_1, \dots, \sigma_k\}$ local basis of E on U

$$\bar{\partial}_E \sigma_i = T_{ij} \otimes \sigma_j \quad T_{ij} \text{ (0,1) form}$$

$$\bar{\partial}_E^2 = 0 \Rightarrow 0 = \bar{\partial}_E (T_{ij} \otimes \sigma_j) = \bar{\partial} T_{ij} \otimes \sigma_j - T_{ik} \wedge \bar{T}_{kj} \otimes \sigma_j$$

$$\Rightarrow \bar{\partial} T_{ij} = T_{ik} \wedge \bar{T}_{kj} \quad \text{in simplicity } \bar{\partial} T = T \wedge T.$$

[Lemma $T = (T_{ij})$ is a $GL_k(\mathbb{C})$ valued (0,1)-form on U

$\bar{\partial} T = T \wedge T$, then $\forall P \in U, \exists U' \subset U \quad f: U' \rightarrow GL_k(\mathbb{C})$

$$\text{s.t. } \bar{\partial} f + f T = 0$$

]

Then on U' , define $S_j = f_{j\ell} \sigma_\ell$

$$\begin{aligned} \bar{\partial} S_j &= \bar{\partial} f_{j\ell} \otimes \sigma_\ell + f_{j\gamma} T_{\gamma\ell} \otimes \sigma_\ell \\ &= (\bar{\partial} f + f T)_{j\ell} \otimes \sigma_\ell = 0 \end{aligned}$$

pf of lemma: $N = U \times \mathbb{C}^k$ $\{z^\alpha\}_{\alpha=1}^n$ a local coordinate of U . $\{w^i\}_{i=1}^k$ the complex coordinate in $\mathbb{C}^k$

we want to fix an almost complex structure $\tilde{J}$

$\Leftrightarrow$ a subbundle of $\Lambda_{\mathbb{C}}^1 M$ as a $(1,0)$ -subbundle

$$\Lambda_{\mathbb{C}}^{1,0} N = \{dZ^\alpha, dw^i - T_{i\bar{\ell}} w_{\bar{\ell}} \mid 1 \leq \alpha \leq n, 1 \leq i \leq k\} \text{ (check independence)}$$

$$J \text{ is integrable} \Leftrightarrow d\Gamma(\Lambda_{\mathbb{C}}^{1,0} N) \subset \Gamma(\Lambda_{\mathbb{C}}^{1,0} N \wedge \Lambda_{\mathbb{C}}^1 N)$$

$$d(dZ^\alpha) = 0$$

$$d(dw^i - T_{i\bar{\ell}} w_{\bar{\ell}}) = -dT_{i\bar{\ell}} w_{\bar{\ell}} + T_{i\bar{\ell}} \wedge dw_{\bar{\ell}}$$

$$= -\partial T_{i\bar{\ell}} w_{\bar{\ell}} - \bar{\partial} T_{i\bar{\ell}} w_{\bar{\ell}} + T_{i\bar{\ell}} \wedge dw_{\bar{\ell}}$$

$$\stackrel{\bar{\partial} T = T \wedge \bar{\partial} T}{=} -\partial T_{i\bar{\ell}} w_{\bar{\ell}} - (T_{i\bar{s}} \wedge T_{s\bar{\ell}}) w_{\bar{\ell}} + T_{i\bar{\ell}} \wedge dw_{\bar{\ell}}$$

$$= -\partial T_{i\bar{\ell}} w_{\bar{\ell}} + T_{i\bar{\ell}} \wedge (dw_{\bar{\ell}} - T_{\ell\bar{s}} w_{\bar{s}})$$

$$\in \Gamma(\Lambda_{\mathbb{C}}^{1,0} N \wedge \Lambda_{\mathbb{C}}^1 N)$$

$$\{z^\alpha, u^j\} \text{ on } u' \subset U \quad du^j = \psi_{ji} (dw^i - T_{i\bar{\ell}} w_{\bar{\ell}}) + \psi_{j\alpha} dz^\alpha$$

$$0 = d^2 u^j = d\psi_{ji} \wedge (dw^i - T_{i\bar{\ell}} w_{\bar{\ell}}) + \psi_{ji} (-dT_{i\bar{\ell}} w_{\bar{\ell}} + T_{i\bar{\ell}} \wedge dw_{\bar{\ell}}) + d\psi_{j\alpha} \wedge dz^\alpha$$

$$\text{restrict to } \{w_i = 0\} \quad f_{ji}(z) \stackrel{\circ}{=} \psi_{ji}(z, 0)$$

$$\bar{\partial} f_{ji} \wedge dw^i + f_{ji} T_{i\bar{\ell}} \wedge dw_{\bar{\ell}} = 0$$

$$\text{then } \bar{\partial} f_{ji} + f_{i\bar{\ell}} T_{\ell\bar{i}} = 0$$

$\neq$

Chapter 3 Vector Bundle.

3.1 Connections on Complex Vector Bundle

Def. (E, M, π) complex vector bundle. A connection on

E is $\mathbb{C}$ -linear map $D: \Gamma(E) \rightarrow \Gamma(\wedge^1 E)$

$$\text{s.t. } D(f\sigma) = df \otimes \sigma + f \cdot D\sigma \quad \begin{matrix} f \in C^1(M) \\ \sigma \in \Gamma(E) \end{matrix}$$

Remark (1) D can also be extended to

$$D: \Gamma(\wedge^k E) \rightarrow \Gamma(\wedge^{k+1} E)$$

$$D(\theta \otimes \sigma) = d\theta \otimes \sigma + (-1)^k \theta \wedge D\sigma \quad \theta \in \Gamma(\wedge^k M)$$

$$(2) (D_1 - D_2)(f\sigma) = f(D_1 - D_2)\sigma \Rightarrow D_1 - D_2 \in \Gamma(\wedge^1(E^* \otimes E)) \\ = \Gamma(\wedge^1(\text{End}(E)))$$

(3) M complex manifold $D = D^{1,0} + D^{0,1}$

$$D^{1,0}: \Omega^{p,q}(E) \rightarrow \Omega^{p+1,q}(E)$$

$$D^{0,1}: \Omega^{p,q}(E) \rightarrow \Omega^{p,q+1}(E)$$

Def. The curvature F_D of D is $\text{End}(E)$ -valued 2 form

defined by $F_D = D \circ D$

$$F_D(X, Y)\sigma = D(D\sigma)(X, Y)$$

$$= D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X, Y]}\sigma$$

$$De_\beta = e_\alpha A_\beta^\alpha \quad A_\beta^\alpha \in \Lambda^1 M \text{ - connection 1-form}$$

$$A = (A_\beta^\alpha)_{k \times k}$$

$$D(e_1, \dots, e_k) = (e_1, \dots, e_k) (A_\beta^\alpha)$$

$$x \in E \quad x = (e_1, \dots, e_k) \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix} \Rightarrow DX = (e_1, \dots, e_k) (d+A) \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}$$

$$\text{locally } D = d + A$$

$$F_D X = D((e_1, \dots, e_k) (d+A) \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}) = (e_1, \dots, e_k) \left\{ A \wedge [(d+A)X] + d[(d+A)X] \right\}$$

$$= (e_1, \dots, e_k) (dA + A \wedge A) X$$

$$F_D = dA + A \wedge A$$

Def (Hermitian Structure) $\pi: E \rightarrow M$ complex rank k

Vector bundle A Hermitian structure H on E

is a smooth field of Hermitian inner products on

the fibers of E

Prop Every complex manifold admits Hermitian structure

local $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ $\{f_\alpha\}_{\alpha \in I}$ is the P.O.U.

$\forall p \in U_\alpha$, define $H_\alpha(X, Y) = \pi_\alpha(\psi_\alpha(X)) \overline{\pi_\alpha(\psi_\alpha(Y))}^T$

$$H = \sum_{\alpha \in I} f_\alpha H_\alpha$$

Now given (E, H) , we say D is H -connection.

$$\text{i.e. } X(H(e_1, e_2)) = H(D_X e_1, e_2) + H(e_1, D_X e_2)$$

Def (Chern connection) $(E, \bar{\partial}_E)$ holomorphic vector

bundle over complex manifold M . H a Hermitian metric

A Chern connection D is a H -connection which is compatible with $\bar{\partial}_E$. i.e. $D^{0,1} = \bar{\partial}_E$

Prop $(E, \bar{\partial}_E)$ is a holomorphic vector bundle with

a Hermitian metric, then $\exists!$ Chern connection $D_{H, \bar{\partial}_E}$

Pf Define $I_H: E \rightarrow E^*$

$$I_H(e_1)(e_2) = H(e_2, e_1)$$

$\bar{\partial}_E$ induces a holomorphic structure on E^*

$$\bar{\partial}_E^* \theta(e) = \bar{\partial}(\theta(e)) - \theta(\bar{\partial}e)$$

$$\text{set } D_X^{1,0} e = I_H^{-1} \circ (\bar{\partial}_E^*)_X \circ I_H(e) \quad D_X e = (\bar{\partial}_E)_X e + D_X^{1,0} e$$

is a Chern connection

Uniqueness: D is a Chern connection, then $D^{0,1} = \bar{\partial}_E$

then $\forall x \in T^{1,0}M, e_1, e_2 \in T(x)$

$$\bar{x} H(e_1, e_2) = H(D_{\bar{x}} e_1, e_2) + H(e_1, D_x e_2)$$

$$= H((\bar{\partial}_x)_{\bar{x}} e_1, e_2) + \underbrace{H(e_1, D_x^{1,0} e_2)}_{I_H(D_x^{1,0} e_1)(e_1)}$$

$$\Rightarrow I_H(D_x^{1,0} e_2)(e_1) = \bar{x}(I_H(e_2)(e_1)) - I_H(e_2)(\bar{\partial}_{\bar{x}} e_1) \\ = (\bar{\partial}_{\bar{x}} I_H(e_2)) e_1 \quad \neq$$

Now for locally holomorphic basis $\{e_1, \dots, e_k\}$ $\bar{\partial}_x e_\alpha = 0$

$$D_H e_\alpha = e_\beta A^\beta_\alpha$$

$$H_{\alpha\beta} = \langle e_\beta, e_\alpha \rangle_H = \overline{\langle e_\alpha, e_\beta \rangle_H} \quad H = (H_{\alpha\beta})$$

$$d\bar{H} = d \left\langle \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1, \dots, e_k) \right\rangle_H = \left\langle D \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1, \dots, e_k) \right\rangle_H \\ + \left\langle \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, D(e_1, \dots, e_k) \right\rangle_H$$

$$= \underbrace{A^T \bar{H}}_{(1,0)} + \underbrace{\bar{H} A}_{(0,1)}$$

$$\partial \bar{H} = A^T \bar{H} \\ \Rightarrow A = H^{-1} \circ \partial H$$

$$\bar{F}_H \triangleq \bar{F}_{D_H} = dA_H + A_H \wedge A_H$$

$$= \underbrace{d(H^{-1} \partial H)}_{\partial + \bar{\partial}} + H^{-1} \partial H \wedge H^{-1} \partial H$$

$$\partial H^{-1} = -H^{-1} \partial H \cdot H^{-1}$$

$$= \bar{\partial}(H^{-1} \partial H)$$

Given two Hermitian H, K $h = K^{-1}H \in \text{End}(E)$

where $(he_1, e_2)_K = \langle e_1, e_2 \rangle_H$

$$\bar{I} \quad h(e_1 \cdots e_k) = (e_1 \cdots e_k) (h^\alpha_\beta)$$

$$\langle h \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}, (e_1, \dots, e_k) \rangle_K = H \Rightarrow h^T = H \cdot K^{-1}$$
$$h^T K$$
$$h = K^{-1} H]$$

Chern connection: $A_{i-1} = d + A_{i-1}$ $D_k = d + A_k$

$$A_H = H^{-1} \partial H = h^{-1} k^T \partial(kh) \\ = A_k + h^{-1} (\partial h + k^T \partial k h - h \circ k^T \partial k)$$

$$P_H - P_K = h^{-1} P_K^{(0)} h = h^{-1} \partial_K h$$

$$\begin{aligned} F_H &= \bar{\partial}(H^{-1} \partial H) = \bar{\partial}(k^{-1} \partial k + h^{-1} \partial_k h) \\ &= F_k + \bar{\partial}(h^{-1} \partial_k h) \end{aligned}$$

$$\text{tr } F_H = \text{tr } F_K + \bar{\partial} \partial \log \det h$$

Chapter 4 Kähler manifold

4.1 Almost Hermitian Manifold

Def. A Hermitian metric g on an almost complex manifold (M, J) is a Riemannian metric g

$$\text{s.t. } g(X, Y) = g(JX, JY) \quad \forall X, Y \in TM$$

The fundamental form is defined $\omega(X, Y) = g(JX, Y)$

If $d\omega = 0$, (M, J, g) will be called an almost Kähler manifold

Remk. ① Extend g to $T^{\mathbb{C}}M$ satisfies

$$(i) \quad g(\bar{z}_1, \bar{z}_2) = \overline{g(z_1, z_2)} \quad \forall z_1, z_2 \in T^{\mathbb{C}}M$$

$$(ii) \quad g(z, \bar{z}) > 0 \quad \forall z \in T^{\mathbb{C}}M - \{0\}$$

$$(iii) \quad g(z_1, z_2) = 0 \quad \forall z_1, z_2 \in T^{1,0}M$$

② TM is in particular a complex vector bundle:

$\forall p \in M$ nbhd $U \ni p$ frame $e_1, \dots, e_m, J e_1, \dots, J e_m$

$$\psi_U = \pi^{-1}(U) \rightarrow U \times \mathbb{C}^m$$

$$x^i e_i + y^j J e_j = x \mapsto (p, (x^1 + \sqrt{-1} y^1, \dots, x^m + \sqrt{-1} y^m))$$

$$T^{1,0}M \subseteq TM: \quad e_i \mapsto \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}J e_i)$$

$$J e_i \mapsto \frac{\sqrt{-1}}{\sqrt{2}}(e_i - \sqrt{-1}J e_i)$$

$$\forall X \in TM \quad X \mapsto \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$$

③ $H(X, Y) \stackrel{\circ}{=} (g - \sqrt{-1}\omega)(X, Y)$ Hermitian structure
on the complex vector bundle TM

$$Z = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX) \quad W = \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)$$

$$H(Z, W) = g(Z, \bar{W})$$

Conversely, if a Hermitian structure H on $T^{1,0}M \subseteq TM$,

we have a hermitian metric g on TM

$$g(X, Y) \stackrel{\circ}{=} \operatorname{Re}(H(\frac{1}{\sqrt{2}}(X - \sqrt{-1}JX), \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)))$$

$$g(JX, JY) = g(X, Y)$$

$$\omega(X, Y) = \operatorname{Re}(H(\frac{\sqrt{-1}}{\sqrt{2}}(X - \sqrt{-1}JX), \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY)))$$

④ Every almost complex manifold admits Hermitian metric.

Volume form:

$$z^a = x^a + i y^a \quad dV_g = \sqrt{\det(g_{i,j})} dx^1 \wedge dy^1 \cdots \wedge dx^m \wedge dy^m$$

Ex 11

$$\frac{\omega^m}{m!} = \frac{1}{m!} (\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta) \wedge \cdots \wedge (\sqrt{-1} g_{\alpha_m \bar{\beta}_m} dz^{\alpha_m} \wedge d\bar{z}^{\beta_m})$$

$$= \frac{(\sqrt{-1})^m}{m!} g_{\alpha_1 \bar{\beta}_1} \cdots g_{\alpha_m \bar{\beta}_m} dz^{\alpha_1} \wedge d\bar{z}^{\beta_1} \cdots \wedge dz^{\alpha_m} \wedge d\bar{z}^{\beta_m}$$

$$= \frac{(\sqrt{-1})^m}{m!} \sum_{\alpha_1, \dots, \alpha_m} \frac{\text{sgn}(\alpha_1, \dots, \alpha_m)}{(-1)^{\frac{m(m-1)}{2}}} \det(g_{\alpha\beta})$$

$$= (\sqrt{-1})^m \det(g_{\alpha\bar{\beta}}) dz^1 \wedge d\bar{z}^1 \wedge dz^m \wedge d\bar{z}^m$$

$$= \underbrace{2^m \det(g_{\alpha\bar{\beta}})}_{\sqrt{\det(g_{i,j})}} dx^1 \wedge dy^1 \cdots \wedge dx^m \wedge dy^m$$

4.2 Kähler Metric

Def Let g be a Hermitian metric on a complex manifold (M, J) . If $dw=0$, then g is called a

Kähler metric. (M, J, g) is called a Kähler manifold.

Levi-Civita connection: metric g

$$\nabla g: T(TM) \rightarrow \mathcal{L}'(TM)$$

$$(1) \nabla g = 0$$

$$(2) \nabla_X Y - \nabla_Y X = [X, Y]$$

Lemma (M, J, g) an almost Hermitian manifold
and ∇g , then (M, J, g) is Kähler $\Leftrightarrow \nabla J = 0$.

p.f. $\Leftarrow \forall X, Y \in TM$

$$\begin{aligned} N^J(X, Y) &= \nabla_X Y - \nabla_Y X + \underbrace{J(\nabla_{JX} Y - \nabla_{JY} X)}_{\substack{\nabla_{JX} JY - \nabla_{JY} JX}} \\ &\quad + \underbrace{J(\nabla_X JY - \nabla_{JY} X)}_{\substack{\nabla_X JY + \nabla_{JY} JX}} \\ &= -(\underbrace{\nabla_{JX} J}_{\Delta})Y + (\underbrace{\nabla_{JY} J}_{\Delta})X - \underbrace{J(\nabla_Y J)X}_{\substack{+ J(\nabla_X J)Y}} \\ \nabla J = 0 &\Rightarrow N^J = 0 \quad \text{i.e. } J \text{ is integrable} \end{aligned}$$

$$\begin{aligned} (\nabla_X \omega)(Y, Z) &= \nabla_X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) \\ &= \nabla_X(g(JY, Z)) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z) \\ &= g((\nabla_X J)Y, Z) = 0. \end{aligned}$$

$$\begin{aligned} d\omega(X, Y, Z) &= (\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(Z, X) + (\nabla_Z \omega)(X, Y) \\ &= g((\nabla_X J)Y, Z) - g((\nabla_Y J)X, Z) + g((\nabla_Z J)X, Y) \quad (*) \end{aligned}$$

$\Rightarrow$ Replace X by JX , Y by JY .

and add up the two equations

$\neq$

(M, J, g) Kähler $\Leftrightarrow \nabla J = 0$

ω Kähler form: $\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$

real (1,1) form $\Rightarrow \omega = \sqrt{-1} \partial \bar{\partial} u$ $g_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}$

Prop. (M, J, g) Kähler $\forall x \in M \exists$ local complex coordinate $(z^1, \dots, z^n)$ s.t. $g_{\alpha\bar{\beta}}(x) = \frac{1}{2} \delta_{\alpha\beta}$ $dg_{\alpha\bar{\beta}}(x) = 0$

pf. Diagonalize $(g_{\alpha\bar{\beta}}) = \frac{1}{2} (\delta_{\alpha\beta})$ at x $(z^1, \dots, z^n)$

$$\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \quad g_{\alpha\bar{\beta}}(x) = \frac{1}{2} \delta_{\alpha\beta}$$

Near x $g_{\alpha\bar{\beta}} = \frac{1}{2} \delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} z^\gamma + \psi_{\alpha\bar{\beta}\bar{\gamma}} \bar{z}^\gamma + o(|z|)$

$$g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}} \Rightarrow \psi_{\alpha\bar{\beta}\bar{\gamma}} = \overline{\psi_{\beta\bar{\alpha}\gamma}}$$

$$\begin{aligned} 0 = d\omega &= \sqrt{-1} \left(\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} dz^\gamma + \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}^\gamma} d\bar{z}^\gamma \right) \wedge dz^\alpha \wedge d\bar{z}^\beta \\ &= \sqrt{-1} (\psi_{\alpha\bar{\beta}\gamma} dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^\beta + \psi_{\alpha\bar{\beta}\bar{\gamma}} d\bar{z}^\gamma \wedge dz^\alpha \wedge d\bar{z}^\beta) + o(1) \theta \end{aligned}$$

$$\psi_{\alpha\bar{\beta}\gamma} = \psi_{\gamma\bar{\beta}\alpha} \quad \psi_{\alpha\bar{\beta}\bar{\gamma}} = \psi_{\alpha\bar{\gamma}\bar{\beta}}$$

$$z^\alpha = w^\alpha - \frac{1}{2} \sum \psi_{\gamma\bar{\alpha}\bar{\beta}} w^\beta \bar{w}^\gamma \quad \text{inverse function theorem}$$

$$dz^\alpha = dw^\alpha - \frac{1}{2} (\psi_{\gamma\bar{\alpha}\bar{\beta}} dw^\beta \bar{w}^\gamma + \psi_{\gamma\bar{\alpha}\beta} w^\beta d\bar{w}^\gamma) = dw^\alpha - \psi_{\gamma\bar{\alpha}\bar{\beta}} w^\beta d\bar{w}^\gamma$$

$$\text{Then } \omega = \sqrt{-1} \left(\frac{1}{2} \delta_{\alpha\bar{\beta}} + o(|w|) \right) dw^\alpha \wedge d\bar{w}^\beta$$

$$d\omega = 0$$

#

4.3 Chern connection on Hermitian Manifold

$$(M, J, g) \quad TM \simeq T^{1,0}M \quad \bar{\partial} \text{ holomorphic structure on } T^{1,0}M \\ x \mapsto \frac{1}{2}(x - J \cdot Jx) \quad \bar{\partial}\left(\frac{\partial}{\partial \bar{z}_\alpha}\right) = 0$$

$$\bar{\partial}' \text{ holomorphic structure on } TM \\ \bar{\partial}'(x) \mapsto \bar{\partial}\left(\frac{1}{2}(x - J \cdot Jx)\right)$$

$$\text{Define } \bar{\partial}^D \gamma(x) = \bar{\partial}_x^D \gamma = \frac{1}{2}(\nabla_x \gamma + J \nabla_{Jx} \gamma - J(\nabla_{\gamma} J)x)$$

One can check: (i) Leibniz rule

$$(ii) \bar{\partial} \gamma = 0 \Leftrightarrow \mathcal{L}_{\gamma} J = 0$$

$$\Leftrightarrow (\mathcal{L}_{\gamma} J)(x) = 0 \xrightarrow{\text{check}} J(\bar{\partial}_x^D \gamma) = 0 \Leftrightarrow \bar{\partial}_x^D \gamma = 0$$

$$\text{Thus } \bar{\partial} = \bar{\partial}^D$$

$$f: TM \rightarrow T^{1,0}M$$

$$f(Jx) = J \cdot f(x)$$

$$x \mapsto \frac{1}{2}(x - J \cdot Jx)$$

$$f^{-1}(Jz) = J f^{-1}(z)$$

$$\text{On } TM, \quad H'(x, \gamma) = H(f(x), f(\gamma)) = g(f(x), \overline{f(\gamma)}) \\ = g(x, \gamma) - J \cdot g(Jx, \gamma)$$

$$\bar{\partial}'(\gamma) = f^{-1} \bar{\partial}(f(\gamma))$$

$$D_H: \text{Chern connection on } T^{1,0}M$$

$$D_H H = 0$$

$$(D_H^{0,1}) = \bar{\partial}$$

$$\nabla': \text{Chern connection on } TM$$

$$\nabla' \gamma = f^{-1}(D_H f(\gamma))$$

$$\text{One has: } \nabla' g = 0, \quad \nabla' J = 0, \quad (T \nabla')^{(1,1)} = 0$$

Check: $(\nabla'_X J)(Y) = \nabla'_X (JY) - J(\nabla'_X Y)$
 $= f^{-1}(D_{H_X} f(JY)) - J(f^{-1} D_{H_X} f(Y))$
 $= f^{-1}(J \cdot D_{H_X} f(Y)) - f^{-1}(J \cdot D_{H_X} f(Y)) = 0$

$$\begin{aligned} (T^{\nabla'})'(X, Y) &= \frac{1}{4} T^{\nabla'}(X - JY, Y + JX) \\ &= T^{D_H}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\gamma}\right) \\ &= \underbrace{D_H \frac{\partial}{\partial \bar{z}^\gamma}}_0 \frac{\partial}{\partial z^\alpha} - \underbrace{D_H \frac{\partial}{\partial z^\alpha}}_0 \frac{\partial}{\partial \bar{z}^\gamma} - \underbrace{\left[\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\gamma}\right]}_0 = 0 \end{aligned}$$

Prop g is Kähler iff $\nabla' = \nabla$

p.f. $\Leftarrow \nabla' = \nabla$. $\nabla' J = 0$ then $\nabla J = 0$
 $\Rightarrow \nabla J = 0 \quad \nabla g = 0 \Rightarrow \nabla \omega = 0$

Thus $\nabla H = 0$
 $T^{\nabla} = 0 \Rightarrow \nabla = \nabla'$

$\square$

4.4 Curvature of Kähler Manifold

Let D be a connection on E .

$$D \circ D \stackrel{\text{locally}}{=} D(dA + A \wedge A)$$

$$\begin{aligned} &\stackrel{D=d+A}{=} d(dA + A \wedge A) + A \wedge (dA + A \wedge A) \\ &\quad - (dA + A \wedge A) \wedge A \\ &= dA \wedge A - A \wedge dA \end{aligned}$$

$$\begin{aligned} \text{Ric}(X, Y) &= \text{Tr} \{ W \mapsto R(W, X)Y \} \\ &= \langle R(e_i, X)Y, e_j \rangle g^{ij} \end{aligned}$$

$$R(X, Y, JZ, JW) = R(JX, JY, Z, W) = R(JX, JY, JZ, JW)$$

$$R(JX, JY, Z) = R(X, Y)Z$$

$$\text{Ric}(JX, JY) = \sum_{i=1}^{2n} R(Je_i, X, JJe_i, Y)$$

Def. Ricci form ρ is defined by $\rho(X, Y) = \text{Ric}(JX, Y)$
($\forall X, Y \in TM$)

rank ρ is closed $2\rho(X, Y) \stackrel{\text{check}}{=} \text{Tr}(R(X, Y) \circ J)$

$$\begin{aligned} d\rho &= \frac{1}{2} d \text{Tr}(R(X, Y) \circ J) \\ &= \frac{1}{2} d \text{Tr}(\nabla(R(X, Y) \circ J)) = 0 \end{aligned}$$

Holomorphic Sectional Curvature

$G: \{X, JX\} \rightarrow J$ -invariant plane

$$H(G) = \frac{R(X, JX, X, JX)}{|X|^4}$$

Def. Given two J -invariant planes G, G'

$$H(G, G') = R(X, JX, Y, JY)$$

Remark (i) One can check it's well defined

$$(ii) H(G, G) = H(G)$$

$$(iii) R(X, JX, Y, JY) \stackrel{\text{Bianchi I}}{=} -R(JX, Y, X, JY) - R(Y, X, JX, JY)$$

acted by J

$$= R(X, JY, X, JY) + R(X, Y, JX, JY)$$

$(e_1, \dots, e_n, J e_1, \dots, J e_n)$ orthonormal basis for $T_p M$

$$\begin{aligned} \text{Ric}(X, Y) &= R(e_\alpha, X, e_\alpha, Y) + R(J e_\alpha, X, J e_\alpha, Y) \\ &\stackrel{\text{Bianchi I}}{=} R(e_\alpha, J e_\alpha, X, Y) \end{aligned}$$

$\text{Ric}(X, X)$ can be decided by the holomorphic

bisectional curvature.

Moreover, bisectional curvature positive $\Rightarrow$ Ricci Curvature positive

4.5 In Local Coordinates

(M, g, J) Kähler

$$\{z^1, \dots, z^n\} \quad \nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \frac{\partial}{\partial z^\alpha} = \nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = 0$$

$$\nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = \Gamma_{\alpha\beta}^\gamma \frac{\partial}{\partial z^\gamma} + \underbrace{\Gamma_{\alpha\beta}^{\bar{\gamma}} \frac{\partial}{\partial \bar{z}^{\bar{\gamma}}}}_{=0}$$

$$\nabla_{\frac{\partial}{\partial \bar{z}^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = \Gamma_{\alpha\beta}^{\bar{\gamma}} \frac{\partial}{\partial \bar{z}^{\bar{\gamma}}} \quad \begin{array}{l} \text{Thus } \nabla J = 0 \\ \text{Thus } \nabla \text{ maps } (1,0) \text{ to } (1,0) \end{array}$$

$$\frac{\partial g_{\alpha\beta}}{\partial z^\gamma} = \left\langle \nabla_{\frac{\partial}{\partial z^\gamma}} \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right\rangle \Rightarrow \Gamma_{\alpha\beta}^\gamma = g^{\gamma\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha}$$

$$\Gamma_{\alpha\bar{\beta}}^{\bar{\gamma}} = \overline{\Gamma_{\alpha\beta}^\gamma}$$

$$R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right)X = R\left(\frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right)X = 0$$

$$\begin{aligned} R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) \frac{\partial}{\partial z^\gamma} &= R_{\alpha\bar{\beta}\gamma}^\delta \frac{\partial}{\partial z^\delta} \\ &= \nabla_{\frac{\partial}{\partial z^\alpha}} \underbrace{\nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \frac{\partial}{\partial z^\gamma}}_0 - \nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\gamma} \\ &= -\nabla_{\frac{\partial}{\partial \bar{z}^\beta}} \left(\Gamma_{\alpha\gamma}^\delta \frac{\partial}{\partial z^\delta} \right) \\ &= -\frac{\partial}{\partial \bar{z}^\beta} \left(g^{\delta\bar{\tau}} \frac{\partial g_{\gamma\bar{\tau}}}{\partial z^\alpha} \right) \frac{\partial}{\partial z^\delta} \end{aligned}$$

$$R_{\alpha\bar{\beta}\bar{\gamma}\gamma} = R_{\alpha\bar{\beta}\gamma}^\delta g_{\delta\bar{\gamma}} = -g_{\delta\bar{\gamma}} \frac{\partial}{\partial \bar{z}^\beta} \left(g^{\delta\bar{\tau}} \frac{\partial g_{\gamma\bar{\tau}}}{\partial z^\alpha} \right)$$

$$R_{\alpha\bar{\beta}}^{\delta} = R_{\alpha\bar{\beta}\gamma\delta} g^{\delta\bar{\gamma}} = -\frac{\partial}{\partial \bar{z}^{\beta}} \left(g^{\delta\bar{\gamma}} \frac{\partial g_{\delta\bar{\gamma}}}{\partial z^{\alpha}} \right) \\ = -\frac{\partial}{\partial \bar{z}^{\beta}} \frac{\partial}{\partial z^{\alpha}} (\log \det(g_{S\bar{T}}))$$

$$R\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}\right) = R\left(\frac{\partial}{\partial z^{\alpha}}, e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, e_{\gamma}\right) + R\left(\frac{\partial}{\partial z^{\alpha}}, J e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, J e_{\gamma}\right) \\ = -\sqrt{-1} R\left(\frac{\partial}{\partial z^{\alpha}}, e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, J e_{\gamma}\right) + \sqrt{-1} R\left(\frac{\partial}{\partial z^{\alpha}}, J e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, e_{\gamma}\right) \\ = -\sqrt{-1} R\left(\frac{\partial}{\partial z^{\alpha}}, e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, J e_{\gamma}\right) - \sqrt{-1} R\left(e_{\gamma}, \frac{\partial}{\partial \bar{z}^{\beta}}, \frac{\partial}{\partial z^{\alpha}}, J e_{\gamma}\right) \\ \stackrel{\text{Blanchi I}}{=} \sqrt{-1} R\left(\frac{\partial}{\partial \bar{z}^{\beta}}, \frac{\partial}{\partial z^{\alpha}}, e_{\gamma}, J e_{\gamma}\right) \\ = -R\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}, \frac{1}{\sqrt{2}}(e_{\gamma} - \sqrt{-1} J e_{\gamma}), \frac{1}{\sqrt{2}}(e_{\gamma} + \sqrt{-1} J e_{\gamma})\right)$$

$T^{1,0}M$ unitary basis

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(e_1 - \sqrt{-1} J e_1) \\ \vdots \\ \frac{1}{\sqrt{2}}(e_n - \sqrt{-1} J e_n) \end{pmatrix} = (\psi_{st}) \begin{pmatrix} \frac{\partial}{\partial z^s} \\ \vdots \\ \frac{\partial}{\partial z^n} \end{pmatrix} \\ \Rightarrow (g_{\alpha\bar{\beta}}) = (\psi_{st})^{-1} (\overline{\psi_{st}})^{-1}$$

$$R_{\alpha\bar{\beta}} = Ric\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}\right) = -R\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}, \frac{\partial}{\partial z^{\gamma}}, \frac{\partial}{\partial \bar{z}^{\tau}}\right) \\ \quad \quad \quad \psi_{\gamma s} \overline{\psi_{\tau t}} \\ = R_{\alpha\bar{\beta}\tau s} g^{s\bar{\tau}} \\ = R_{\alpha\bar{\beta}s}^s \\ = - \frac{\partial^2 \log \det(g_{S\bar{T}})}{\partial z^{\alpha} \partial \bar{z}^{\beta}}$$

$$\rho(JX, JY) = \rho(X, Y) \Rightarrow \rho \text{ is } (1,1).$$

$$\begin{aligned} \rho &= \rho\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) dz^\alpha \wedge d\bar{z}^\beta = \text{Ric}\left(J\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) dz^\alpha \wedge d\bar{z}^\beta \\ &= -\sqrt{-1} \frac{\partial^2 \log \det(g_{s\bar{t}})}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta \\ &= -\sqrt{-1} \partial \bar{\partial} \log \det(g_{s\bar{t}}) \end{aligned}$$

Holomorphic Bisectional Curvature:

$$G_\alpha: J\text{-invariant plane} \quad \left\{ \frac{\partial}{\partial x^\alpha}, J \frac{\partial}{\partial x^\alpha} \right\}$$

$$\begin{aligned} H(G_\alpha, G_\beta) &= R\left(\frac{\partial}{\partial x^\alpha}, J \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, J \frac{\partial}{\partial x^\beta}\right) \\ &= -4 R\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial z^\beta}, \frac{\partial}{\partial \bar{z}^\beta}\right) \end{aligned}$$

Def. A Kähler manifold (M, J, g) is said to be of constant bisectional curvature if

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\lambda (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}})$$

4.6 Examples

1. $M = \mathbb{C}^n$ $\omega = \frac{\sqrt{-1}}{2} dZ^\alpha \wedge d\bar{Z}^\alpha$ flat

2. $M = \mathbb{CP}^n$ $U_0 = \{[1, z^1, \dots, z^n]\} \subset \mathbb{CP}^n$
 $\omega_g = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log(1 + |z|^2)$
 $= \frac{\sqrt{-1}}{2} \left(\frac{dZ^\alpha \wedge d\bar{Z}^\beta}{1 + |z|^2} \delta_{\alpha\beta} - \frac{\bar{z}^\alpha z^\beta dZ^\alpha \wedge d\bar{Z}^\beta}{(1 + |z|^2)^2} \right)$

$SU(n+1)$ acts transitively on $\mathbb{CP}^n$. $\tau \in SU(n+1)$
 $\tau^* g = g.$

$$\omega_g^h = \left(\frac{\sqrt{-1}}{2}\right)^n \frac{(dZ^\alpha \wedge d\bar{Z}^\alpha)^n}{(1 + |z|^2)^{n+1}}$$

$$R_{\alpha\bar{\beta}} = - \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log\left(\frac{1}{(1 + |z|^2)^{n+1}}\right) = (n+1) g_{\alpha\bar{\beta}} \quad \boxed{\text{Einstein}}$$

$$R_{\alpha\bar{\beta}\gamma\bar{\gamma}} = - (g_{\alpha\bar{\beta}} g_{\gamma\bar{\gamma}} + g_{\alpha\bar{\gamma}} g_{\gamma\bar{\beta}}) \quad \text{constant bi sectional curvature } +1$$

3. $M = B^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$ $\omega_g = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log(1 - |z|^2)$
 constant bi sectional curvature -1

Chapter 5 Laplace Operator

5.1 Hodge $\ast$ -operator

Def. The Hodge $\ast_g$ -operator

$\ast: \Lambda^r M \rightarrow \Lambda^{m-r} M$ is defined by

$$\omega \wedge \ast \tau = \langle \omega, \tau \rangle_g dV_g \quad \forall \omega, \tau \in \Lambda^r M$$

$$[\theta, \gamma] \in \Lambda^1 M \quad \langle \theta, \gamma \rangle_g = \theta\left(\frac{\partial}{\partial x^i}\right) \gamma\left(\frac{\partial}{\partial x^i}\right) g^{ii}$$

$$\langle \theta_1 \otimes \dots \otimes \theta_r, \gamma_1 \otimes \dots \otimes \gamma_r \rangle_g = \langle \theta_1, \gamma_1 \rangle_g \dots \langle \theta_r, \gamma_r \rangle_g$$

$$\theta_1 \wedge \theta_2 = \theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1$$

$$\theta_1 \wedge \theta_2(X, Y) = \det \begin{pmatrix} \theta_1(X) & \theta_1(Y) \\ \theta_2(X) & \theta_2(Y) \end{pmatrix}$$

$$\langle \omega, \tau \rangle_g = \sum_{i_1 < \dots < i_r} \omega^{i_1 \dots i_r} \tau_{i_1 \dots i_r}$$

$$= \frac{1}{r!} \sum \omega^{i_1 \dots i_r} \tau_{i_1 \dots i_r}$$

$$\tau = \tau_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$\omega^{i_1 \dots i_r} = g^{i_1 j_1} \dots g^{i_r j_r} \omega_{j_1 \dots j_r}$$

Remark (1) $\left\{ \frac{\partial}{\partial x^i} \right\}$ $\{dx^i\}$

$$\ast \tau = \sum_{j_{r+1} < \dots < j_m} \ast \tau_{j_{r+1} \dots j_m} dx^{j_{r+1}} \wedge \dots \wedge dx^{j_m}$$

$$\ast \tau_{j_{r+1} \dots j_m} = \sum_{i_1 < \dots < i_r} \gamma_{i_1 \dots i_r j_{r+1} \dots j_m} \tau^{i_1 \dots i_r}$$

(2) $\{e_i\}$ orthonormal basis of TM

$\{\theta^i\}$ dual basis

$$T = \sum_{i_1 < \dots < i_r} \tilde{T}_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \theta^{i_r}$$

$$\gamma = \theta^1 \wedge \dots \wedge \theta^m$$

$$\star T = \sum_{j_1 < \dots < j_m} \star \tilde{T}_{j_1 \dots j_m} \theta^{j_1+1} \wedge \dots \wedge \theta^{j_m}$$

$$\star \tilde{T}_{j_1 \dots j_m} = \sum_{i_1 < \dots < i_r} \delta_{i_1 \dots i_r j_1 \dots j_m}^{1 \dots m} \tilde{T}_{i_1 \dots i_r}$$

$$(3) \star 1 = dVg \quad \star dVg = 1$$

$$\star \star \alpha = (-1)^{r(m-r)} \alpha$$

$$\star(\alpha + \beta) = \star \alpha + \star \beta \quad \star(f\alpha) = f \star \alpha$$

$$\langle \star \omega, \star \tau \rangle_g = \langle \omega, \tau \rangle_g$$

Def $d: T(\wedge^r M) \rightarrow T(\wedge^{r+1} M)$ formal adjoint

$$d^*: T(\wedge^r M) \rightarrow T(\wedge^{r-1} M)$$

$$d^* \omega = (-1)^{m(r-1)+1} \star d \star \omega = -g^{ij} \left(\nabla_{\frac{\partial}{\partial x^j}} \omega \right) \left(\frac{\partial}{\partial x^i} \right)$$

$$\tau \in T(\wedge^{r-1} M) \quad \omega \in T(\wedge^r M)$$

$$\langle d\tau, \omega \rangle_g dVg = d\tau \wedge \star \omega = d(\tau \wedge \star \omega) - (-1)^{r-1} \tau \wedge d\star \omega$$

$$= d(\tau \wedge \star \omega) - (-1)^{r-1+(m-r+1)(r-1)} \tau \wedge \star \star d\star \omega$$

$$= d(\tau \wedge \star \omega) - (-1)^{m(r-1)} \langle \tau, \star d\star \omega \rangle dVg$$

$$= d(\tau \wedge \star \omega) + \langle \tau, d^* \omega \rangle dVg$$

$$\Rightarrow \int_M \langle d\tau, \omega \rangle dVg = \int_M \langle \tau, d^* \omega \rangle dVg$$

Eg. α 1-form X dual vector field

$$\alpha = \alpha_i dx^i \quad X = \alpha^i \frac{\partial}{\partial x^i} \quad \alpha^i = g^{ij} \alpha_j$$

$$\langle X, Y \rangle_g = \alpha(Y) \quad \forall Y \in TM$$

$$d^* \alpha = -g^{ij} \left(\nabla_{\frac{\partial}{\partial x^i}} \alpha \right) \left(\frac{\partial}{\partial x^j} \right) = -g^{ij} \alpha_{j,i}$$

$$= -\operatorname{div}(X).$$

$$\Delta_g : \Gamma(\wedge^r M) \rightarrow \Gamma(\wedge^r M)$$

$$\Delta_g = -dd^* - d^*d$$

Remark (1) $d \circ \Delta = \Delta \circ d$ $d^* \circ \Delta = \Delta \circ d^*$

$$*\Delta = \Delta*$$

$$(2) \int_M \langle \Delta \omega, \tau \rangle dV_g = \int_M \langle \omega, \Delta \tau \rangle dV_g$$

(3) M closed (i.e. compact, without boundary)

$$\Delta \omega = 0 \Leftrightarrow d\omega = 0, d^* \omega = 0$$

(4) f is a C^2 function.

$$\Delta f = -d^* df = g^{ij} \left(\nabla_{\frac{\partial}{\partial x^i}} df \right) \left(\frac{\partial}{\partial x^j} \right)$$

$$= g^{ij} \nabla df \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

$$= g^{ij} f_{,ji} = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right)$$

(5) trace Laplace operator

$$\text{tr}_g \nabla^2 : \Pi(\Lambda^r M) \rightarrow \Pi(\Lambda^r M)$$

$$\text{tr}_g \nabla^2 \alpha = g^{ij} \left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \alpha - \nabla_{\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}} \alpha \right)$$

Weitenböck Formula:

$$\Delta \alpha(X_1, \dots, X_r) = \text{tr}_g \nabla^2 \alpha(X_1, \dots, X_r) - \sum_{t=1}^r (-1)^t \sum_{i=1}^m (R(e_i, X_t) \alpha)(e_i, X_1, \dots, \hat{X}_t, \dots)$$

$$\alpha \in \Lambda^r M \Rightarrow \Delta \alpha(Y) = (\text{tr}_g \nabla^2 \alpha)(Y) - \text{Ric}(X, Y)$$

X is dual of α

α harmonic form $\Leftrightarrow \Delta \alpha = 0$

$$\begin{aligned} \Delta \underbrace{|\alpha|_g^2}_f &= g^{ij} \left(\nabla_{\frac{\partial}{\partial x^i}} df \right) \left(\frac{\partial}{\partial x^j} \right) \\ &= g^{ij} \left[\left(\frac{\partial}{\partial x^i} (df(\frac{\partial}{\partial x^j})) \right) - df \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right] \\ &= g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) |\alpha|_g^2 \\ &\stackrel{\text{compute}}{=} 2|\nabla \alpha|^2 + 2 \text{Ric}(\alpha^\#, \alpha^\#) \end{aligned}$$

$\text{Ric} \geq 0$. M closed. then $\alpha^\# = 0 \Leftrightarrow \alpha = 0$

5.2 Complex Laplace Operator on Hermitian Manifold

$$(M^{2n}, J, g) \quad \omega(X, Y) = g(JX, Y)$$

$$\theta \wedge * \tau = \langle \theta, \tau \rangle_g \frac{\omega^n}{n!}$$

$$* \omega^p = \frac{p!}{(n-p)!} \omega^{n-p}$$

$$*: \Lambda^{p,q}(M) \rightarrow \Lambda^{n-q, n-p}(M)$$

Def. The Lefschetz operator L_ω associate to ω

$$\text{is } L_\omega: \Lambda^p M \rightarrow \Lambda^{p+2} M$$

$$\theta \mapsto \theta \wedge \omega.$$

The contraction operator

$$\Lambda_\omega: \Lambda^p M \rightarrow \Lambda^{p-2} M \quad \text{adjoint of } L_\omega$$

$$\langle \Lambda_\omega \theta, \tau \rangle = \langle \theta, L_\omega \tau \rangle$$

$$\Lambda_\omega \theta = (-1)^p * L_\omega * \theta$$

Remark ① Λ_ω maps $\Lambda^{p,q} M$ to $\Lambda^{p-1, q-1} M$

② L_ω, Λ_ω are real operators

$$\overline{L_\omega(\theta)} = L_\omega(\bar{\theta}), \quad \overline{\Lambda_\omega(\theta)} = \Lambda_\omega \bar{\theta}$$

③ $\theta \in \Lambda^2(M)$. $\Lambda_\omega \theta \cdot 1 = \langle \theta, \omega \rangle_g$

$$\begin{aligned} \Lambda_\omega \theta \frac{\omega^n}{n!} &= \langle \theta, \omega \rangle_g \frac{\omega^n}{n!} \\ &= \theta \wedge * \omega = \theta \wedge \frac{\omega^{n-1}}{(n-1)!} \end{aligned}$$

$$\wedge \omega \omega = 0$$

$$\mathcal{L}_\omega^{(n-1)} \theta = \theta \wedge \omega^{n-1} = \frac{1}{n} \wedge_\omega \theta \omega^n = (n-1)! \wedge_\omega \theta \frac{\omega^n}{n!}$$

$$\Lambda_0^P(M) = \{ \theta \in \Lambda^P(M) \mid \wedge_\omega \theta = 0 \} \quad \text{primitive forms}$$

$$\Lambda_0^{P,Q}(M) = \Lambda_0^{P+Q}(M) \cap \Lambda^{P,Q}(M)$$

$$\begin{aligned} \theta \in \Lambda_0^{2P}(M) \quad g(\theta, \omega^P) = 0 &\Leftrightarrow g(\theta, \mathcal{L}_\omega^P) = 0 \\ &\Leftrightarrow g(\wedge_\omega^P \theta, 1) = 0 \Leftrightarrow \mathcal{L}_\omega^{n-P} \theta = 0 \end{aligned}$$

$$\begin{aligned} \theta \wedge * \omega^P &= g(\theta, \omega^P) \frac{\omega^n}{n!} \\ &= \frac{P!}{(n-P)!} \theta \wedge \omega^{n-P} = \frac{P!}{(n-P)!} \mathcal{L}_\omega^{n-P}(\theta) \end{aligned}$$

$$\Lambda^2(M) = \Lambda_0^2(M) \oplus \mathbb{C} \omega = \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M) \oplus \Lambda^{0,2}(M) \oplus \mathbb{C} \omega$$

$$\theta \in \Lambda^1(M) \quad * \theta = \frac{-1}{(n-1)!} \mathcal{L}_\omega^{n-1}(\mathcal{J}(\theta))$$

$$* \theta^{1,0} = -\frac{\sqrt{-1}}{(n-1)!} \theta^{1,0} \wedge \omega^{n-1} \quad * \theta^{0,1} = \frac{\sqrt{-1}}{(n-1)!} \theta^{0,1} \wedge \omega^{n-1}$$

$$\theta^{1,0} \wedge * \overline{\theta^{1,0}} = g(\theta^{1,0}, \overline{\theta^{1,0}}) \frac{\omega^n}{n!} = |\theta^{1,0}|_H^2 \frac{\omega^n}{n!}$$

"

$$\frac{\sqrt{-1}}{(n-1)!} \theta^{1,0} \wedge \overline{\theta^{1,0}} \wedge \omega^{n-1}$$

$$\Rightarrow \sqrt{-1} \wedge_\omega \theta^{1,0} \wedge \overline{\theta^{1,0}} = |\theta^{1,0}|_H^2$$

$$\sqrt{-1} \wedge_\omega \theta^{0,1} \wedge \overline{\theta^{0,1}} = -|\theta^{0,1}|_H^2$$

$$d = \partial + \bar{\partial}$$

$$\begin{array}{c} \partial^*: \wedge^{p,q} \rightarrow \wedge^{p+1,q} \\ \parallel \\ -* \bar{\partial} * \end{array}$$

$$\begin{array}{c} \bar{\partial}^*: \wedge^{p,q} \rightarrow \wedge^{p,q+1} \\ \parallel \\ -* \partial * \end{array}$$

$$(\bar{\partial}\theta, \tau) = (\theta, \partial^* \tau) \quad (\partial\theta, \tau) = (\theta, \bar{\partial}^* \tau)$$

$$\Delta^{\partial} = -\partial\bar{\partial}^* - \bar{\partial}^*\partial$$

$$\Delta^{\bar{\partial}} = -\bar{\partial}\partial^* - \partial^*\bar{\partial}$$

Lemma $\partial^* \theta = \sqrt{-1} \wedge_{\omega} \bar{\partial} \theta + \frac{\sqrt{-1}}{(n-1)!} * (\bar{\partial} (\omega_g^{n-1}) \wedge \theta) \quad \theta \in \wedge^{1,0}_n$
 $\bar{\partial}^* \theta = -\sqrt{-1} \wedge_{\omega} \partial \theta - \frac{\sqrt{-1}}{(n-1)!} * (\partial (\omega_g^{n-1}) \wedge \theta) \quad \theta \in \wedge^{0,1}_n$

Prop (Kähler Identities) (M, J, g) Kähler

$$(1) [\bar{\partial}, L_{\omega}] = [\partial, L_{\omega}] = 0 \quad [\bar{\partial}^*, L_{\omega}] = [\partial^*, L_{\omega}] = 0$$

$$(2) [\bar{\partial}^*, L_{\omega}] = \sqrt{-1} \partial \quad [\partial^*, L_{\omega}] = -\sqrt{-1} \bar{\partial}$$

$$[L_{\omega}, \bar{\partial}] = -\sqrt{-1} \partial^* \quad [L_{\omega}, \partial] = \sqrt{-1} \bar{\partial}^*$$

$$(3) \Delta^{\partial} = \Delta^{\bar{\partial}} = \frac{1}{2} \Delta \quad \Delta \text{ commutes with } \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L_{\omega}, \Lambda_{\omega}$$

5.3 Generalization to Bundle Valued Form

E complex bundle (M, J, g) ω

$\hookrightarrow$ Hermitian metric on E

extends to $H: \Lambda^p(E) \times \Lambda^q(E) \rightarrow \Lambda^{p+q}(E)$

$$H(\theta \otimes s, \eta \otimes t) = H(s, t) \theta \wedge \bar{\eta}$$

$$g(\theta \otimes s, \eta \otimes t) = g(\theta, \bar{\eta}) \cdot H(s, t)$$

$$H(a, *b) = g(a, b) dVg$$

$$g(\wedge \omega c, a) = g(c, \lrcorner \omega a)$$

$F \in \Lambda^p(\text{End } E)$ F^{*H} H -adjoint of F

$$\int F^2 dVg = \text{tr}(F \wedge * F^{*H})$$

$D_A = d + A$ H -unitary connection in E

$$D_A = \partial_A + \bar{\partial}_A$$

$$D_A^* = -*D_A* \quad \bar{\partial}_A^* = -*\partial_A*$$

$$\Delta_A = -D_A^* D_A - D_A D_A^*$$

$$\partial_A^* = -*\bar{\partial}_A*$$

In Kähler case $a \in A^{1,0}(E)$

$$\partial_A^* a = \sqrt{-1} \wedge \omega \bar{\partial}_A a$$

$$a \in A^{0,1}(E)$$

$$\bar{\partial}_A^* a = -\sqrt{-1} \wedge \omega \partial_A a$$

$$a \in A^0(E)$$

$$\Delta_A^{\bar{\partial}} = -\bar{\partial}_A^* \bar{\partial}_A = \sqrt{-1} \wedge \omega \partial_A \bar{\partial}_A$$

$$\Delta_A^{\partial} = -\sqrt{-1} \wedge \omega \bar{\partial}_A \partial_A$$

$$F_A = \bar{\partial}_A^2 + \partial_A^2 + \partial_A \bar{\partial}_A + \bar{\partial}_A \partial_A$$

$$\Rightarrow \Delta_A^{\bar{\partial}} - \Delta_A^{\partial} = \underbrace{\sqrt{-1} \Lambda_{\omega} F_A}_{\text{mean curvature}}$$

$$\Delta_A = \Delta_A^{\bar{\partial}} + \Delta_A^{\partial} = \sqrt{-1} \Lambda_{\omega} (\partial_A \bar{\partial}_A - \bar{\partial}_A \partial_A)$$

$$= 2\Delta_A^{\partial} + \sqrt{-1} \Lambda_{\omega} F_A$$

$$= 2\Delta_A^{\bar{\partial}} - \sqrt{-1} \Lambda_{\omega} F_A$$

Chapter 6 Hodge Theory

6.1 De Rham Cohomology Group

(M, g) compact oriented Riemannian manifold

$$Z^r(M, \mathbb{C}) \quad B^r(M, \mathbb{C}) \quad H_{DR}^r(M, \mathbb{C}) = \frac{Z^r(M, \mathbb{C})}{B^r(M, \mathbb{C})}$$

$$\text{Thm. (De Rham)} \quad H^k(M, \mathbb{C}) \cong H_{DR}^k(M, \mathbb{C})$$

$$\text{Def. } \mathcal{H}^r(M, g) = \{\eta \in A^r(M) : \Delta \eta = 0\}$$

Thm (Hodge Decomposition) $\forall 0 \leq r \leq n$

$\dim H^r(M, g) < \infty$ and we have the orthonormal direct sum

$$\begin{aligned} A^r(M) &= \Delta(A^r(M)) \oplus \mathcal{H}^r \\ &= dd^*(A^r(M)) \oplus d^*d(A^r(M)) \oplus \mathcal{H}^r \\ &= d(A^{r-1}(M)) \oplus d^*(A^{r+1}(M)) \oplus \mathcal{H}^r \end{aligned}$$

Consequently, $\Delta w = \alpha$ has solution $\Leftrightarrow \alpha$ is orthogonal to $\mathcal{H}^r$

Pf global inner product $(\eta_1, \eta_2) = \int_M \eta_1 \wedge * \eta_2$

$$\Delta \text{ self-adjoint} \Rightarrow \Delta(A^r(M)) \perp \mathcal{H}^r$$

Functional Analysis's viewpoint:

A solution $\Delta w = \eta \iff$ linear functional on $A'(M)$

$$l(\beta) = (w, \beta)$$

$$\text{then } l(\Delta \theta) = (w, \Delta \theta) = (\eta, \theta)$$

A weak solution is a bounded linear functional

$$l: A'(M) \rightarrow \mathbb{R} \quad \text{s.t. } l(\Delta \theta) = (\eta, \theta)$$

Thm (Regularity Thm) $\eta \in A'(M)$ $\hookrightarrow$ a weak solution of $\Delta w = \eta$. Then $\exists w \in A'(M)$ s.t. $l(\beta) = (w, \beta)$ i.e. $\Delta w = \eta$

Thm (Compactness Thm) $\{\eta_n\}$ a sequence in $A'(M)$

$$\text{s.t. } \|\eta_n\| \leq C, \|\Delta \eta_n\| \leq C.$$

Then $\exists$ subsequence of $\{\eta_n\}$ that is a Cauchy sequence.

(1) $H^r(M, g)$ is finite dimensional

If not, choose orthonormal sequence $\{\eta_n\}$

$$\Gamma_2 = \lim_{n \rightarrow \infty} \|\eta_{n+1} - \eta_n\| = 0 \quad (\text{by compactness Thm})$$

Contradiction!

(2) $w_1, \dots, w_k$ orthonormal basis of $H^r(M)$

$$\omega = \omega^\perp + \sum_{i=1}^K \langle \omega, \omega_i \rangle \omega_i \quad \omega^\perp \in (\mathcal{H}^r)^\perp$$

$$A^r(M) = (\mathcal{H}^r)^\perp \oplus \mathcal{H}^r$$

$$(3) \exists c > 0 \text{ s.t. } \|\eta\| \leq c \|\Delta \eta\| \quad (\forall \eta \in (\mathcal{H}^r)^\perp)$$

$$\text{If not } \exists \eta_i \in (\mathcal{H}^r)^\perp \text{ s.t. } \|\eta_i\| = 1 \quad \|\Delta \eta_i\| \rightarrow 0$$

Then $\exists$ subsequence $\{\eta_i\}$ is Cauchy

Then (η_j, ψ) is Cauchy $(\psi \in A^r(M))$

$$L(\psi) = \lim_{j \rightarrow \infty} (\eta_j, \psi) \text{ is bounded } \|L(\psi)\| \leq \|\psi\|$$

$$L(\Delta \psi) = \lim_{j \rightarrow \infty} (\Delta \eta_j, \psi) = 0 \Rightarrow L \text{ is a weak solution of } \Delta \omega = 0$$

$$\text{By regularity then } \exists \omega \in A^r(M) \text{ s.t. } L(\psi) = (\omega, \psi) \quad \Delta \omega = 0$$

$$\eta_j \xrightarrow{\text{Cauchy}} \gamma \in L^2$$

$$\text{Then } \forall \psi \in A^r(M) \quad (\gamma - \omega, \psi) = 0 \Rightarrow \gamma = \omega \quad \begin{matrix} \uparrow \\ \omega \in (\mathcal{H}^r(M))^\perp \end{matrix} \quad \text{Contradiction!}$$

$$(4) \Delta(A^r(M)) = (\mathcal{H}^r)^\perp$$

$$(i) \Delta(A^r(M)) \subset (\mathcal{H}^r)^\perp \text{ trivial}$$

$$(ii) \text{ Given } \eta \in (\mathcal{H}^r)^\perp$$

$$\text{Def } L: \Delta(A^r(M)) \rightarrow \mathbb{R}$$

$$L(\Delta \psi) = (\eta, \psi)$$

$$(\eta \in (\mathcal{H}^r)^\perp \Rightarrow \text{well defined})$$

Bounded:
$$\begin{aligned} \|(\phi\psi)\| &= \|L(\Delta(\psi^\perp + \pi(\psi)))\| \\ &= \|L(\phi\psi^\perp)\| = |(\eta, \psi^\perp)| \\ &\leq \|\eta\| \|\psi^\perp\| \leq C \|\eta\| \|\phi\psi\| \end{aligned}$$

Hahn-Banach Thm $\Rightarrow \exists$ continuous extension of l to $A^1(M)$

$\Rightarrow l$ is a weak solution $d\omega = \eta$.

Then by regularity Thm $\exists \omega \in \mathcal{H}^1(M)$ $d\omega = \eta$

#

For closed form ω , i.e. $d\omega = 0$

$$\omega = d\omega_1 + \omega^{1-1} + d^*\omega_2$$

$$(d^*\omega_2, d^*\omega_2) = (d^*\omega, \omega) = (\omega, d\omega) = 0$$

$$\text{Thus } d^*\omega_2 = 0 \Rightarrow \omega = d\omega_1 + \omega^{1-1}$$

$$\text{If } d^*\omega = 0 \Rightarrow \omega = d^*\omega_2 + \omega^{1-1}$$

Prop (Hodge Isomorphism)
$$\begin{aligned} \psi: H^r &\rightarrow H_{DR}^r(M, \mathbb{C}) \\ \omega &\mapsto [\omega] \end{aligned}$$

is an isomorphism.

Cor $\dim H_{DR}^{lc}(M, \mathbb{C}) < \infty$. (M compact orientable)

$$* : A^r(M) \rightarrow A^{m-r}(M) \quad *^2 = (-1)^{m(m-r)}$$

$$*\Delta = \Delta* \quad \omega \text{ is harmonic} \Leftrightarrow *\omega \text{ is harmonic}$$

Prop(Poincaré duality) $\mathcal{H}^r(M, g) \cong \mathcal{H}^{m-r}(M, g)$

$$b_r(M) = \dim H_{DR}^r(M, \mathbb{C}) = b_{m-r}(M)$$

6.2 Dolbeault Theory

(M, J, g) compact Hermitian manifold

$$\bar{\partial} : A^{p,q}(M) \rightarrow A^{p,q+1}(M)$$

Dolbeault cohomology group $H_1^{p,q}(M, J) = \frac{Z^{p,q}(M, J)}{B^{p,q}(M, J)}$

It depends on the complex structure J .

$$\mathcal{H}^{p,q}(M) = \{ \omega \in A^{p,q}(M) \mid \Delta \bar{\partial} \omega = 0 \}$$

$$\omega \text{ is } \bar{\partial}\text{-harmonic} \Leftrightarrow \bar{\partial} \omega = \bar{\partial}^* \omega = 0$$

Thm (Dolbeault Decomposition)

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \bar{\partial}^* A^{p,q+1}(M) \oplus \bar{\partial} A^{p,q-1}(M)$$

$$\bar{\partial} \omega = 0 \Rightarrow \omega = \bar{\partial} \omega_1 + \omega^H$$

Cor. ① A (p,q) form is holomorphic $\Leftrightarrow \bar{\partial}$ harmonic

② $\psi: \mathcal{H}^{p,q}(M) \rightarrow H^{p,q}(M, \mathbb{C})$ is isomorphism
 $\omega \mapsto [\omega]$

Def (Hodge number) $h^{p,q} = \dim_{\mathbb{C}} \mathcal{H}^{p,q}(M)$

Prop (Serre Duality) $\mathcal{H}^{p,q}(M) \cong \mathcal{H}^{n-p, n-q}(M)$

Pf $\bar{*}: A^{p,q}(M) \rightarrow A^{n-p, n-q}(M)$
 $\bar{*}\omega = * \bar{\omega}$

$$\begin{aligned} \bar{*} \Delta^{\bar{\partial}}(\omega) &= * (\overline{\partial \partial^* + \partial^* \bar{\partial}}) \omega = * (\partial \partial^* + \partial^* \bar{\partial}) \bar{\omega} \\ &= - * (\partial^* \bar{\partial} * + * \bar{\partial} \partial) \bar{\omega} \\ &= \bar{\partial}^* \bar{\partial}(\bar{*}\omega) - *^2 \bar{\partial} \partial \bar{\omega} = \Delta^{\bar{\partial}}(\bar{*}\omega) \end{aligned}$$

Thus $\bar{*}: \mathcal{H}^{p,q}(M) \rightarrow \mathcal{H}^{n-p, n-q}(M)$ is isomorphism $\neq$

In Kähler case $\Delta = 2\Delta^{\bar{\partial}}$

$$\mathcal{H}^{p,q}(M) \subset \mathcal{H}^{p+q}(M)$$

$$H^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$$

$$\Delta(\bar{\omega}) = \overline{\Delta \omega} \quad \therefore \mathcal{H}^{p,q}(M) \rightarrow \mathcal{H}^{q,p}(M)$$

Prop. $b_k = \sum_{p+q=k} h^{p,q}$ $h^{p,q} = h^{q,p}$ $h^{p,p} \geq 1$ ($1 \leq p \leq n$)
 for Kähler manifold.
 $\downarrow$
 Kähler form $\omega \in \mathcal{H}^{1,1}$
 ($\nabla J = 0 \Rightarrow \nabla \omega = 0$)

k is odd $\Rightarrow b_k$ is even

k is even $\Rightarrow b_k \geq 1$.

e.g. Hopf manifold $S^1 \times S^{2n+1}$ is non-Kähler

Lemma (Global $\partial\bar{\partial}$) (M, J, g) Kähler. $\phi, \phi_2 \in A''(M, \mathbb{C})$
 ϕ_1 is cohomologous to ϕ_2 ($\phi_1 = \phi_2 + d\psi$, ψ real) $\wedge A'(M, \mathbb{R})$

Then $\exists f \in C^\infty(M, \mathbb{R})$ st. $\phi_1 - \phi_2 = \bar{\partial} \partial f$

Pf $\psi = \psi^{1,0} + \psi^{0,1}$ $\partial \psi^{1,0} = \bar{\partial} \psi^{0,1} = 0$

Consider equation $-\frac{1}{2} \Delta u = \bar{\partial}^* \bar{\partial} u = \bar{\partial}^* \psi^{0,1}$

$\bar{\partial}^*(\psi^{0,1} - \bar{\partial} u) = 0$ $\left[\int_M \bar{\partial}^* \psi^{0,1} \omega^n = 0 \right.$
 $\Rightarrow$ has solution]

$\bar{\partial}(\psi^{0,1} - \bar{\partial} u) = 0$

One can prove that $\partial(\psi^{0,1} - \bar{\partial} u) = 0$

Then $\phi_1 - \phi_2 = d\psi = \partial \psi^{0,1} + \bar{\partial} \psi^{1,0} = \partial \bar{\partial} u + \bar{\partial} \partial u$
 $= 2\bar{\partial} \partial \operatorname{Im} u = \bar{\partial} \partial \bar{f} \neq$

Chapter 7 Chern Classes

[Chern-weil Theory]

V complex vector space

k -multilinear symmetric map $\tilde{P}: V \times V \cdots V \rightarrow \mathbb{C}$ $S^k(V)^*$

$B \in V, \tilde{P} \in S^k(V)^*$ $P(B) = \tilde{P}(B, B, \dots, B)$

homogeneous polynomial of degree $k: S^k(V)$

$$P \in S^k(V) \quad \tilde{P}(B_1, \dots, B_k) = \frac{(-1)^k}{k!} \sum_{j=1}^k \sum_{i_1 < \dots < i_j} (-1)^j P(B_{i_1} + \dots + B_{i_j}) \in S^k(V)^*$$

Consider $V = \mathfrak{gl}(r, \mathbb{C})$

Def. $\tilde{P} \in S^k(\mathfrak{gl}(r, \mathbb{C}))^*$ is called invariant if

$$\forall g \in GL(r, \mathbb{C}) \quad A_i \in \mathfrak{gl}(r, \mathbb{C})$$

$$\tilde{P}(gA_1g^{-1}, \dots, gA_kg^{-1}) = \tilde{P}(A_1, \dots, A_k)$$

$$P(gAg^{-1}) = P(A) \longrightarrow I_k(\mathfrak{gl}(r, \mathbb{C}))$$

Eg. E complex vector bundle. $\text{rank}(E) = r$.

extend $\tilde{P} \in I_k(\mathfrak{gl}(r, \mathbb{C}))$ to $A^k(M, \text{End}(E))$

Prop. $\tilde{\varphi} \in \tilde{I}_k(\mathfrak{gl}(r, \mathbb{C}))$ then for any vector bundle E of rank r and any partition $m = i_1 + \dots + i_k$

$\exists$ an induced k -linear map

$$\tilde{\varphi} = (\wedge_{\mathcal{M}}^{i_1} \otimes \text{End}(E)) \times \dots \times (\wedge_{\mathcal{M}}^{i_k} \otimes \text{End}(E)) \rightarrow \wedge_{\mathbb{C}}^m \mathcal{M}$$

$$\tilde{\varphi}(\alpha_1 \otimes \theta_1, \dots, \alpha_k \otimes \theta_k) = (\alpha_1 \wedge \dots \wedge \alpha_k) \tilde{\varphi}(\theta_1, \dots, \theta_k)$$

Lemma $\forall \eta_j \in A^{i_j}(\mathcal{M}, \text{End}(E))$

$$d\tilde{\varphi}(\eta_1, \dots, \eta_k) = \sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} \tilde{\varphi}(\eta_1, \dots, D\eta_j, \dots, \eta_k)$$

Pf. $D = d + A \quad D\eta = d\eta + [A, \eta]$

$$d\tilde{\varphi}(\eta_1, \dots, \eta_k) = \sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} \tilde{\varphi}(\eta_1, \dots, D\eta_j - [A, \eta_j], \dots, \eta_k)$$

Let $\eta_j = \alpha_j \otimes \theta_j \quad A = \beta \otimes B$

$$\sum_{j=1}^k (-1)^{\sum_{l=1}^{j-1} i_l} \tilde{\varphi}(\eta_1, \dots, [A, \eta_j], \dots, \eta_k) \xrightarrow{\beta \wedge \alpha_j \otimes [B, \theta_j]} \frac{d}{dt} (e^{tB} \theta_j e^{-tB}) \Big|_{t=0}$$

$$= \sum_{j=1}^k \beta \wedge \alpha_1 \wedge \dots \wedge \alpha_{j-1} \wedge \alpha_{j+1} \wedge \dots \wedge \alpha_k \tilde{\varphi}(\theta_1, \dots, [B, \theta_j], \dots, \theta_k)$$

$$= \beta \wedge \dots \wedge \alpha_k \frac{d}{dt} \Big|_{t=0} \tilde{\varphi}(e^{tB} \theta_1 e^{-tB}, \dots, e^{tB} \theta_k e^{-tB})$$

invariant
0

#

$\varphi(F_D) = \tilde{\varphi}(F_D \cdots F_D) \in A_{\mathbb{C}}^{2k}(M)$ is closed

$$d\varphi(F_D) = \sum_j \tilde{\varphi}(F_D, \dots, D F_D, \dots, F_D) = 0$$

Lemma $\varphi \in I_k(\mathfrak{gl}(r, \mathbb{C}))$ D, D' are two connections on E , then $[\varphi(F_D)] = [\varphi(F_{D'})] \in H_{DR}^{2k}(M, \mathbb{C})$

Pf. $D' = D + A$ $A \in A^1(\text{End } E)$

$$\begin{aligned} F_{D'} &= (D+A)^2 = D^2 + D \circ A + A \circ D + A \wedge A \\ &= F_D + D(A) + A \wedge A \end{aligned}$$

$$\varphi(F_{D+tA}) = \varphi(F_D) + kt \tilde{\varphi}(F_D, \dots, F_D, D(A)) + O(t^2)$$

$$\begin{aligned} \frac{d}{dt} \varphi(F_{D+tA})|_{t=t_0} &= \frac{d}{ds} \varphi(F_{D+t_0A+sA})|_{s=0} = k \tilde{\varphi}(F_{D_{t_0}}, \dots, F_{D_{t_0}}, D_{t_0}(A)) \\ &= kd \tilde{\varphi}(F_{D_{t_0}}, \dots, F_{D_{t_0}}, A) \end{aligned}$$

$$\varphi(F_{D'}) - \varphi(F_D) = \int_0^1 \frac{d}{dt} \varphi(F_{D+tA}) dt$$

#

Chern-weil homomorphism

$$I_k(\mathfrak{gl}(n, \mathbb{C})) \rightarrow H_{DR}^{2k}(M, \mathbb{C})$$

$$\det(I + tB) = 1 + \sum_{k=1}^r t^k \underline{\Phi}_k(B)$$

$$\varphi \mapsto [\varphi(F_D)]$$

$$\left(\frac{F-1}{2\pi} \right)^k \frac{1}{k!} = \sum_{j_1, \dots, j_k}^{i_1, \dots, i_k} F_{j_1}^{i_1} \wedge \dots \wedge F_{j_k}^{i_k}$$

$$\text{chern form: } C_k(E, D) = \underline{\Phi}_k\left(\frac{F-1}{2\pi} F_D\right) \in A_{\mathbb{C}}^{2k}(M)$$

$$\text{chern class: } C_k(E) = [C_k(E, D)] \in H_{DR}^{2k}(M, \mathbb{C})$$

chern character $\text{tr}(e^{tB}) = P_0(B) + tP_1(B) + \dots$

$$\begin{aligned} k\text{-th chern: } Ch_k(E) &= P_k\left(\frac{F_1}{2\pi} F_D\right) \in A_{\mathbb{C}}^{2k}(M) \\ &= \text{tr} \left(\frac{F_1}{2\pi} F_D \wedge \dots \wedge \frac{F_1}{2\pi} F_D \right) \end{aligned}$$

Rmk. ① All the characteristic classes are real.

complex vector bundle. H . Hermitian connection D
s.t. $DH=0$

$$D=d+A \quad A^{*H} = -A$$

$$(FA)^* = (dA + A \wedge A)^{*H} = \bar{H}^{-1}(dA)\bar{H} - \bar{H}^{-1}\bar{A}^T \wedge \bar{A}^T \bar{H}$$

$$dH = \bar{A}^T H + H \bar{A} \Rightarrow dH \cdot H^{-1} = A^T + H \bar{A} H^{-1}$$

$$H^{-1} dH = H^{-1} \bar{A}^T H + A$$

$$\Rightarrow (FA)^* = -\bar{F}A \quad (J^{-1}FA)^{*H} = J^{-1}\bar{F}A$$

By invariance, choose orthonormal basis of E

$$\bar{F}_A^T = -\bar{F}A$$

$$\begin{aligned} \text{then } c(E, D) &= \det\left(I + \frac{F_1}{2\pi} F_D\right) = \overline{\det\left(I + \frac{F_1}{2\pi} F_D\right)} \\ &= \overline{c(E, D)}. \end{aligned}$$

$$\text{thus } c_k(E) \in H_{DR}^{2k}(M, \mathbb{R})$$

② If E is a holomorphic bundle, one can choose

$$\begin{aligned} \text{chern connection } D. \quad F_D &\in A^{1,1}(M, \text{End}(E)) \quad c_k(E, D) \in A^{k,k}(M) \\ &\Rightarrow c_k(E) \in H^{2k}(M, \mathbb{R}) \cap H^{k,k}(M) \end{aligned}$$

Prop E_1, E_2 are complex vector bundles, then

- (a) $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$
 (b) $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$

$$E_1 \otimes E_2: D^{\otimes}(S_1 \otimes S_2) = D_1 S_1 \otimes S_2 + S_1 \otimes D_2 S_2$$

$$D^{\otimes} = D_1 \otimes Id_{E_2} + Id_{E_1} \otimes D_2$$

$$\Rightarrow F_{D^{\otimes}} = F_{D_1} \otimes Id_{E_2} + Id_{E_1} \otimes F_{D_2}$$

$$E^*: D(e_1, \dots, e_r) = (e_1, \dots, e_r) A, \quad D^*(\theta^1, \dots, \theta^r) A^*$$

$$D^* \theta^i(e_j) = -\theta^i(D e_j) = -\theta^i(A_j^k e_k) = -A_j^i$$

$$A_j^{*k} \theta^k(e_j) \Rightarrow A_j^{*j} = -A_j^j \quad A^* = -A^T$$

$$F_D^* = dA^* + A^* \wedge A^* = -dA^T + A^T \wedge A^T = -F_D^T$$

$$\Rightarrow c_j(E^*) = (-1)^j c_j(E)$$

Prop $f: M \rightarrow N$ (E, D) on N f^*E is pull back

bundle. Then locally the connection $f^*D = d + f^*(A)$

$$F_{f^*D} = f^*F_D$$

$$c(f^*E) = f^*c(E)$$

One can check:

$$(i) \text{ch}(E) = \text{rank}(E) + \text{ch}_1(E) + \text{ch}_2(E) + \text{ch}_3(E) + \dots$$

$$= \text{rank}(E) + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3(E) - 3c_1(E)c_2(E)}{6} + \dots$$

$$(ii) c_i(E \otimes L) = \sum_{j=0}^i \binom{\text{rank}(E)-j}{i-j} c_j(E) c_1(L)^{i-j} \quad L \text{ is a line bundle}$$

$$\text{ch}(E_1 \otimes E_2) = \text{ch}(E_1) \text{ch}(E_2)$$

Def. The Chern class of a complex manifold (M, J, g)

$$\text{is } c_k(M) \triangleq c_k(TM) \in H^{2k}(M, \mathbb{R})$$

Rmk. (M, J, g) Kähler. ∇ Levi-Civita

$$c_1(M) = \left[\frac{F_1}{2\pi} \text{tr} F_0 \right]$$

$$\{z^\alpha\} \quad g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right)$$

$$\nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial \bar{z}^\beta} = \Gamma_{\beta\alpha}^\gamma \frac{\partial}{\partial z^\gamma}$$

$$\Gamma_{\beta\alpha}^\gamma = g^{\gamma\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha}$$

$$\sqrt{-1} \text{tr} F_0 = F_1 \bar{\partial} \partial \log \det(g_{\alpha\bar{\beta}}) = \rho \quad (\text{Ricci form})$$

$$\Rightarrow \rho_\omega \in 2\pi c_1(M)$$

Chapter 8 Sheaf Theory

8.1 Presheaves and Sheaves

Def. Presheaf, sheaf, module-sheaf, analytic sheaf

Thm. $(M, \mathcal{S})$ be a connected $\mathcal{S}$ -manifold, then

(isomorphism class of) $\mathcal{S}$ -bundle of M

$\longleftrightarrow$ (isomorphism class of) locally free sheaves of $\mathcal{S}$ -modules

Def. Coherent sheaf

8.2 Čech Cohomology

8.3 Resolution of Sheaf

8.4 Cohomology Theory

Chapter 9 Stability of Holomorphic Vector Bundles

9.1 Determinant Bundle

E holomorphic bundle on Kähler manifold (M, ω)

$$\omega\text{-degree } \deg_{\omega}(E) = \int_M c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = c_1(E) [\omega]^{n-1}$$

$$\omega\text{-slope } \mu_{\omega}(E) = \frac{\deg_{\omega}(E)}{\text{rank}(E)}$$

$$\det(E) = \wedge^r E \quad \text{line bundle}$$

$$\text{Prop } c_1(E) = c_1(\det(E))$$

Pf. D a connection on E it induces $\hat{D}$ on $\det(E)$

$$\hat{D}(e_1 \wedge \dots \wedge e_r) = (De_1) \wedge \dots \wedge e_r + \dots + e_1 \wedge \dots \wedge (De_r)$$

$$F_{\hat{D}}(e_1 \wedge \dots \wedge e_r) = (F_D e_1) \wedge \dots \wedge e_r + \dots + e_1 \wedge \dots \wedge (F_D e_r)$$

$$= \text{tr } F_D (e_1 \wedge \dots \wedge e_r) \Rightarrow F_{\hat{D}} = \text{tr } F_D \quad \neq$$

For Riemann surface M , E is stable if

for every proper subbundle E' of E $\text{rank } E' < \text{rank } E$

$$\text{we have } \mu(E') < \mu(E)$$

$\mathcal{F}$ coherent sheaf

$$\mathcal{F}^* = \text{Hom}(\mathcal{F}, \mathcal{O})$$

$$\mathcal{G}: \mathcal{F} \rightarrow \mathcal{F}^{**}$$

natural homomorphism

$\text{Ker } \mathcal{G} \hookrightarrow \mathcal{F}$: coherent subsheaf. $\mathcal{T}(\mathcal{F})$: torsion subsheaf

$\mathcal{T}(\mathcal{F}) = 0$: torsion free

$\mathcal{G}$ bijective: reflexive

Prop (1) locally free $\rightarrow$ reflexive $\rightarrow$ torsion free

(2) coherent subsheaf of a torsion-free sheaf is also torsion free

(3) For any coherent sheaf $\mathcal{F}$, $\mathcal{F}^*$ is reflexive

(4) $\mathcal{F}$ torsion free, then $\mathcal{F}$ is locally free on $M \setminus \Sigma$

Σ is an analytic set of codim at least 2

(5) $\mathcal{F}$ reflexive,

$\mathcal{F}^* \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{F}^{**} \xrightarrow{\mathcal{G}} \mathcal{F}$

Def. $\mathcal{F}$ is normal, if every open set $U \subset M$ and

any analytic subset $\Sigma \subset U$ of codim at least 2,

$\Gamma(U, \mathcal{F}) \rightarrow \Gamma(U - \Sigma, \mathcal{F})$ is an isomorphism

Prop (1) reflexive $\Leftrightarrow$ torsion free + normal

(2) $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ exact

$\mathcal{F}$ reflexive, $\mathcal{F}''$ torsion free, then $\mathcal{F}'$ normal torsion free

Def. $0 \rightarrow \mathcal{E}_n \rightarrow \dots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F}_n \rightarrow 0$ is a locally free coherent resolution. $\mathcal{E}_i$ be the bundle corresponding to $\mathcal{E}_i$, then define $\det \mathcal{F}_n = \bigotimes_{i=0}^n (\det \mathcal{E}_i)^{(-1)^i}$

Rank $\mathcal{F}$ torsion-free coherent, then $\det \mathcal{F} = (\wedge^r \mathcal{F})^{**}$

$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ ω -degree $\deg_\omega(\mathcal{F}) = \deg_\omega(\det \mathcal{F})$

ω -slope $\mu_\omega(\mathcal{F}) = \frac{\deg_\omega(\mathcal{F})}{\text{rank}(\mathcal{F})}$

Def. $(E, \bar{\partial}_E)$ holomorphic bundle on Kähler manifold

(M, J, g) $(E, \bar{\partial}_E)$ is ω -stable if for

every proper coherent subsheaf $\mathcal{F}$ with torsion free quotient

$$\mu_\omega(\mathcal{F}) < \mu_\omega(E)$$

9.2 Degree of Coherent Sheaves

$\mathcal{F} \subset E \rightarrow \text{bundle}$
 $\downarrow$
 subsheaf
 with torsion
 free quotient

$$0 \rightarrow \mathcal{F} \rightarrow E \rightarrow \mathcal{Q} \rightarrow 0$$

$\mathcal{F}, \mathcal{Q}$ locally free in $M \setminus \Sigma$

Σ closed analytic subset $\text{codim} \geq 2$

$$0 \rightarrow \mathcal{F}|_{M \setminus \Sigma} \rightarrow E|_{M \setminus \Sigma} \rightarrow \mathcal{Q}|_{M \setminus \Sigma} \rightarrow 0$$

H : Hermitian metric on E

$$\pi_H: E|_{M \setminus \Sigma} \rightarrow \mathcal{F}$$

$$\pi \in \mathcal{P}(\text{End}(E)) \quad \pi^2 = \pi = \pi^{*H}$$

$$(\text{Id} - \pi) \bar{\partial}_E \pi = 0$$

$$\pi_{H-1} \in L^2(M, \text{End}(E))$$

H_F : induced metric on $\mathcal{F}|_{M \setminus \Sigma}$.

$$\deg_\omega(\mathcal{F}) = \int_M c_1(\det \mathcal{F}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\stackrel{?}{=} \int_{M \setminus \Sigma} \frac{F_H}{2\pi} \text{tr}(\bar{F}_H \mathcal{F}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\stackrel{\text{Gauss-Codazzi} (*)}{=} \int_{M \setminus \Sigma} \text{tr}(\pi_{H-1} \frac{F_H}{2\pi} \bar{F}_{H-1}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$= \frac{1}{2\pi} \int_{M \setminus \Sigma} |\bar{\partial}_E \pi_{H-1}|^2 \frac{\omega^n}{n!}$$

$$= \int_{M \setminus \Sigma} \frac{1}{2\pi} \left[\text{tr}(\pi_{H-1} F_H \wedge_\omega \bar{F}_{H-1}) - |\bar{\partial}_E \pi_{H-1}|^2 \right] \frac{\omega^n}{n!}$$

Gauss-Codazzi equation:

$$\mathcal{S} = \mathcal{F}|_{M \setminus \Sigma}$$

$$0 \rightarrow \mathcal{S} \xrightarrow{i} E \xrightarrow{P} Q \rightarrow 0 \quad \text{on } M \setminus \Sigma$$

$x \mapsto [x]$

H-metric

H_S induce metric

$$0 \leftarrow S \xleftarrow{\pi_H} E \xleftarrow{P^{*H}} Q \leftarrow 0$$

$x^\perp \leftarrow [x]$

$$H_Q: H_Q([x], [y]) = H(P^*(x), P^*(y)) = H(x^\perp, y^\perp)$$

$\bar{\partial}_S$: induce holomorphic structure

$$\bar{\partial}_Q: Y \in \Gamma(E) \quad \bar{\partial}_Q[Y] = [\bar{\partial}_E Y]$$

D: H-connection on E

induce connection $D^\perp$ on subbundle S

$$D^\perp X = (DX)^\perp$$

$B(X) = (DX)^\perp$ is a tensor [check $B(fX) = fB(X)$]

$B \in \Gamma(\wedge^1(M) \otimes S^{\perp*} \otimes S)$: second fundamental form

For $\xi \in \Gamma(S^\perp)$ $D^\perp \xi = (D\xi)^\perp$ induce connection on $S^\perp$

$$A(\xi) = (D\xi)^\perp \quad A(f\xi) = Af(\xi)$$

$$A \in \Gamma(\wedge^1(M) \otimes (S^\perp)^* \otimes S)$$

$$\begin{aligned}
 0 = d(H(X, \xi)) &= H(DX, \xi) + H(X, D\xi) \\
 &= H((DX)^\perp, \xi) + H(X, (D\xi)^T) \\
 &= H(B(X), \xi) + H(X, A(\xi))
 \end{aligned}$$

$$A = -B^* H$$

$$\begin{aligned}
 f_H: S \oplus Q &\rightarrow E \\
 (X, [Y]) &\mapsto X + Y^{\perp H}
 \end{aligned}$$

$$f_H^*(D) = \begin{pmatrix} D_S & -B^* \\ B & D_Q \end{pmatrix} \Rightarrow f_H^*(F_D) = f^*(D^2) = f^*(D) \circ f^*(D)$$

$$\begin{aligned}
 f^*(F_D) &= \begin{pmatrix} D_S & -B^* \\ B & D_Q \end{pmatrix}^2 = \begin{pmatrix} D_S^2 - B^* \wedge B & -D_S B^* - B^* \circ D_Q \\ B \circ D_S + D_Q \circ B & D_Q^2 - B \wedge B^* \end{pmatrix} \\
 &= \begin{pmatrix} F_{D_S} - B^* \wedge B & -D_Q^* \otimes_S B^* \\ D_S^* \otimes_Q B & F_{D_Q} - B \wedge B^* \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 f^*(D) \left(\overbrace{e_1, \dots, e_S}^S, \overbrace{e_{S+1}, \dots, e_r}^Q \right) &= (e_1, \dots, e_S, e_{S+1}, \dots, e_r) \begin{pmatrix} A_S & -B^* \\ B & A_Q \end{pmatrix} \\
 D_S &= d + A_S \quad D_Q = d + A_Q
 \end{aligned}$$

$$f^*(\overline{\partial}_E) = \begin{pmatrix} \overline{\partial}_S & \gamma \\ 0 & \overline{\partial}_Q \end{pmatrix} \quad (\overline{\partial}_E Y^\perp)^T = \gamma(Y^\perp) \in S$$

$$\gamma = 0: \text{ we say } (S, \overline{\partial}_S) \oplus (Q, \overline{\partial}_Q) = (\tilde{E}, \overline{\partial}_E)$$

$$\gamma = A^{0,1} \quad \overline{\partial}_E = D_H^{0,1} \quad B^{0,1} = 0 \Rightarrow A^{1,0} = 0$$

D_S is the chern connection on S with $\overline{\partial}_E, H_E$
 $(Q) \quad (Q)$

Recall: $f^*(F_{D_H}) = \begin{pmatrix} F_{D_S} - B^* \wedge B & -DB^* \\ DB & F_{D_Q} - B \wedge B^* \end{pmatrix}$

$$B = (Id - \pi) \partial_H \pi$$

Pf. $B(X) = (Id - \pi)(D_H X) = (Id - \pi)(\partial_H + \overline{\partial}_E)X$
 $= (Id - \pi) \partial_H X$

$$\partial_H(\pi X) - \pi(\partial_H X) = (Id - \pi) \partial_H \pi X$$

A computation gives $F_{D_S} = f^*(\pi \circ F_{D_H}) + \overline{\partial}_E \pi \wedge \partial_H \pi$

$$\int \text{tr} F_{D_S} = \int \text{tr}(\pi \circ F_H) - |\overline{\partial}_E \pi|_H^2$$

Thus we've proved (*)

$$(E, \bar{\partial}_E) = (E_1, \bar{\partial}_{E_1}) \oplus (E_2, \bar{\partial}_{E_2})$$

E_1, E_2 stable with same slope
then we say E is polystable

Prop. Any stable bundle is simple.

(i.e. E admits no global holomorphic endomorphism other than homotheties)

pf: $G: E \rightarrow E$ holomorphic endomorphism is not a multiple of identity.

Then $\exists p \in M$ $\exists$ an eigenvalue λ

$$G - \lambda \text{Id}$$

kernel E' $\rightarrow$ proper subbundle
image E''

$$E = E' \oplus E'' \Rightarrow \deg(E) = \deg(E') + \deg(E'')$$

$$\mu(E) = \frac{\text{rank } E'}{\text{rank } E} \mu(E') + \frac{\text{rank } E''}{\text{rank } E} \mu(E'') \leq \mu(E) \quad \times \quad \#$$

Hermitian - Einstein metric H .

$$\underbrace{F \wedge \omega F^{-1}}_2 = \lambda \text{Id}_E \quad (*)$$

mean curvature

I Hermitian - Yang-Mills functional $\int_M |F_D|_H^2 \frac{\omega^n}{n!}$

In Kähler case $(*)$ gives the critical point]

Donaldson - Uhlenbeck - Yau's Thm

$\exists$ Hermitian - Einstein metric $\Leftrightarrow (E, \bar{\partial}_E)$ polystable

Thm (Kobayashi - Lübke) $(E, \bar{\partial}_E, H)$ is H - E bundle, then $(E, \bar{\partial}_E)$ is polystable

$$\text{Pf } F \wedge \omega F^{-1} = \lambda \text{Id} \quad \lambda = \mu(E) \cdot \frac{2\pi}{\int_M \frac{\omega^n}{n!}}$$

Any $F \rightarrow E$ coherent subsheaf

$$\mu(F) = \frac{1}{\text{rank}(F)} \int_M \frac{F^{-1}}{2\pi} \text{tr } F_{H_F} \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$= \frac{1}{\text{rank}(\mathcal{F})} \int_M \left(\frac{1}{2\pi} \text{tr} \pi_H \cdot \Lambda_{\omega} F_H - |\bar{\partial}_E \pi_H|^2 \right) \frac{\omega^n}{n!}$$

$$\leq \frac{1}{\text{rank}(\mathcal{F})} \int_M \frac{1}{2\pi} \text{tr} \pi_H = \mu(E) \Rightarrow \text{semistable}$$

Choose the smallest subsheaf $\mathcal{F} \rightarrow E$ s.t. $\mu(\mathcal{F}) = \mu(E)$

$$\mathcal{F} \text{ stable} \quad |\bar{\partial}_E \pi_H| = 0 \Rightarrow |\partial_H \pi_H| = 0$$

$$\Rightarrow D_H \pi_H = 0 \quad \text{i.e. } \pi_H \text{ is parallel on } M \setminus \Sigma$$

By Hartog's Thm. π_H can be extended to M .

(*) $\Rightarrow \mathcal{F}$ is subbundle of E , then repeat the process

(*) : It suffices to show $\text{rank}(\pi_H) = \text{rank } \mathcal{F}$

We prove that the eigenvalues of π_H are constant

$$\begin{aligned} d\tilde{\varphi}(\pi_H, \dots, \pi_H) &= \tilde{\varphi}(d(\pi_H), \dots, \pi_H) + \dots + \tilde{\varphi}(\pi_H, \dots, d(\pi_H)) \\ &\stackrel{= D\pi_H \cdot [\Lambda_{\omega}, \pi_H]}{=} \tilde{\varphi}(D\pi_H, \dots, \pi_H) + \dots + \tilde{\varphi}(\pi_H, \dots, D\pi_H) \\ &= 0 \end{aligned}$$

#

$$\begin{aligned} \mathcal{F} \hookrightarrow E \\ (\text{bundle}) \Rightarrow \begin{cases} \pi_H \in L^2_1(M, \text{End}(E)) \\ C^\infty \text{ in the complement of an analytic set} \\ \text{with codim} \geq 2 \\ (*) \left\{ \begin{array}{l} \pi = \pi^2 = \pi^* \quad (\text{Id} - \pi) \bar{\partial}_E \pi = 0 \end{array} \right. \end{cases} \end{aligned}$$

Lemma (Uhlenbeck-Yau) (E, h) holomorphic bundle
with C^∞ Hermitian metric (M, ω) compact Kähler

And π satisfies $(*)$ almost everywhere

Then $\exists$ coherent sheaf $\mathcal{F} \subset \mathcal{O}(E)$ and an analytic
set $\Sigma \subset M$ with $\text{codim} \geq 2$, s.t.

- (i) $\pi|_{M \setminus \Sigma}$ is C^∞
- (ii) $\pi = \pi^2 = \pi^*$. $(\text{Id}_E - \pi) \circ \bar{\partial}_E \pi = 0$ on $M \setminus \Sigma$
- (iii) $\mathcal{F}|_{M \setminus \Sigma} = \text{Im } \pi|_{M \setminus \Sigma}$ is a holomorphic subbundle of $E|_{M \setminus \Sigma}$

Prop E admits an approximate H-E metric.

i.e. $\forall \varepsilon > 0 \quad \exists H_\varepsilon$ s.t.

$$\max_M |F_H \wedge \omega + F_{H_\varepsilon} - \lambda \text{Id}_E|_{H_\varepsilon} < \varepsilon$$

then we have the following Bogomolov type inequality

$$\int_M \left(2C_2(E) - \frac{r-1}{r} C_1(E) \wedge C_1(E) \right) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0.$$

9.3 Bogomolov Type Inequality

$$4\pi^2 \int_M \left(2C_2(E) - \frac{r-1}{r} C_1(E) \wedge C_1(E) \right) \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$\stackrel{(*)}{=} \int_M \left(\text{tr}(F_H \wedge \bar{F}_H) - \frac{1}{r} \text{tr} \bar{F}_H \wedge \text{tr} F_H \right) \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$\left[\text{Ch}_2(E) = \frac{C_1^2(E) - 2C_2(E)}{2} = \frac{1}{2} \text{tr} \left(\left(\frac{F_H}{2\pi} \right)^2 \right) \right]$$

$$= \int_M \text{tr}(F_H^\perp \wedge F_H^\perp) \wedge \frac{\omega^{n-2}}{(n-2)!} \quad F_H^\perp = F_H - \frac{1}{r} \text{tr} F_H \otimes \text{Id}_E$$

$$\stackrel{(*)}{=} \int_M \left(|F_H^\perp|^2 - |\wedge \omega F_H^\perp|^2 \right) \frac{\omega^n}{n!}$$

$$(*) - \text{tr}(F_H \wedge \bar{F}_H) \wedge \frac{\omega^{n-2}}{(n-2)!}$$

take coordinate

$$H(e_\alpha, e_\beta) = \delta_{\alpha\beta}$$

$$g_{i\bar{j}} = \delta_{ij} \quad \omega = \sum dz^i \wedge d\bar{z}^{\bar{j}}$$

$i=j$ then $k \neq l$

$$= \sum_{i \neq k} F_{\alpha i \bar{i}}^\beta F_{\beta k \bar{k}}^\alpha dz^i \wedge d\bar{z}^{\bar{i}} \wedge dz^k \wedge d\bar{z}^{\bar{k}} \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$+ \sum_{i \neq j} F_{\alpha i \bar{j}}^\beta F_{\beta j \bar{i}}^\alpha dz^i \wedge d\bar{z}^{\bar{j}} \wedge dz^j \wedge d\bar{z}^{\bar{i}} \wedge \frac{\omega^{n-2}}{(n-2)!} \quad \begin{matrix} i \neq j \text{ then } k=j \\ l=i \end{matrix}$$

$$= \left(\sum_{i,j} |F_{\alpha i \bar{j}}^\beta|^2 - \left| \sum_i F_{\alpha i \bar{i}}^\beta \right|^2 \right) \frac{\omega^n}{n!}$$

$$= \left(|F_H|^2 - |\wedge \omega F_H|^2 \right) \frac{\omega^n}{n!}$$

Miyazaka-Yau Inequality:

(M, ω) Kähler. $2\pi C_1(M) = \lambda[\omega]$ $\lambda = \pm 1$ $\dim_{\mathbb{C}} M = n$

If M admits Kähler-Einstein metric $\rho(\omega) = \lambda\omega$.

then $4\pi^2 \int_M \left(2C_2(M) - \frac{n}{n+1} C_1(M) \wedge C_2(M) \right) \cdot \frac{[\lambda^{-1}C_1(M)]^{n-2}}{(n-2)!} \geq 0$

Chapter 10 Line Bundle

10.1 The Picard Group.

Def The set of holomorphic line bundle on (M, J) forms a group under tensor product, duality. This is called the Picard group $\text{Pic}(M)$.

Prop isomorphism $\pi: \text{Pic}(M) \rightarrow H^1(M, \mathcal{O}^*)$

Pf. Line bundle L transition $\{g_{\alpha\beta}\}$ with $g_{\alpha\alpha} = 1$, $g_{\alpha\beta} g_{\beta\alpha} = 1$, $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$

$\{g_{\alpha\beta}\}$ define 1-cocycle in $C^1(\mathcal{U}, \mathcal{O}^*)$

$$\pi(L) = [g_{\alpha\beta}] \in H^1(\mathcal{U}, \mathcal{O}^*)$$

(1) well-defined: $\{g'_{\alpha\beta}\}$ $e_\alpha = f_\alpha e'_\alpha$ $e_\beta = f_\beta e'_\beta$

$$\begin{aligned} \text{then } e_\alpha &= g_{\alpha\beta} e'_\beta \\ e'_\alpha &= g'_{\alpha\beta} e'_\beta \end{aligned} \Rightarrow \frac{g_{\alpha\beta}}{g'_{\alpha\beta}} = \frac{f_\alpha}{f_\beta} \quad \frac{g}{g'} = \delta f$$

(2) homeomorphic

(3) surjective: $[g_{\alpha\beta}] \in H^1(\mathcal{U}, \mathcal{O}^*)$

define a line bundle $\bigsqcup_{\alpha} U_{\alpha} \times \mathbb{C} / \sim$.

$$\begin{aligned} (x, \xi) \sim (y, \zeta) &\Leftrightarrow x=y \\ \bigcap_{\alpha} U_{\alpha} \times \mathbb{C} &\quad \bigcap_{\beta} U_{\beta} \times \mathbb{C} \quad \frac{x}{g_{\alpha\beta}} \xi = \zeta \end{aligned}$$

14) injective. If $\pi(L)=0$

$$[\langle g_{\alpha\beta} \rangle] = 0 \Leftrightarrow g = \delta f. \quad g_{\alpha\beta} = \frac{f_{\alpha}}{f_{\beta}}(x)$$

$$\bigcup_{\alpha} U_{\alpha} \times \mathbb{C}/n$$

$$s_{\alpha}^{(x)} = [\langle x, f_{\alpha}(x) \rangle] \in L|_{U_{\alpha}}$$

$$s_{\beta}^{(x)} = [\langle x, f_{\beta}(x) \rangle] \in L|_{U_{\beta}}$$

$$\forall x \in U_{\alpha} \cap U_{\beta} \quad (*) \text{ gives } s_{\beta}(x) = s_{\alpha}(x)$$

$\Rightarrow \exists$ a global nonzero section $\Rightarrow L \cong M \times \mathbb{C}$

#

10.2 Line Bundle, first Chern Class

Prop. L is a holomorphic line bundle. then

$$c_1(L) \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$$

$$\text{Ex. } K_M = \bigwedge^n T^{*(1,0)} M \quad (\text{canonical line bundle})$$

$$= \bigwedge^{n,0} M = \det(\bigwedge^{1,0} M)$$

$$K_M^{-1} = \bigwedge^n T^{1,0} M \quad (\text{anti-canonical})$$

$$c_1(K_M^{-1}) = c_1(M)$$

Def. A line bundle L is positive if $\exists$ a metric

H on L with curvature $\bar{\Gamma}_H$ is positive (1,1)-form

Prop. M Kähler η is real closed (1.1) form

with $\frac{1}{2\pi} [1] = c_1(L) \in H_{dR}^2(M, \mathbb{R})$

then $\exists$ a metric on L with curvature form $F_H = \eta$.

pf Let H be a Hermitian metric on L

$$\text{s.t. } [F_H] = [1]$$

$$\Rightarrow F_H = \eta + \partial\bar{\partial}\varphi$$

$$\begin{aligned} \text{Let } H' = e^\varphi H. \text{ then } F_{H'} &= F_H + \partial\bar{\partial}(e^{-\varphi} \partial e^\varphi) \\ &= F_H + \partial\bar{\partial}\varphi = \eta. \quad \# \end{aligned}$$

10.3 Kodaira Vanishing Theorem

(M, J, g) Kähler E holomorphic bundle with Hermitian metric H .

$$\text{Let } \Phi, \Psi \in \Omega^{p,q}(M, E) = \Gamma(\wedge^{p,q} T^*M \otimes E) \quad \begin{matrix} (z^1, \dots, z^n) \\ (e_1, \dots, e_r) \end{matrix}$$

$$\Phi = \Phi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_1, \dots, \bar{\beta}_q}^i e_i \otimes dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}$$

$$g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) \quad H_{ij} = H(e_i, e_j)$$

$$\langle \phi, \psi \rangle = P! \, q! \, \phi_{\alpha_1 \dots \alpha_P \bar{\beta}_1 \dots \bar{\beta}_q} \overline{\psi_{\gamma_1 \dots \gamma_P \bar{\delta}_1 \dots \bar{\delta}_q}} H_{\alpha_1 \gamma_1} g^{\alpha_2 \bar{\gamma}_2} \dots g^{\alpha_P \bar{\gamma}_P} g^{\beta_1 \bar{\delta}_1} \dots g^{\beta_q \bar{\delta}_q}$$

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle \frac{\omega^n}{n!}$$

∇ induced connection on $\Lambda^{p,q}(M) \otimes E$ by $\begin{cases} \text{Levi-Civita of } (M, g) \\ \text{Chern connection } (E, H) \end{cases}$

Lemma For $\phi \in \Gamma(\Lambda^{p,q}(M) \otimes L)$ a holomorphic basis

$$\bar{\partial}_L \phi = (-1)^p \nabla_{\bar{\beta}} \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q} e \otimes d\bar{z}^A \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}$$

$$(\bar{\partial}_L^{*H} \phi)_{A \bar{\beta}_1 \dots \bar{\beta}_q} = (-1)^{p+1} q g^{\alpha \bar{\beta}} (\nabla_{\alpha} + H^{\gamma} \partial_{\alpha} H_{\gamma}) \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q}$$

Pf. $\bar{\partial}_L \phi = (-1) \underbrace{\frac{\partial \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q}}{\partial \bar{z}^{\beta}}}_{\parallel} e \otimes \underbrace{d\bar{z}^{\beta} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}}_{\parallel \nabla_{\bar{\beta}} \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q}}$

$$p!(q-1)! \nabla_{\bar{\beta}} \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q} + \sum_K \underbrace{\nabla_{\bar{\beta}_K}^{\alpha} \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q}}_{\downarrow} \underbrace{\frac{1}{\alpha} \dots \frac{1}{\beta_q}}_{k\text{-th}} e \otimes d\bar{z}^{\beta} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q} = 0$$

$$(\bar{\partial}_L^{*H} \phi) = (\phi, \bar{\partial}_L \psi) \quad H(e, e) = e^{-h}$$

$$\langle \phi, \bar{\partial}_L \psi \rangle = (-1)^{p+1} P! \, q! \, \phi_{A \bar{\beta}_1 \dots \bar{\beta}_q} \overline{\nabla_{\bar{\beta}} \psi_{B \bar{\alpha}_1 \dots \bar{\alpha}_q}} g^{AB} g^{\beta_1 \bar{\alpha}_1} \dots g^{\beta_q \bar{\alpha}_q} e^{-h}$$

$$= (-1)^{p+1} P! \, q! \left(\underbrace{\nabla_{\bar{\beta}} (\phi_{A \bar{\beta}_1 \dots \bar{\beta}_q} \psi_{B \bar{\alpha}_1 \dots \bar{\alpha}_q} g^{AB} g^{\beta_1 \bar{\alpha}_1} \dots g^{\beta_q \bar{\alpha}_q})}_{\text{div}(x)} e^{-h} \right)$$

$$\text{div}(x) \quad \int_M \text{div}(x) = 0$$

$$-g^{\beta\bar{\beta}_1} (\nabla_{\beta} \phi_{\bar{A}\bar{B}\dots\bar{B}_q} - \partial_{\beta} h \phi_{\bar{A}\bar{B}\dots\bar{B}_q}) \overline{\psi_{\bar{B}_{\bar{q}}\dots\bar{B}_q}} g^{AB} g^{\delta_1\bar{A}_1} g^{\delta_2\bar{A}_2} \dots g^{\delta_q\bar{A}_q} e_{\bar{A}_1}^{\dots\bar{A}_q}$$

compare with $(\bar{\partial}_L^{\star+1} \phi, \psi)$ $\neq$

$$\text{Rmk. } (\bar{\partial}_L \phi)_{\bar{A}\bar{B}\bar{B}_1\dots\bar{B}_q} = \frac{1}{q+1} (-1)^P (\nabla_{\bar{\beta}} \phi_{\bar{A}\bar{B}\bar{B}_1\dots\bar{B}_q} - \sum_{k=1}^q \nabla_{\bar{B}_k} \phi_{\bar{A}\bar{B}\bar{B}_1\dots\bar{B}_{k-1}\bar{B}_{k+1}\dots\bar{B}_q})$$

$\downarrow$
k-th

In the following $\phi \in \Omega^q(M, L)$

$$\begin{aligned} (\bar{\partial}_L^{\star} \bar{\partial}_L \phi)_{\bar{B}_1\dots\bar{B}_q} &= -(q+1) g^{\alpha\bar{\beta}} (\nabla_{\alpha} + H^{-1} \partial_{\alpha} H) (\bar{\partial}_L \phi)_{\bar{B}_1\dots\bar{B}_q} \\ &= -g^{\alpha\bar{\beta}} \left\{ (\nabla_{\alpha} + H^{-1} \partial_{\alpha} H) \nabla_{\bar{\beta}} \phi_{\bar{B}_1\dots\bar{B}_q} - \sum_{k=1}^q (\nabla_{\alpha} + H^{-1} \partial_{\alpha} H) \nabla_{\bar{B}_k} \phi_{\bar{B}_1\dots\bar{B}_{k-1}\bar{B}_{k+1}\dots\bar{B}_q} \right\} \end{aligned}$$

$$\begin{aligned} (\bar{\partial}_L \bar{\partial}_L^{\star} \phi)_{\bar{B}_1\dots\bar{B}_q} &= \frac{1}{q} \sum_{k=1}^q (-1)^{k-1} \nabla_{\bar{A}_k} (\bar{\partial}_L^{\star} \phi)_{\bar{B}_1\dots\bar{A}_{k-1}\bar{A}_{k+1}\dots\bar{B}_q} \\ &= - \sum_{k=1}^q (-1)^{k-1} \nabla_{\bar{B}_k} \left(g^{\alpha\bar{\beta}} (\nabla_{\alpha} + H^{-1} \partial_{\alpha} H) \phi_{\bar{B}_1\dots\bar{B}_{k-1}\bar{B}_{k+1}\dots\bar{B}_q} \right) \\ &= - \sum_{k=1}^q g^{\alpha\bar{\beta}} \nabla_{\bar{A}_k} \nabla_{\alpha} \phi_{\bar{B}_1\dots\bar{A}_{k-1}\bar{B}\bar{A}_{k+1}\dots\bar{B}_q} \\ &\quad - \sum_{k=1}^q g^{\alpha\bar{\beta}} \partial_{\bar{B}_k} (H^{-1} \partial_{\alpha} H) \phi_{\bar{B}_1\dots\bar{B}\dots\bar{B}_q} \\ &\quad - \sum_{k=1}^q g^{\alpha\bar{\beta}} H^{-1} \partial_{\alpha} H \nabla_{\bar{B}_k} \phi_{\bar{B}_1\dots\bar{B}\dots\bar{B}_q} \end{aligned}$$

$$\Delta \bar{\partial} = -(\bar{\partial}_L \bar{\partial}_L^* + \bar{\partial}_L^* \bar{\partial}_L)$$

$$-(\Delta \bar{\partial} \phi)_{\bar{\beta}_1 \dots \bar{\beta}_g} = -g^{\alpha \bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_g} - g^{\alpha \bar{\beta}} H^{-1} \partial_\alpha H \nabla_{\bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_g} \\ + g^{\alpha \bar{\beta}} \underbrace{\sum_{k=1}^g [\nabla_\alpha, \nabla_{\bar{\beta}_k}] \phi_{\bar{\beta}_1 \dots \bar{\beta}_g}}_{(*)} + \sum_{k=1}^g g^{\alpha \bar{\beta}} (F_L)_{\alpha \bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_g}$$

$$\nabla_\alpha \phi_{\bar{\beta}_1 \dots \bar{\beta}_g} = \partial_\alpha \phi_{\bar{\beta}_1 \dots \bar{\beta}_g}$$

$$\nabla_{\bar{\beta}_k} \phi_{\bar{\beta}_1 \dots \bar{\beta}_g} = \bar{\partial}_{\beta_k} \phi_{\bar{\beta}_1 \dots \bar{\beta}_g} - \overline{T_{\beta_k, \beta}^\gamma} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta}_g} - \overline{T_{\beta_k \beta}^\gamma} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta}}$$

Take normal coordinate $T_{\alpha \beta}^\gamma(0) = 0$

$$(*) = \sum_{k=1}^g g^{\alpha \bar{\beta}} \underbrace{(-\partial_\alpha (\overline{T_{\beta_k, \beta}^\gamma}))}_{\overline{R_{\beta \alpha}^\gamma \beta_k}} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta}_g} - \underbrace{\partial_\alpha (\overline{T_{\beta_k \beta}^\gamma})}_{\overline{R_{\beta \alpha}^\gamma \beta_k}} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta} \dots \bar{\beta}_g} \quad (\star)$$

$$= \sum_{k=1}^g R_{\beta \beta_k} g^{\beta \bar{\gamma}} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta}_g}$$

$$(\star) = \sum_{k=1}^g \sum_{l \neq k} g^{\alpha \bar{\beta}} g^{\gamma \bar{\gamma}} \frac{R_{\beta_l \alpha \bar{\gamma} \beta_k}}{R_{\beta_l \alpha \bar{\gamma} \beta_k}} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta} \dots \bar{\beta}_g} \\ = \sum_{k=1}^g \sum_{l < k} g^{\alpha \bar{\beta}} g^{\gamma \bar{\gamma}} \frac{R_{\beta_l \alpha \bar{\gamma} \beta_k}}{R_{\beta_l \alpha \bar{\gamma} \beta_k}} \phi_{\bar{\beta}_1 \dots \bar{\gamma} \dots \bar{\beta} \dots \bar{\beta}_g} \\ + \sum_{k=1}^g \sum_{k < l} g^{\alpha \bar{\beta}} g^{\gamma \bar{\gamma}} \frac{R_{\beta_l \alpha \bar{\gamma} \beta_k}}{R_{\beta_l \alpha \bar{\gamma} \beta_k}} \phi_{\bar{\beta}_1 \dots \bar{\beta} \dots \bar{\gamma} \dots \bar{\beta}_g} = 0.$$

In general

$$-\Delta \bar{\partial} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} = -g^{\alpha \bar{\beta}} (\nabla_\alpha + H^{-1} \partial_\alpha H) \nabla_{\bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} \\ + \sum_{k=1}^q (R_{\alpha \bar{\beta}_k} + (F_H)_{\alpha \bar{\beta}_k}) g^{\alpha \bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} \quad (*)$$

(assume ϕ $\bar{\partial}$ -harmonic)

$$\Rightarrow (-\Delta \bar{\partial} \phi, \phi) = q! \int_M -\Delta \bar{\partial} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} \phi^{\bar{\beta}_1 \dots \bar{\beta}_q} e^{-h} \frac{\omega^n}{n!}$$

$$\begin{aligned} &= \int_M \underbrace{-g^{\alpha \bar{\beta}} \nabla_\alpha (\nabla_{\bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} \phi^{\bar{\beta}_1 \dots \bar{\beta}_q} e^{-h})}_{\text{div}(X)} \frac{\omega^n}{n!} \\ &\quad + \int_M g^{\alpha \bar{\beta}} \nabla_{\bar{\beta}} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q} \nabla_\alpha \phi^{\bar{\beta}_1 \dots \bar{\beta}_q} e^{-h} \\ &\quad + \int_M * \frac{\omega^n}{n!} \end{aligned}$$

$$R_{\alpha \bar{\beta}} + (F_H)_{\alpha \bar{\beta}} > 0$$

$$= \int_M |\nabla^{0,1} \phi|^2 + * \frac{\omega^n}{n!}$$

$$\Rightarrow \phi = 0.$$

Thm (Kodaira) Let $L \rightarrow M$ be a holomorphic bundle with a Hermitian metric, s.t.

$$F = (Ric_{\alpha \bar{\beta}} + (F_H)_{\alpha \bar{\beta}}) dz^\alpha \wedge d\bar{z}^\beta > 0$$

$$\text{then } H^q(M, \mathcal{O}(L)) = 0 \quad (q \geq 1)$$

Cor (Kodaira Vanishing Theorem) Let L be a positive bundle of M then

$$H^q(M, \mathcal{O}(K \otimes L)) = H^{n,q}(M, L) = 0 \quad \forall q \geq 1$$

Pf. $C(K \otimes L, \omega \otimes \eta) = -\rho_\omega + \bar{\partial}_L \eta$ $\neq$

Cor⁽¹⁾ If $L \otimes K^{-1}$ is positive, then $H^q(M, \mathcal{O}(L)) = 0 \quad (q > 0)$

(2) If L^* is positive, then $H^q(M, \mathcal{O}(L)) = 0 \quad q \leq n-1$

Serre Duality:

$$H^{0,q}(M, L) \cong H^{n,n-q}(M, L) \cong H^{n-q}(M, \Omega^{n,0}(L^*)) \cong H^{n-q}(M, \mathcal{O}(K \otimes L^*))$$

Thm (Nakano) L positive, then $H^{p,q}(M, L) = 0 \quad p+q \geq n+1$

Nakano Identity: $\varphi \in \Omega^{p,q}(M, L)$. H Hermitian metric.

$$\begin{aligned} \text{then } \left((\bar{\partial}_L \wedge \omega \bar{\partial}_L - \bar{\partial}_L \wedge \omega) \varphi, \varphi \right) &= \|\bar{\partial}_L \varphi\|_L^2 + \|\bar{\partial}_L^* \varphi\|_L^2 \\ &\quad - \|\bar{\partial}_L \varphi\|^2 - \|\bar{\partial}_L^* \varphi\|^2 \end{aligned}$$

Pf. Kähler's identity $[\wedge \omega, \bar{\partial}_L] = -\bar{\partial}_L \bar{\partial}_L^*$
 $[\wedge \omega, \bar{\partial}_L] = \bar{\partial}_L \bar{\partial}_L^*$

$$F_H \varphi = (\partial_H \bar{\partial}_L + \bar{\partial}_L \partial_H) \varphi$$

$$\begin{aligned} (F_H \wedge \omega F_H \varphi) &= (F_H \wedge \omega (\partial_H \bar{\partial}_L + \bar{\partial}_L \partial_H) \varphi, \varphi) \\ &= \underbrace{(F_H (\omega \partial_H - \partial_H \omega) \bar{\partial}_L^* \varphi, \varphi)}_{\partial_H^*} + (F_H \partial_H \wedge \omega \bar{\partial}_L \varphi, \varphi) \\ &\quad + \underbrace{(F_H (\omega \bar{\partial}_L - \bar{\partial}_L \omega) \partial_H \varphi, \varphi)}_{\partial_H^*} + (F_H \bar{\partial}_L \wedge \omega \partial_H \varphi, \varphi) \end{aligned}$$

$$\begin{aligned} (-F_H \wedge \omega F_H \varphi, \varphi) &\stackrel{\text{similar}}{=} (F_H \partial_H \partial_H^* \varphi, \varphi) - (F_H \partial_H \wedge \omega \bar{\partial}_L \varphi, \varphi) \\ &\quad - (F_H \bar{\partial}_L \bar{\partial}_L^* \varphi, \varphi) - (F_H \bar{\partial}_L \wedge \omega \partial_H \varphi, \varphi) \neq \end{aligned}$$

If $\Delta \bar{\partial}_L \varphi = 0$, then $\bar{\partial}_L \varphi = \bar{\partial}_L^* \varphi = 0 \Rightarrow (F_H \wedge \omega F_H - F_H F_H \wedge \omega) \varphi, \varphi \geq 0$

$$\begin{aligned} \text{Let } \omega = F_H F_H &\Rightarrow ((\omega \lrcorner \omega - L \omega \wedge \omega) \varphi, \varphi) \\ &= ((n-p-q) \varphi, \varphi) \geq 0 \Rightarrow \text{for } p+q \geq n+1 \\ &\quad \varphi = 0 \end{aligned}$$

Thus we proved Nakano's Thm.

Rank M Kähler, $C_1(L) \geq 0$, $\text{rank } C_1(L) \geq k$.

then $H^{p,q}(M, L) = 0$ for $p+q \geq 2n-k+1$

10.4 Line Bundle from Divisors

analytic hypersurface V of M (M compact complex manifold)

(1) $V \subset M$ compact; $\exists$ finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$

U_α is biholomorphic to the unit ball of $\mathbb{C}^n$

(2) $U_\alpha \cap V$ is defined by the zeros of a holomorphic function f_α

If V can't be written as the union of two nonempty hypersurfaces

and f_α can be chosen to vanish to order 1,

then V is called irreducible.

Def. A divisor D on M is a formal finite linear

combination $D = \sum d_i V_i$ $d_i \in \mathbb{Z}$
 V_i : irreducible hypersurface

$$\text{Div}(M) \rightarrow \text{Pic}(M)$$

$$D = \sum d_i V_i \mapsto \bigotimes_i (\mathcal{L}_{V_i})^{d_i}$$

10.5 Kodaira Embedding Theorem

M compact complex L positive line bundle

M must be an algebraic manifold.

$$H^0(M, \mathcal{O}(L^k)) = \Gamma(M, L^k)$$

$$\dim \Gamma(M, L^k) = N(k) + 1$$

$\varphi = \{\varphi_0, \dots, \varphi_N\}$ form a basis of the holomorphic sections

e is a frame of L_x .

$$\eta_k: x \mapsto [\varphi_0(e), \dots, \varphi_N(e)] \in \mathbb{CP}^N$$

Thm: $\exists k_0 \in \mathbb{Z}^+$ s.t. $\forall k \geq k_0$ η_k is a well-defined embedding

Generalization:

Thm $f: X \rightarrow Y$ smooth holomorphic fibration, all the fibers

compact Y Kähler If $\exists$ a line bundle on X

s.t. $\forall y \in Y$ $L_{f^{-1}(y)}$ is positive. Then $\exists$ a holomorphic

vector bundle $\pi_E: E \rightarrow Y$ and an embedding $j: X \rightarrow P(E^*)$

$$\text{s.t. } f = \pi_{P(E^*)} \circ j.$$

Chapter 11 H-E metrics on holomorphic Bundle

11.1 H-E connection and Yang-Mills connection

(M, ω) Kähler E complex vector bundle

H_0 : Hermitian metric on E .

$\mathcal{A}_{H_0} = \{ \text{connections compatible with } H_0 \}$

$$\mathcal{A}_{H_0}^{1,1} : F_D^{0,2} = F_D^{2,0} = 0 \quad (F_D^{*,H_0} = -F_D)$$

"unitary integrable connections"

Rmk. $(E, \bar{\partial}E)$ holomorphic bundle. $\begin{cases} D_A \text{ is } H_0\text{-unitary} \\ D_A^{0,1} = \bar{\partial}E \end{cases}$

conversely, given $D_A \in \mathcal{A}_{H_0}^{1,1}$, $D_A^{0,1}$ defines a holomorphic structure on E .
 $(D_A^{0,1})^{*,H_0} = D_A^{1,0}$

?

$\mathcal{L}^{1,1}(E) = \{ \text{holomorphic bundle structures on } E \}$

$\downarrow$
 $\mathcal{A}_{H_0}^{1,1}$

$\mathcal{GL}(E) = \text{Aut}(E) = \text{group of } C^\infty \text{ bundle automorphisms of } E$

$\mathcal{gl}(E) = \{ C^\infty \text{ section of } \text{Ind}(E) = E \otimes E^* \}$

unitary gauge group $\mathcal{G} = \{G \in GL(E) \mid G^{*H_0} G = \text{Id}\}$

$\mathcal{G}$ acts on A_{H_0}

$$\mathcal{G} \times A_{H_0} \rightarrow A_{H_0}$$

$$\begin{aligned} (G, D_A) &\mapsto D_{G(A)} \stackrel{\text{def}}{=} G \circ D_A \circ G^{-1} \\ &= (G \circ D_A - D_A \circ G) G^{-1} + D_A \\ &= D_A - (D_A G) G^{-1} \end{aligned}$$

$$\begin{aligned} \text{locally } D_A &= d + A & D_{G(A)} &= d + \overline{G(A)} \\ & & &= G \circ (d + A) \circ G^{-1} \\ & & &= \underline{d - (dG) G^{-1} + GA \circ G^{-1}} \end{aligned}$$

Check $D_{G(A)}$ compatible with H_0

$$\begin{aligned} (D_{G(A)})^{*H_0} &= (G^{-1})^{*H_0} \circ (D_A)^{*H_0} \circ (G)^{*H_0} \\ &= G \circ D_A \circ G^{-1} = D_{G(A)} \end{aligned}$$

$$\langle D_{G(A)} X, Y \rangle_{H_0} + \langle X, D_{G(A)} Y \rangle_{H_0}$$

$$= \langle G \circ D_A \circ G^{-1} X, Y \rangle_{H_0} + \langle X, G \circ D_A \circ G^{-1} Y \rangle$$

$$\begin{aligned} G^{*H_0} &= G^{-1} \\ &= \langle D_A \circ G^{-1} X, G^{-1} Y \rangle + \langle G^{-1} X, D_A \circ G^{-1} Y \rangle = d \langle X, Y \rangle \end{aligned}$$

Complexification $\mathcal{L}^{\mathbb{C}}(E, H_0) \xrightarrow{\quad} \mathcal{L}^{\mathbb{C}} \times A_{H_0}^{\prime\prime} \rightarrow A_{H_0}^{\prime\prime}$
 not necessarily unitary $(G, D_A) \mapsto D_{G(A)}$

$$D_A = \bar{\partial}_A + \partial_A \quad D_{G(A)} = \bar{\partial}_{G(A)} + \partial_{G(A)}$$

$$\bar{\partial}_{G(A)} = G \circ \bar{\partial}_A \circ G^{-1}$$

$$G(A) \text{ is } H_0\text{-unitary} \Leftrightarrow (\bar{\partial}_{G(A)})^{*H_0} = \partial_{G(A)}$$

$$\text{thus } \partial_{G(A)} = (G^{*H_0})^{-1} \circ \partial_A \circ G^{*H_0}$$

$$A_{H_0}^{\prime\prime}(E) \longleftrightarrow \mathcal{H}^{\prime\prime}(E) \quad \bar{\partial}_2 = G \circ \bar{\partial}_1 \circ G^{-1}$$

$$\begin{array}{ccc} \sigma \downarrow & & \downarrow G \\ A_{H_0}^{\prime\prime}(E) & \longleftrightarrow & \mathcal{H}^{\prime\prime}(E) \end{array}$$

The space of equivalent classes of holomorphic structures of E is $A_{H_0}^{\prime\prime} / \mathcal{L}^{\mathbb{C}}$

D-U-Y Thm. $(E, \bar{\partial}_E)$ stable $\Rightarrow \exists!$ H-E metric H

$$h = H \circ^{-1} | \cdot | > 0 \quad \exists G \in \mathcal{L}^{\mathbb{C}} \text{ s.t. } G^{*H_0} \circ G = h$$

$$F_{D_{G(A)}} = G \circ F_{D_H} \circ G^{-1}$$

$$D_{A_0} = D_{(\bar{\partial}_E, H_0)}$$

$$D_{H^{-1}} = D_{(\bar{\partial}_E, H)} \notin A_{H_0}$$

$$D_{G(A_0)} = G \circ \bar{\partial}_{A_0} \circ G^{-1} + (G^{*H_0})^{-1} \circ \partial_A \circ G^* H_0$$

$$F_{D_{G(A_0)}} \stackrel{(*)}{=} G \circ F_{D_{H_0}} \circ G^{-1}$$

$$\Rightarrow \exists \lambda \in \mathbb{C} \quad F_{D_{G(A_0)}} = \lambda \text{Id}$$

$$D_{G(A_0)} \in \mathcal{A}_{H_0}^{''}$$

$$(*) \quad F_{D_{G(A_0)}} = \bar{\partial}_{G(A_0)} \partial_{G(A_0)} + \partial_{G(A_0)} \bar{\partial}_{G(A_0)}$$

$$= G \circ \bar{\partial}_{A_0} \circ G^{-1} \circ (G^*)^{-1} \circ \partial_{A_0} \circ G^* + (G^*)^{-1} \circ \partial_{A_0} \circ G^* \circ G \circ \bar{\partial}_{A_0} \circ G^{-1}$$

$$= G \circ \bar{\partial}_{A_0} \circ \underbrace{h^{-1} \circ \partial_{A_0} \circ h}_{h^{-1}(\partial_{A_0} h)} \circ G^{-1} + G \circ h^{-1} \circ \partial_{A_0} \circ h \circ \bar{\partial}_{A_0} \circ G^{-1}$$

$$= G \circ (\bar{\partial}_{A_0} \circ \partial_{A_0} + \partial_{A_0} \circ \bar{\partial}_{A_0} + \bar{\partial}_{A_0} (h^{-1}(\partial_{A_0} h)) \circ G^{-1}$$

$$= G \circ F_{D_{H_0}} \circ G^{-1}$$

$$\begin{cases} D_{H_0} = D_{A_0} + h^{-1}(\partial_{A_0} h) \\ F_{D_{H_0}} = F_{H_0} + \bar{\partial}_{A_0} (h^{-1} \partial_{A_0} h) \end{cases}$$

Thus if $D_{A_0} \in \mathcal{A}_{H_0}^{''}$ define a stable structure $(E, \bar{\partial}_{A_0})$

$\exists$ a H-E connection $D_{G(A_0)}$ in the $\mathcal{L}^E$ -orbit of A_0 .

Conversely if $D_{A_0} \in \mathcal{A}_{H_0}^{''}$ is H-E, then $(\bar{E}, \bar{\partial}_{A_0})$

is polystable.

In 9.3, we gave Yang-Mills functional

$$\int_M |\bar{F}_A|^2 \frac{\omega^n}{n!} = \int_M |\bar{F}_1 \wedge \omega \bar{F}_A|^2 \frac{\omega^n}{n!} + 4\pi^2 \int_M (2C_2(E) - C_1(E) \wedge C_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$= \int_M \underbrace{|\bar{F}_1 \wedge \omega \bar{F}_A - \lambda \text{Id}|^2 + \lambda^2 \text{rank}(E)}_{\text{free of the choice of } A} \frac{\omega^n}{n!} + \dots$$

H-E connection $\longrightarrow$ absolute minimum of Y-M functional

$$\frac{d}{dt} \int_M |\bar{F}_{A_t}|^2 \frac{\omega^n}{n!} \Big|_{t=0}$$

$$F_{A_t} = (D_{A_0} + \theta_t)^2$$

$$= F_{A_0} + D_{A_0} \theta_t + \theta_t \wedge \theta_t$$

$$2 \int_M \text{Re} \langle D_{A_0} \frac{d\theta}{dt}, F_{A_0} \rangle \frac{\omega^n}{n!} = 2 \text{Re} \int_M \langle \frac{d\theta}{dt}, D_{A_0}^* F_{A_0} \rangle \frac{\omega^n}{n!}$$

$$\Rightarrow D_{A_0}^* F_{A_0} = 0$$

Kähler case

$$\bar{\partial} A^* = -F_1 [\wedge \omega, \partial A]$$

$$\partial A^* = F_1 [\wedge \omega, \bar{\partial} A]$$

$$F_1 \wedge \omega F_A = \lambda \text{Id} \Rightarrow \bar{\partial} A^* F_A = \partial A^* F_A = 0$$

11.2 Donaldson's heat flow. H-Y-M flow

H-Y-M flow: $H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1} \wedge_{\omega} F_H - \lambda \text{Id})$

Y-M flow: $\frac{\partial A}{\partial t} = -D^* F_A$

$$\frac{\partial H}{\partial t} | (x, Y) = \langle H^{-1} \frac{\partial H}{\partial t} (x), Y \rangle \quad H^{-1} \frac{\partial H}{\partial t} \in T^*(\text{End}(E))$$

$$(E, \bar{\partial}_E)$$

$$h(t) = H_0^{-1} H(t) \in GL(E)$$

$$\partial_H - \partial_{H_0} = h^{-1} \partial_{H_0} h$$

$$F_H - F_{H_0} = \bar{\partial}_E (h^{-1} \partial_{H_0} h)$$

$$H^{-1} \frac{\partial H}{\partial t} = h^{-1} \frac{\partial h}{\partial t}$$

$$= -2\sqrt{-1} \wedge_{\omega} (\bar{F}_{H_0} + \bar{\partial}_E (h^{-1} \partial_{H_0} h)) + 2\lambda \text{Id}$$

$$\Rightarrow \frac{\partial h}{\partial t} = -2\sqrt{-1} \wedge_{\omega} \bar{\partial}_E \partial_{H_0} h + 2\sqrt{-1} \bar{\partial}_E h \cdot h^{-1} \partial_{H_0} h - 2\sqrt{-1} h \wedge_{\omega} \bar{F}_{H_0} + 2\lambda h$$

$$\begin{cases} h(t) |_{t=0} = \text{Id}_E \end{cases}$$

$$h^{*H_0} = h$$

Line Bundle case: $h = e^f \quad \frac{\partial f}{\partial t} = \Delta f + 2\sqrt{-1} \wedge_{\omega} (\bar{\partial} f \wedge \partial f) + (*)$

Rmk H-Y-M flow is a nonlinear strictly parabolic equation.

Y-M flow:

(E, H₀)

$$\begin{cases} \frac{\partial A}{\partial t} = -D_A^* F_A \\ A|_{t=0} = A_0 \in \mathcal{A}_{H_0}'' \end{cases}$$

(E, $\bar{\partial} A_0$) is holomorphic.

D_{A_0} is the chern connection

If H-Y-M has a long-time solution, then $\exists!$ solution of Y-M in the complex gauge group of $A_0 \in \mathcal{A}_{H_0}''$.

$$G(t) \in \mathcal{G}_1^{\mathbb{C}} \quad G(t)^{*H_0} \circ G(t) = h(t) = H_0^{-1} H(t)$$

$H(t)$: solution of H-Y-M

$$\tilde{A}(t) = G(t)(A_0) \quad \text{then } F_{D_{G(t)(A_0)}} = G(t) \circ F_{D_{H(t)}} \circ G(t)^{-1}$$

$$G^*(t)^{-1} \frac{\partial h}{\partial t} G^{-1}(t) = (G^*)^{-1} \frac{\partial G^*}{\partial t} + \frac{\partial G}{\partial t} \cdot G^{-1}$$

"

$$-2F_1 G \wedge_{\omega} F_{D_H} \circ G^{-1} + 2\lambda \text{Id}_{\mathbb{C}}$$

"

$$-2F_1 \wedge_{\omega} \bar{F}_{\tilde{A}(t)} + 2\lambda \text{Id}_{\mathbb{C}}$$

$$\frac{\partial \tilde{A}(t)}{\partial t} = \frac{\partial}{\partial t} (G \circ \bar{\partial}_{A_0} \circ G^{-1} + (G^*)^{-1} \circ \partial_{A_0} \circ G^*)$$

$$= -\bar{\partial}_{\tilde{A}} \left(\frac{\partial G}{\partial t} \circ G^{-1} \right) + \partial_{\tilde{A}} \left((G^*)^{-1} \frac{\partial G^*}{\partial t} \right)$$

$$= \frac{1}{2} (\partial_{\tilde{A}} - \bar{\partial}_{\tilde{A}}) \left(\frac{\partial G}{\partial t} \circ G^{-1} + (G^*)^{-1} \frac{\partial G^*}{\partial t} \right) - \frac{1}{2} (\partial_{\tilde{A}} + \bar{\partial}_{\tilde{A}}) \underbrace{\left(\frac{\partial G}{\partial t} \circ G^{-1} - (G^*)^{-1} \frac{\partial G^*}{\partial t} \right)}_{-2\alpha(1+)}$$

$$= -F_1(\partial_A - \bar{\partial}_A) \wedge \omega \bar{F}_A + \frac{1}{2}(\partial_A + \bar{\partial}_A)(\alpha(t))$$

$$\alpha^*(t) = -\alpha(t)$$

Let $S(t) \in \mathcal{L}$ be the unique solution to

$$\frac{dS}{dt} = S\alpha \quad S(0) = I$$

$$\text{Let } A(t) = S(t)(\check{A}(t))$$

$$\frac{\partial A(t)}{\partial t} = \frac{\partial}{\partial t}(S \circ D_{A(t)} \circ S^{-1})$$

$$= -F_1(\partial_{A(t)} - \bar{\partial}_{A(t)}) \wedge \omega \bar{F}_{A(t)} = -D_A^* \bar{F}_A$$

$$\frac{\partial \bar{\partial}_A}{\partial t} = F_1 \bar{\partial}_A \wedge \omega \bar{F}_A$$

$$\frac{\partial \partial_A}{\partial t} = -F_1 \bar{\partial}_A \wedge \omega \bar{F}_A$$

Uniqueness: If $\hat{A}(t)$ is another solution

$$\frac{\partial \hat{A}(t)}{\partial t} = -D_{\hat{A}}^* \bar{F}_{\hat{A}} = -F_1(\partial_{\hat{A}} - \bar{\partial}_{\hat{A}}) \wedge \omega \bar{F}_{\hat{A}}$$

$$\text{Let } g(t) \in \mathcal{L}^{\mathbb{C}} \quad \frac{\partial g}{\partial t} g^{-1} = (F_1 \wedge \omega \bar{F}_A + \lambda \text{Id}). \quad g(0) = \text{Id}_{\mathbb{C}}$$

$$\begin{aligned} \text{then } \frac{\partial}{\partial t}(g^{-1}(\hat{A}(t))) &= \frac{\partial}{\partial t}(g^{-1} \circ \bar{\partial}_{\hat{A}} \circ g + g^* \circ \partial_{\hat{A}} \circ (g^*)^{-1}) \\ &= 0 \quad \Rightarrow \quad \hat{A}(t) = g(t)(A_0) \end{aligned}$$

$$g^{*H_0} g = \hat{h}(t) = H_0^{-1} \hat{h}(t)$$

$$\hat{h}(t) = H_0 \hat{h}(t)$$

$$\bar{F}_{\hat{A}(t)} = g_0 \bar{F}_{D_{\hat{A}(t)}} \circ g^{-1}$$

$$\Rightarrow \begin{cases} \hat{h} - \frac{\partial \hat{h}}{\partial t} = -2(\hat{F} - \omega \hat{F}_0 \hat{h}(t) - \lambda \text{Id}) \\ \hat{h}|_{t=0} = \text{Id} \end{cases}$$

by uniqueness of H-R-M,

$$\hat{h}(t) = h(t)$$

$$g^*g = \hat{h} = h$$

$$\tilde{S} = g \circ \phi^{-1} \Rightarrow \begin{cases} \tilde{S}^* \tilde{S} = \text{Id} \\ \tilde{S}(0) = \text{Id}_E \end{cases}$$

$$\frac{\partial}{\partial t} \tilde{S} = \frac{\partial}{\partial t} \left(\frac{1}{2} \tilde{S} + \frac{1}{2} \tilde{S}^{*-1} \right)$$

$$= \frac{1}{2} \frac{\partial \tilde{S}}{\partial t} - \frac{1}{2} \tilde{S}^{*-1} \frac{\partial \tilde{S}^*}{\partial t} \tilde{S}^{*-1}$$

$$= \frac{1}{2} \tilde{S} \left(-\frac{\partial \phi}{\partial t} \cdot \phi^{-1} + (\phi^*)^{-1} \frac{\partial \phi^*}{\partial t} \right)$$

$$\Rightarrow \frac{d}{dt} \tilde{S} = \tilde{S} \alpha(t)$$

$$\tilde{S}(0) = \text{Id}_E$$

$$\text{thus } \tilde{S}(t) = S(t)$$

$$\text{then } \hat{A}(t) = \tilde{S}(t) \phi(t)(A_0) = A(t)$$

~~≠~~

Donaldson Stable $(E, \bar{\partial}_E) \Rightarrow h(t)$ is a solution of H-R-M

$$h(t) \rightarrow h_{\infty} \quad t \rightarrow \infty$$



H-E metric

11.3 Basic Estimates

$$\begin{cases} H^{-1} \frac{\partial H}{\partial t} = -2(F_1 \wedge \omega F_1 - \lambda \text{Id}) \\ H(t)|_{t=0} = H_0 \end{cases}$$

$$h(t) = H \bar{\partial}^* H(t)$$

Lemma. $(\frac{\partial}{\partial t} - \Delta) \text{tr}(F_1 \wedge \omega F_1 - \lambda \text{Id}) = 0$

$$(\frac{\partial}{\partial t} - \Delta) |F_1 \wedge \omega F_1 - \lambda \text{Id}|_H^2 = -4 |\bar{\partial}_E (F_1 \wedge \omega F_1)|_H^2 \leq 0$$

Pf. $\Phi(H) = F_1 \wedge \omega F_1 - \lambda \text{Id} = F_1 \wedge \omega (F_{H_0} + \bar{\partial}_E(h^{-1} \partial H_0 h)) - \lambda \text{Id}$

$$\frac{\partial}{\partial t} \Phi(H) = F_1 \wedge \omega \bar{\partial}_E \left(-h^{-1} \frac{\partial h}{\partial t} h^{-1} \partial H_0 h + h^{-1} \partial_{H_0} (h \circ h^{-1} \frac{\partial h}{\partial t}) \right)$$

$$= F_1 \wedge \omega \bar{\partial}_E \left(-h^{-1} \frac{\partial h}{\partial t} h^{-1} \partial H_0 h + h^{-1} \partial H_0 h \cdot h^{-1} \frac{\partial h}{\partial t} + \partial H_0 (h^{-1} \frac{\partial h}{\partial t}) \right)$$

$$= F_1 \wedge \omega \bar{\partial}_E \partial H (h^{-1} \frac{\partial h}{\partial t})$$

$$\begin{aligned} \frac{\partial}{\partial t} \text{tr} \Phi(H) &= F_1 \wedge \omega \text{tr}(\bar{\partial}_E \partial H (h^{-1} \frac{\partial h}{\partial t})) \\ &= F_1 \wedge \omega \bar{\partial} \partial \text{tr}(h^{-1} \frac{\partial h}{\partial t}) \end{aligned}$$

$$= -2 F_1 \wedge \omega \bar{\partial} \partial \text{tr}(\Phi(H)) = \Delta \text{tr}(\Phi(H))$$

$$\Delta |\Phi|^2 = -2 F_1 \wedge \omega \bar{\partial} \partial \text{tr}(\Phi H^{-1} \bar{\Phi}^T H)$$

$$= \sum_{i=1}^n 2 \partial_i \bar{\partial}_i \langle \Phi, \Phi \rangle_H$$

$$= 2 \partial_i (\langle D_{\bar{\partial}_i}^{0,1} \Phi, \Phi \rangle + \langle \Phi, D_{\partial_i}^{1,0} \Phi \rangle)$$

$$= 2 (\langle D_{\bar{\partial}_i}^{0,1} \Phi, D_{\bar{\partial}_i}^{0,1} \Phi \rangle + \langle D_{\partial_i}^{1,0} \Phi, D_{\partial_i}^{1,0} \Phi \rangle)$$

$$\begin{aligned} |\partial_H \Phi|^2 \\ \uparrow = |\partial_E \Phi|^2 \end{aligned}$$

$$+ \underbrace{\langle D_{\partial_i}^{1,0} D_{\bar{\partial}_i}^{0,1} \Phi, \bar{\Phi} \rangle + \langle \bar{\Phi}, D_{\bar{\partial}_i}^{0,1} D_{\partial_i}^{1,0} \Phi \rangle}$$

$$2 \langle [F_H(\partial_i, \bar{\partial}_i), \Phi], \bar{\Phi} \rangle + 4 \operatorname{Re} \langle \bar{\Phi}, D_{\bar{\partial}_i}^{0,1} D_{\partial_i}^{1,0} \Phi \rangle$$

$$\frac{\partial}{\partial t} \|\Phi\|_H^2 = \frac{\partial}{\partial t} \operatorname{tr} \Phi H^{-1} \bar{\Phi}^T H = 2 \operatorname{Re} \langle \frac{\partial}{\partial t} \Phi, \bar{\Phi} \rangle + 2 \langle [\Lambda_\omega F_H, \bar{\Phi}], \Phi \rangle$$

#

Lemma Stable bundle admits unique H-E metric

pf If H, K are two H-E metric

$$h = k^{-1} H \quad F_H - F_K = \bar{\partial}_E(h^{-1} \partial_K h)$$

$$0 = \operatorname{tr}(h \bar{\partial}_E (\Lambda_\omega F_H - \Lambda_\omega F_K))$$

$$= -\frac{1}{2} \Delta \operatorname{tr} h + \underbrace{\operatorname{tr}(-\bar{\partial}_E \Lambda_\omega \bar{\partial} h h^{-1} \partial_K h)}_{0 \leq} \Rightarrow \Delta \operatorname{tr} h \geq 0$$

Thus $\operatorname{tr} h = \text{const}$ (compact without boundary)

$\bar{\partial} h = 0$ $\xrightarrow{h \text{ is automorphism}}$

stable $\Rightarrow$ simple $\Rightarrow h = \mu \operatorname{Id}$

#

$H(t), K(t)$ two families of metrics $h(t) = K(t) + H(t)$

$$\textcircled{1} \left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h = 2 \text{tr}(-\sqrt{-1} \Lambda_{\omega} \bar{\partial} h h^{-1} \partial_k h) - 2 \text{tr}(\sqrt{-1} h (\Lambda_{\omega} \bar{F}_k - \Lambda_{\omega} \bar{F}_k)) \\ - \text{tr}(K^{-1} H \underbrace{H^{-1} \frac{\partial H}{\partial t}} - \underbrace{K^{-1} \frac{\partial K}{\partial t}} K^{-1} H)$$

$$\textcircled{2} \left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h^{-1} = 2 \text{tr}(-\sqrt{-1} \Lambda_{\omega} \bar{\partial} h^{-1} h \partial_k h) - 2 \text{tr}(\sqrt{-1} h^{-1} (\Lambda_{\omega} \bar{F}_k - \Lambda_{\omega} \bar{F}_k)) \\ - \text{tr}(H^{-1} K \underbrace{K^{-1} \frac{\partial K}{\partial t}} - \underbrace{H^{-1} \frac{\partial H}{\partial t}} H^{-1} K)$$

$$\textcircled{3} 2(\text{tr} h)^{-1} \text{tr}(-\sqrt{-1} \Lambda_{\omega} \bar{\partial} h h^{-1} \partial_k h) - (\text{tr} h)^{-2} |d \text{tr} h|^2 \geq 0$$

$$\textcircled{4} 2(\text{tr} h^{-1})^{-1} \text{tr}(-\sqrt{-1} \Lambda_{\omega} \bar{\partial} h^{-1} h \partial_k h^{-1}) - (\text{tr} h^{-1})^{-2} |d \text{tr} h^{-1}|^2 \geq 0$$

$$\Rightarrow 2(\text{tr} h + \text{tr} h^{-1})^{-1} \{-\sqrt{-1} \Lambda_{\omega} \bar{\partial} h h^{-1} \partial_k h - \sqrt{-1} \Lambda_{\omega} \bar{\partial} h^{-1} h \partial_k h^{-1}\} \\ \geq \frac{\text{tr} h |d \text{tr} h|^2 + (\text{tr} h^{-1}) |d \text{tr} h^{-1}|^2}{(\text{tr} h + \text{tr} h^{-1}) \text{tr} h \text{tr} h^{-1}}$$

$$\geq \frac{1}{(\text{tr} h + \text{tr} h^{-1})^2} |d(\text{tr} h + \text{tr} h^{-1})|^2 \quad \textcircled{5}$$

Pf of $\textcircled{3}$ normal coordinates $\{z^1, \dots, z^m\}$

$$\langle e_i, \dots, e_r \rangle \quad \langle e_i, e_j \rangle_{H_0} = \delta_{ij}$$

Then for H , WLOG H is diagonal at P , i.e. h

$$\frac{1}{2} \frac{|d \operatorname{tr} h|^2}{(\operatorname{tr} h)^2} = \sum_{i=1}^n \left| \sum_{\alpha=1}^r \bar{\partial}_i h_{\alpha}^{\alpha} \right|^2 \left(\sum_{\alpha=1}^r h_{\alpha}^{\alpha} \right)^{-2}$$

$$= \sum_{i=1}^n \left(\sum_{\alpha=1}^r h_{\alpha}^{\alpha} \right)^{-2} \left| \sum_{\alpha=1}^r (\bar{\partial}_i h_{\alpha}^{\alpha}) (h_{\alpha}^{\alpha})^{-\frac{1}{2}} (h_{\alpha}^{\alpha})^{\frac{1}{2}} \right|^2.$$

$$\text{Schwarz} \leq \sum_{i=1}^n \left(\sum_{\alpha=1}^r |\bar{\partial}_i h_{\alpha}^{\alpha}|^2 (h_{\alpha}^{\alpha})^{-1} \right) \left(\sum_{\alpha=1}^r h_{\alpha}^{\alpha} \right)^{-1}.$$

$$\operatorname{tr}(-\mathbb{F}_1 \wedge \omega \bar{\partial} h \cdot h^{-1} \partial_k h) = \sum_{i=1}^n \bar{\partial}_i h_{\beta}^{\alpha} (h^{-1})_{\alpha}^{\gamma} \partial_i h_{\gamma}^{\beta}$$

$$= \sum_{i=1}^n \sum_{1 \leq \alpha, \beta \leq r} \bar{\partial}_i h_{\beta}^{\alpha} (h_{\alpha}^{\alpha})^{-1} \partial_i h_{\alpha}^{\beta}$$

$$= \sum_{i=1}^n \sum_{\alpha, \beta} |\bar{\partial}_i h_{\beta}^{\alpha}|^2 (h_{\alpha}^{\alpha})^{-1} \geq \sum_{i=1}^n \left| \sum_{\alpha=1}^r \bar{\partial}_i h_{\alpha}^{\alpha} \right|^2 (h_{\alpha}^{\alpha})^{-1}$$

$$\geq \frac{1}{2} \frac{|d \operatorname{tr} h|^2}{\operatorname{tr} h}$$

#

Def Donaldson's distance:

$$\mathcal{G}(H, K) = \operatorname{tr} H^{-1} K + \operatorname{tr} K^{-1} H - 2 \operatorname{rank} E$$

then $\mathcal{G}(H, K) \geq 0$.

$$\mathcal{G}(H, K) = 0 \Leftrightarrow H = K.$$

Lemma. $H(t), K(t)$ the solutions of H-R-M flow,

$$\text{then } \left(\partial - \frac{\partial}{\partial t} \right) \mathcal{G}(H(t), K(t)) \geq 0.$$

If $H(0) = K(0)$, then $H(t) = K(t)$.

pf. $h(t) = k^{-1}(t)h(t)$

$$\left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h = 2 \text{tr} (-F_1 \wedge \omega \bar{\partial} h \cdot h^{-1} \cdot \partial_k h) \stackrel{①}{\geq} 0.$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h^{-1} = 2 \text{tr} (-F_1 \wedge \omega \bar{\partial} h^{-1} \cdot h \cdot \partial_k h^{-1}) \stackrel{②}{\geq} 0 \quad \#$$

lemma. $H(t)$ a solution of H-Y-M flow. $H(0) = H_0$

then (i) $\left(\Delta - \frac{\partial}{\partial t}\right) \log(\text{tr} H_0^{-1} H(t) + \text{tr} H^{-1} H_0) \geq -2 |F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id}|_{H_0}$

(ii) $\Delta \log(\text{tr} H_0^{-1} H + \text{tr} H^{-1} H_0) \geq -2 |F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id}|_{H_0}$

pf. $h(t) = H_0^{-1} H(t)$ ①② gives

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h = & 2 \text{tr} h (F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id}) - 2 \text{tr} h (F_1 \wedge \omega \bar{F}_H - \lambda \text{Id}) \\ & - \text{tr} (H_0^{-1} H(t) \cdot H^{-1} \frac{\partial H}{\partial t}) + 2 \text{tr} (-F_1 \wedge \omega \bar{\partial} h \cdot h^{-1} \cdot \partial_k h) \end{aligned}$$

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right) \text{tr} h^{-1} = & -2 \text{tr} (h^{-1} (F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id})) + 2 \text{tr} (h^{-1} (F_1 \wedge \omega \bar{F}_H - \lambda \text{Id})) \\ & + \text{tr} (H^{-1} \frac{\partial H}{\partial t} \cdot h^{-1}) + 2 \text{tr} (-F_1 \wedge \omega \bar{\partial} h^{-1} \cdot h \cdot \partial_k h^{-1}) \end{aligned}$$

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right) \log(\text{tr} h + \text{tr} h^{-1}) = & (\text{tr} h + \text{tr} h^{-1})^{-1} \left(\Delta - \frac{\partial}{\partial t}\right) (\text{tr} h + \text{tr} h^{-1}) \\ & - \frac{(\text{tr} h + \text{tr} h^{-1})^{-2} |d(\text{tr} h + \text{tr} h^{-1})|^2}{2} \end{aligned}$$

$$= 2(\text{tr} h + \text{tr} h^{-1})^{-1} \left[\text{tr} (h F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id})^* - \text{tr} (h^{-1} (F_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id})) \right]$$

$$+ 2(\text{tr} h + \text{tr} h^{-1}) \left[-F_1 \wedge \omega \bar{\partial} h \cdot h^{-1} \partial H \partial h - F_1 \wedge \omega \bar{\partial} h^{-1} \cdot h \cdot \partial H h^{-1} \right] - * \stackrel{\text{by ⑤}}{\geq} 0$$

$$\geq -2|\bar{F} \wedge \omega \bar{F} H_0 - \lambda \text{Id}|_{H_0}$$

#

11.4 Existence of Long-time Solution of H-Y-M flow

Lemma. If $H(t)$ is the solution of H-Y-M flow

defined for $0 \leq t < T < +\infty$ then $H(t)$ converges in

C^0 -topology to some continuous non-degenerate metric H_T

pf. $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. } \forall s', s'' \in (0, \delta) \quad \sigma(H(s'), H(s'')) < \varepsilon.$

$$H(t) = H(t+s') \quad K(t) = H(t+s'')$$

By the maximum principle of parabolic equation

$$\sup_M \sigma(H(t+s'), H(t+s'')) \leq \sup_M \sigma(H(s'), H(s'')) < \varepsilon.$$

$$\text{then } \forall a, b \in (T-\delta, T) \quad \sup_M \sigma(H(a), H(b)) < \varepsilon.$$

$$\text{thus } t \rightarrow T, \quad H(t) \text{ Cauchy} \quad H(t) \rightarrow H_T$$

$$(\Delta - \frac{\partial}{\partial t}) |\bar{F} \wedge \omega \bar{F} H - \lambda \text{Id}|^2 \geq 0, \text{ thus}$$

$$\sup_M |\bar{F} \wedge \omega \bar{F} H(t) - \lambda \text{Id}|_{H(t)} \leq \sup_M |\bar{F} \wedge \omega \bar{F} H_0 - \lambda \text{Id}|_{H_0} = C_0$$

$$\frac{\partial}{\partial t} \log \text{tr} H_0^{-1} H = \left| \frac{1}{\text{tr} H_0^{-1} H} \text{tr} H_0^{-1} H \quad H^{-1} \frac{\partial H}{\partial t} \right|$$

$$\leq 2 |\bar{F} \wedge \omega \bar{F} H(t) - \lambda \text{Id}|_{H(t)}$$

$$\log \operatorname{tr} H_0^{-1} H(s) - \log r \leq \int_0^s 2 \|F_1 \Lambda_\omega F_{H(t)} - \lambda \operatorname{Id}\|_{H(t)} dt$$

$$\leq 2C_0 s < 2C_0 T$$

$$\Rightarrow \operatorname{tr} H_0^{-1} H(s) \leq r e^{2C_0 T}$$

$$\text{Similarly } \operatorname{tr} H^{-1}(s) H_0 \leq r e^{2C_0 T} \Rightarrow \delta(H_0, H(s)) \leq C_1$$

thus H_T is non-degenerate

#

$$\operatorname{tr}(-F_1 \bar{\partial} h h^{-1} \partial_{H_0} h) \sim |\partial_{H_0} h|_{H_0}^2 = |\bar{\partial} h|_{H_0}^2 \sim |h^{-1} \partial_{H_0} h|_{H_0}^2 \leq |\eta|_{H_0}^2$$

$$(6) \quad \frac{1}{2} \Delta \operatorname{tr} h = \operatorname{tr}(-F_1 \Lambda_\omega \bar{\partial} h h^{-1} \partial_{H_0} h) + \operatorname{tr} F_1 h (\Lambda_\omega \bar{F}_{H_0} - \Lambda_\omega \bar{F}_h)$$

$$(7) \quad (\Delta - \frac{\partial}{\partial t}) \operatorname{tr} h = 2 \operatorname{tr}(-F_1 \bar{\partial} h h^{-1} \partial_{H_0} h) + \operatorname{tr}(h |F_1 \Lambda \bar{F}_{H_0} - \lambda \operatorname{Id}|)$$

$$\geq C_2 |\eta|_{H_0}^2 - \|F_1 \Lambda_\omega \bar{F}_{H_0} - \lambda \operatorname{Id}\| \operatorname{tr} h$$

$$(8) \quad (\Delta - \frac{\partial}{\partial t}) |\eta|^2 \geq |\nabla \eta|^2 - C_4 |\eta|^2 - C_5 \quad C_2, C_4, C_5 \text{ depending only on } \operatorname{tr} h + \operatorname{tr} h^{-1}$$

$$(\Delta - \frac{\partial}{\partial t}) (|\eta|^2 + C_6 \operatorname{tr} h) \geq |\nabla \eta|^2 + |\eta|^2 - C_7$$

$$\text{maximum principle } \Rightarrow \sup_{M \times (0, T)} |\eta|^2 \leq C_8$$

Prop. H-Y-M flow has a unique solution $H(t)$ existing for $0 \leq t < \infty$

11.5 Donaldson's Functional

$$\mu(k, H) = \int_0^1 \int_M \text{tr} \left[(F_H \wedge \omega F_H - \lambda \text{Id}) H^{-1}(s) \frac{\partial H}{\partial s} \right] \frac{\omega^n}{n!} ds$$

where $H(s)$ is a path connecting k and H $H(0) = k$
 $H(1) = H$

Lemma. $\mu(k, H)$ is independent of the choice of H .

Lemma $H(\tau, s)$ $H(\tau, 0) = k$ $H(\tau, 1) = H(\tau)$

$$\begin{aligned} \frac{d}{d\tau} \mu(k, H(\tau)) &= \frac{d}{d\tau} \int_0^1 \int_M \left(\text{tr} (F_H \wedge \omega F_{H(\tau, s)} - \lambda \text{Id}) H^{-1} \frac{\partial H}{\partial s} \right) \frac{\omega^n}{n!} ds \\ &= \int_M \left(\text{tr} (F_H \wedge \omega F_H - \lambda \text{Id}) H^{-1} \frac{\partial H}{\partial \tau} \right) \frac{\omega^n}{n!} \Big|_{s=0}^1 \\ &= \int_M \underbrace{\text{tr} (F_H \wedge \omega F_{H(\tau)} - \lambda \text{Id}) H^{-1} \frac{\partial H}{\partial \tau} \frac{\omega^n}{n!}}_{= -2 \int_M |F_H \wedge \omega F_{H(\tau)} - \lambda \text{Id}|^2} \end{aligned}$$

thus H-Y-M flow is gradient flow of Donaldson's functional

Lemma. ① $\mu(H_0, H') + \mu(H', H'') = \mu(H_0, H'')$

② $\mu(H, aH) = 0$

Lemma. $H(0)$ is a critical point of $\mu(k, H)$ iff $H(0)$ is a H-E metric and $\mu(k, H)$ attains the absolute minimum at $H(0)$

11.6 Donaldson-Uhlenbeck-Yau's Thm

$$H_0 = Ke^f. \quad \text{tr}(\bar{F}_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id}) \\ = \text{tr}(\bar{F}_1 \wedge \omega (F_k + \partial \bar{\partial} f \text{Id}) - \lambda \text{Id})$$

$$\text{solve } \frac{1}{2} \Delta f = \bar{F}_1 \wedge \omega \partial \bar{\partial} f = \frac{1}{\text{rank}(E)} \text{tr}(\bar{F}_1 \wedge \omega \bar{F}_1 - \lambda \text{Id})$$

thus by conformal transform, we can assume

$$\text{tr}(\bar{F}_1 \wedge \omega \bar{F}_{H_0} - \lambda \text{Id}) = 0.$$

$$H\text{-Y-M flow } \begin{cases} H^{-1} \frac{\partial H}{\partial t} = -2(\bar{F}_1 \wedge \omega \bar{F}_1 - \lambda \text{Id}) \\ H(t)|_{t=0} = H_0. \end{cases}$$

$$\bar{\Phi}(H(t)) = \bar{F}_1 \wedge \omega \bar{F}_{H(t)} - \lambda \text{Id}.$$

$$\text{Recall: } \left(\Delta - \frac{\partial}{\partial t} \right) \text{tr}(\bar{\Phi}(H(t))) = 0.$$

$$\text{then by initial condition } \text{tr}(\bar{\Phi}(H(t))) = \text{tr}(\bar{\Phi}(H_0)) = 0$$

$$\Rightarrow \text{tr} \left(h^{-1} \frac{\partial h}{\partial t} \right) = \frac{\partial}{\partial t} \log \det h \quad \Rightarrow \det h(t) = \det h(0) = 1$$

$$\text{"}$$

$$-2 \text{tr}(\Phi(H(t))) = 0$$

$$\text{Let } S(t) = \log h(t).$$

$$\text{trs} = \log \det h(t) = 0.$$

$$H = H_0 e^S \quad \det(H_0^{-1} H) = 1. \quad \text{trs} = 0.$$

$$\mathcal{U}(H_0, H) = \int_0^1 \int_M \text{tr} \left[(\sqrt{1 + \hbar \omega} \bar{F}_{H(t)} - 1 \text{Id}) S \right] \frac{\omega^n}{n!} dt$$

$$= \int_M \text{tr} (S F_1 \wedge \omega \bar{F}_{H_0}) + \langle \psi(s) (\bar{\partial}_E S), \bar{\partial}_E S \rangle_{H_0} \frac{\omega^n}{n!}$$

Chapter 12 Calabi-Yau Thm

12.1 Calabi-Yau's Thm

Thm (Calabi-Yau) (M, ω_0) Kähler. then for any real (1,1) form $\psi \in 2\pi C_1(M) \exists!$ Kähler metric $\omega \in [\omega_0]$ st. $\rho(\omega) = \psi$.

$$\rho(\omega_0) - \psi = \bar{\partial} \partial F \quad F \in C^\infty(M, \mathbb{R})$$

$$\omega = \omega_0 + \bar{\partial} \partial \psi. \text{ then } \rho(\omega) = \psi \Leftrightarrow \rho(\omega) - \rho(\omega_0) = \psi - \rho(\omega_0)$$

$$\Leftrightarrow -\bar{\partial} \partial \log \frac{\omega^n}{\omega_0^n} = -\bar{\partial} \partial F$$

$$\Leftrightarrow \log \frac{\omega^n}{\omega_0^n} = F + C$$

$$\Leftrightarrow e^F = \frac{(\omega_0 + \bar{\partial} \partial \psi)^n}{\omega_0^n}$$

Complex Monge-Ampère

$$\text{Locally } (z^1, \dots, z^n). \quad \omega_0 = \bar{\partial} \partial (g_0)_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

$$\det((g_0)_{i\bar{j}} + \psi_{i\bar{j}}) = e^F \det((g_0)_{i\bar{j}})$$

$$\text{Let } (g_0)_{i\bar{j}} = (\psi_0)_{i\bar{j}}$$

$$\det((\psi_0 + \psi)_{i\bar{j}}) = e^F \det((g_0)_{i\bar{j}})$$

12.2 Complex M-A Equation

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^F \omega_0^n$$

$$F=0 \Rightarrow \varphi=c. \text{ solved}$$

$$\frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n} = e^{tF + G_t} \quad (*)$$

$$\int_M (e^{tF + G_t} - 1) \omega_0^n = 0$$

Define $T = \{t \in [0,1] \mid (*) \text{ can be solved}\}$

$$\left. \begin{array}{l} (1) T \neq \emptyset \xrightarrow{T=0} \checkmark \\ (2) T \text{ is open} \\ (3) T \text{ is closed} \end{array} \right\} \Rightarrow T = [0,1]$$

$$(2): B_1 = \{\varphi \in C^{k,\alpha} \mid \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \int_M \varphi \frac{\omega_0^n}{n!} = 0, k \geq 2\}$$

$$B_2 = \{\varphi \in C^{k-2,\alpha} \mid \int_M e^\varphi \frac{\omega_0^n}{n!} = 1\}$$

$$\bar{\Psi}: B_1 \rightarrow B_2, \quad \varphi \mapsto \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n}$$

differential $D\bar{\Psi}_{\varphi_{t_0}}: T_{\varphi_{t_0}} B_1 \rightarrow T_{\bar{\Psi}(\varphi_{t_0})} B_2$

$$u = \frac{\partial \varphi_t}{\partial t} \Big|_{t=t_0}$$

$$T_{\varphi_{t_0}} B_1 = \{u \in C^{k,\alpha} \mid \int_M u \frac{\omega_0^n}{n!} = 0\}$$