Brief Symplectic Topology

Ref «Intro to Symplectic Topology», «J-hol Curves and Quantum Cohomology»
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I. Foundation

1. Hamiltonian Mechanics

Action Integral $I(x)=\int_{t_0}^{t_1} L(t,x,\dot{x})dt$

$$\xrightarrow[\text{method}]{\text{variation}} \frac{d}{dt}\frac{\partial L}{\partial v}=\frac{\partial L}{\partial x} \quad (\text{E-L eq})$$

Legendre condition: $\left(\frac{\partial^2 L}{\partial v_j \partial v_k}\right)$ nonsingular

$\Rightarrow$ under Legendre transform $y_k=\frac{\partial L}{\partial v_k}(x,v) \xrightarrow{\text{IFT}} v_k=G_k(t,x,y)$

Define Hamiltonian $H=\sum_{j=1}^{n} y_j v_j - L \Rightarrow \frac{\partial H}{\partial x_k}=-\frac{\partial L}{\partial x_k} \quad \frac{\partial H}{\partial y_k}=G_k$

$$\xRightarrow{\text{E-L}} \frac{\partial H}{\partial y}=\dot{x}, \quad \frac{\partial H}{\partial x}=-\dot{y} \quad (*)$$

When H independent of t. let $z=(x_1,\cdots x_n, y_1,\cdots y_n)$

$$(*) \Leftrightarrow \dot{z}=-J_0\nabla H(z)=X_H(z)$$

if dependent $H=H_t$ let $\phi_H^{t,t_0}(z_0)=z(t)$

where $z(t)$ fits $(*)$ with $z(t_0)=z_0$

$$\Rightarrow \frac{\partial}{\partial t}\phi_H^{t,t_0}=X_{H_t}\circ\phi_H^{t,t_0}$$

(Hamiltonian flow generated by H)

Fact: ϕ_H^{t,t_0} is a symplectomorphism

$Sp(2n)=\{\Psi\in\mathbb{R}^{2n\times 2n} \mid \Psi^T J_0 \Psi = J_0\}$

Lie group. with Lie algebra $sp(2n)=\{-J_0 S \mid S^T=S\}$

2. Symplectic Topology of Euclidean Space

① Weinstein Conjecture

$H\ \mathbb{R}^{2n} \to \mathbb{R}$ Hamilton. c regular value

Assume $S \triangleq H^{-1}(c)$ compact. $z \in S$ $\quad L_z \triangleq \{ J_0 v \mid v \perp T_z S \} \subset T_z S$

$$\underset{\substack{\uparrow X_H(z) \in L_z}}{\text{characteristic foliation}} \xleftarrow[\text{curve}]{\text{integral}} \text{1-dim distribution}$$

sol. of Ham eq $\leftrightarrow$ leaves. (characteristic)

Thm. (viterbo) Every hypersurface in $\mathbb{R}^{2n}$ with contact type has a closed characteristic

② Non-Squeezing Theorem

$\omega_0 = \sum dx_i \wedge dy_i$ $\qquad B^{2n}(r) = \{ z \in \mathbb{R}^{2n} \mid |z| < r \}$

$$Z^{2n}(r) = \underbrace{B^2(r)}_{\text{symplectic disc}} \times \mathbb{R}^{2n-2} = \{ z \in \mathbb{R}^{2n} \mid \sqrt{x_1^2 + y_1^2} \le r \}$$

Thm (Non-Squeezing) If $\exists$ symplectic embedding $B^{2n}(r) \to Z^{2n}(R) \Rightarrow r \le R$

Symplectic Camel: $\underset{\text{wall}}{W} = \{ y_1 = 0 \}$ $\qquad \underset{\text{hole}}{H_\varepsilon} = \{ |z| < \varepsilon.\ z \in W \}$

B: unit ball

$\{\phi_t\}$ family of symplectic embedding $\quad \begin{cases} \phi_t(B) \subset (\mathbb{R}^{2n} \setminus W) \cup H_\varepsilon \\ \phi_0(B) \subset \{ y_1 < 0 \} \\ \phi_1(B) \subset \{ y_1 > 0 \} \end{cases}$

$\varepsilon \le 1$: non-existence!

3 Linear Symplectic Geometry

(V,ω) symplectic vector space

$Sp(V,\omega)=\{\Psi\in GL(V) \mid \Psi^*\omega=\omega\}$

$\Rightarrow sp(V,\omega)=\{A\in End(V) \mid \omega(A\cdot,\cdot)+\omega(\cdot,A\cdot)=0\}$

Lemma. ① $Sp(2n)\cap O(2n)=Sp(2n)\cap GL(n,\mathbb{C})=O(2n)\cap GL(n,\mathbb{C})=U(n)$

② $\pi_1(U(n))=\pi_1(Sp(2n))=\mathbb{Z}$

Thm (Maslov Index) $\exists!\ \mu$ assigns each loop $\Psi: \mathbb{R}/\mathbb{Z}\to Sp(2n)$

to an integer $\mu(\Psi)$. s.t.

① Ψ_1, Ψ_2 are homotopic $\Leftrightarrow \mu(\Psi_1)=\mu(\Psi_2)$

② $\mu(\Psi_1\Psi_2)=\mu(\Psi_1)+\mu(\Psi_2)$

③ $n=n'+n''$. $Sp(2n')\oplus Sp(2n'')\subseteq Sp(2n)\Rightarrow \mu(\Psi'\oplus\Psi'')=\mu(\Psi')+\mu(\Psi'')$

④ $\Psi(t)=\begin{pmatrix}\cos 2\pi t & -\sin 2\pi t\\ \sin 2\pi t & \cos 2\pi t\end{pmatrix}\Rightarrow \mu(\Psi)=1$

Affine symplectomorphism. $\psi:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ $\quad \psi(z)=\Psi z+z_0$. $\quad \Psi\in Sp(2n)$

$\downarrow$

$ASp(2n)$

Thm. $\psi\in ASp(2n)$ $\psi(B^{2n}(r))\subseteq Z^{2n}(R)\Rightarrow r\le R$

Linear Non-Squeezing Property: $\forall$ linear symplectic ball $B(r)$

linear symplectic cylinder $Z(R)$. then $\Psi B\subset Z\Rightarrow r\le R$

Thm (Affine Rigidity) Ψ, Ψ^{-1} have linear non-squeezing property,

then Ψ is symplectic or anti-symplectic.

Linear Symplectic Width. $A \subset \mathbb{R}^{2n}$ $w_L(A) = \sup\{\pi r^2 \mid \exists \psi \in Asp(\mathbb{R}^{2n}),\ \psi(B^{2n}(r)) \subset A\}$

① (monotonicity): $\psi \in Asp(2n)$ $\psi(A) \subset B \Rightarrow w_L(A) \le w_L(B)$

② $w_L(\lambda A) = \lambda^2 w_L(A)$

Consider an ellipsoid $E = \{z \in \mathbb{R}^{2n} \mid \underbrace{Q(z)}_{\text{positive quadratic}} \le 1\}$ centred at 0

Thm. $\Psi: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ linear. then

Ψ preserves $w_L(E) \iff \Psi$ is symplectic or anti-symplectic.

(pf. Use affine rigidity)

Lemma. $\forall$ compact ellipsoid $E = \{w \in \mathbb{R}^{2n} \mid \sum_{i,j=1}^{2n} a_{ij} w_i w_j \le 1\}$

$\exists \Psi \in Sp(2n)$, $r = (r_1, \cdots, r_n)$ with $0 < r_1 \le \cdots \le r_n$

s.t. $\Psi(E) = E(r) \triangleq \{z \in \mathbb{C}^n \mid \sum_{i=1}^{n} \left|\frac{z_i}{r_i}\right|^2 \le 1\}$

where $r = (r_1, \cdots, r_n)$ is called the symplectic spectrum of E.

(pf omitted)

Thm. $E \subset \mathbb{R}^{2n}$: ellipsoid centred at 0 then

$$w_L(E) = \sup\{w_L(B) \mid B \subset E \text{ is an affine symplectic ball}\}$$
$$= \inf\{w_L(Z) \mid Z \supset E \text{ is an affine symplectic cylinder}\}$$

Especially if E has spectrum $r = (r_i)_{i=1}^n$ then $w_L(E) = \pi r_1^2$

(pf. $\Psi \in Sp(2n)$. $\Psi E = E(r) \Rightarrow \Psi^{-1} B^{2n}(r_1) \subset E \subset \Psi^{-1} Z^{2n}(r_1)$

$$\Rightarrow \inf_{Z \supset E} w_L(Z) \le \pi r_1^2 \le \sup_{B \subset E} w_L(B)$$

Now let $B \subset E$ be an affine symplectic ball of radius r.

then $\Psi B \subset \Psi E \subset Z^{2n}(r_1) \Rightarrow r \le r_1$

$Z \supset E$ - - - - cylinder - - - R

$\parallel$

$\psi(Z^{2n}(R))$

then $B^{2n}(r_1) \subset \Psi E \subset \Psi Z = \Psi(\psi(Z^{2n}(R))) \Rightarrow r_1 \le R$

$$\Rightarrow \sup_{B \subset E} w_L(B) \le \pi r_1^2 \le \inf_{Z \supset E} w_L(Z). \qquad \#)$$

The linear symplectic width extends to symplectic capacity

Def $\mathcal{M}(2n)$. classes of symplectic $2n$-mfld (possibly with bdry)

Then <u>symplectic capacity</u> is a map $c : \mathcal{M}(2n) \to [0, \infty]$ s.t.

① $(M, \omega_M) \hookrightarrow (N, \omega_N)$, then $c(M, \omega_M) \le c(N, \omega_N)$

② $\forall \lambda > 0$. $c(M, \lambda \cdot \omega) = \lambda \cdot c(M, \omega)$

③ $c(B^{2n}(1), \omega_{std}) = c(Z^{2n}(1), \omega_{std}) = 1$

Rmk. If Gromov non-squeezing holds, then

$c_G(M, \omega) = \sup\{r > 0 \mid \exists (B^{2n}(r), \omega_{std}) \hookrightarrow (M, \omega)\}$ is a symplectic capacity

II Persistence Modules

1. Basic Def

Def. A persistence module is a pair (V, π)

$V = \{V_t\}_{t \in \mathbb{R}}$: family of f.d. vector space over $\mathbb{F}$

$\pi = \{\pi_{s,t} : V_s \to V_t\}_{s \le t}$: family of linear maps

s.t. (1) $\forall s \le t \le r \quad \pi_{s,r} = \pi_{t,r} \circ \pi_{s,t}$ (Persistence)

(2) For all but finite $t \in \mathbb{R}$, $\exists$ a nbhd U of t

s.t. $\pi_{s,r}$ is an isomorphism for any $s < r$ in U.

(3) $\forall t \in \mathbb{R}$ and $s \le t$ sufficiently close to t. $\pi_{s,t}$ is an isomorphiom (Semicontinuity)

(4) $\exists s_- \in \mathbb{R}$ s.t. $V_s = 0$ $(\forall s \le s_-)$

$\{P_{s,t} = \mathrm{Im}(\pi_{s,t})\}$: persistent homology

Rmk. ① $\pi_{t,t} = \mathrm{Id}$

② $\exists s_+$ s.t $\forall s \ge s_+$ $V_s \cong V_\infty$.

Ex. ① X closed mfld. $f: X \to \mathbb{R}$ Morse $\quad V_t = H_k(\{f < t\})$

② (X, d) finite $\forall \alpha > 0$. $R_\alpha(X)$: Rips complex

vertices: points in X

σ. k-simplex if $d(x_i, x_j) < \alpha$

$\sigma = [x_0, \cdots x_k]$

$V_\alpha(X) = H_*(R_\alpha(X))$ $\quad \pi_{\alpha\beta} = (i_{\alpha\beta})_*$: Rips Module.

Morphism $A : (V, \pi) \to (V', \pi')$ is a family of maps

$A_t : V_t \to V_t'$ s.t.

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s,t}"] \arrow[d, "A_s"'] \arrow[dr, phantom, "\circlearrowleft"] & V_t \arrow[d, "A_t"] \\ V_s' \arrow[r, "\pi'_{s,t}"'] & V_t' \end{tikzcd}$$

Ex ① (V, π), $\delta \in \mathbb{R}$ $\Rightarrow (V[\delta], \pi[\delta])$ $\quad (V[\delta])_t = V_{t+\delta}$

$(\pi[\delta])_{s,t} = \pi_{s+\delta, t+\delta}$

$\Phi^\delta : (V, \pi) \to (V[\delta], \pi[\delta])$ $\quad \Phi^\delta_t = \pi_{t, t+\delta}$

② $(a, b] \Rightarrow \mathbb{F}(a, b]$ interval module

$$\mathbb{F}(a,b]_t = \begin{cases} \mathbb{F}, & t \in (a, b] \\ 0, & \text{otherwise} \end{cases} \qquad \pi_{s,t} = \begin{cases} Id, & s, t \in (a, b] \\ 0 & \cdots \end{cases}$$

$\mathbb{F}(1,2] \to \mathbb{F}(1,3]$: not morphism

$\mathbb{F}(2,3] \to \mathbb{F}(1,3]$: morphism

Def. (V, π) (W, θ) : δ-interleaved if $\exists$ morphisms $\quad V \xrightarrow{F} W[\delta]$, $\quad W \xrightarrow{G} V[\delta]$

s.t.
$$\begin{tikzcd} V \arrow[r, "F"] \arrow[rr, bend right, "\Phi^{2\delta}_V"'] & W[\delta] \arrow[r, "G[\delta]"] \arrow[d, phantom, "\circlearrowleft"] & V[2\delta] \\ & {} & \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, "G"] \arrow[rr, bend right, "\Phi^{2\delta}_W"'] & V[\delta] \arrow[r, "F[\delta]"] \arrow[d, phantom, "\circlearrowleft"] & W[2\delta] \\ & {} & \end{tikzcd}$$

Rmk. ① (V,π) (W,θ) δ-interleaved $\Leftrightarrow \dim V_\infty = \dim W_\infty$

② ——— . then ··· δ'-interleaved $\forall \delta' > \delta$

③ V,W δ_1 ··· , W,Z δ_2 ···, then V,Z $\delta_1+\delta_2$ ···

Def. $d_{int}(V,W) = \{\inf \delta > 0 \mid (V,\pi)\ (W,\theta)\ \delta\text{-interleaved}\}$

(is actually a metric on isomorphism classes with same V_∞)

Ex ① $d_{int}(\mathbb{F}[a,b], \mathbb{F}[c,d]) = \min\left(\max\left(\frac{b-a}{2}, \frac{d-c}{2}\right), \max(|a-c|, |b-c|)\right)$

② M closed mfld, $f: M \to \mathbb{R}$ Morse $\quad V_t(f) = H_*(\{f<t\})$

$\Rightarrow V(f-\delta) = V(f)[\delta]$

$\forall f,g$ $\delta = \|f-g\|$ $\quad f-\delta \le g \Rightarrow$ natural morphism $F: V(g) \to V(f)[\delta]$

similarly $G: V(f) \to V(g)[\delta]$

$f-2\delta \le g-\delta \le f$

$\downarrow$

$$\begin{tikzcd} V(f) \arrow[r, "G"] \arrow[rr, bend right, "\Phi^{2\delta}_{V(f)}"'] & V(g)[\delta] \arrow[r, "F[\delta]"] & V(f)[2\delta] \end{tikzcd}$$

$\swarrow$

$d_{int}(V(f), V(g)) \le \delta$

$\downarrow$ $V(f) \cong V(\varphi^* f)$

$$d_{int}(V(f), V(g)) \le \inf_{\varphi \in \mathrm{Diff}(M)} \|f - \varphi^* g\|$$

X, Y finite surjective correspondence $C: X \rightrightarrows Y$

is a set $C \subset X \times Y$. s.t. $\mathrm{proj}_X(C) = X$. $\mathrm{proj}_Y(C) = Y$

For $(X, \rho) \overset{C}{\rightrightarrows} (Y, r)$. define $\underset{\text{distortion}}{\mathrm{dis}}(C) = \max\limits_{(x,y),(x',y') \in C} |\rho(x, x') - r(y, y')|$

Gromov-Hausdorff distance: $\underset{\text{actually a metric on the isometry classes of finite metric space}}{d_{GH}}((X, \rho), (Y, r)) = \frac{1}{2} \min\limits_{C} \mathrm{dis}(C)$

Thm. $(X, \rho) \to$ Rips module $V(X, \rho)$

$\Rightarrow d_{GH}((X, \rho), (Y, r)) \geq \frac{1}{2} d_{int}(V(X, \rho), V(Y, r))$

2 Barcodes

Def. A barcode B is a finite multiset of intervals. $\{(\underset{(a,b].\ b \in \mathbb{R}_{\infty}}{I_i}, \underset{\text{multiplicity}}{m_i})\}_{i=1}^{N}$

Thm. (Normal Form) (V, π) persistence module. then $\exists!$ $\underset{\text{barcode of } V\ B(V)}{\underline{\text{barcode}}}$

$\{(I_i, m_i)\}_{i=1}^{N}$ s.t. $V \overset{iso}{=} \bigoplus\limits_{i=1}^{N} \mathbb{F}(I_i)^{m_i}$

Pf omitted here.

$I=(a,b]$ $\quad I^{-\delta} \triangleq (a-\delta, b+\delta]$

B: barcode $\quad B_\varepsilon$: set of bars with length $>\varepsilon$

Def ① A <u>matching</u> between two multisets X, Y is a bijection

$$\begin{array}{ccc} \mu: X' & \to & Y' \\ \cap & & \cap \\ X & & Y \end{array} \qquad X' = \mathrm{coIm}\,\mu, \quad Y' = \mathrm{Im}\,\mu$$

② A <u>δ-matching</u> between B and C is a matching μ

s.t. (i) $B_{2\delta} \subset \mathrm{coIm}\,\mu$ $\quad$ (ii) $C_{2\delta} \subset \mathrm{Im}\,\mu$

(iii) If $\mu(I)=J$, then $I \subset J^{-\delta}$, $J \subset I^{-\delta}$

③ bottleneck distance $d_{bot}(B, C) = \inf\{\delta > 0 \mid \exists \text{ a } \delta\text{-matching } B \to C\}$

actually a distance on space of barcodes with same numbers of infinite rays

Thm (Isometry thm) $V \mapsto B(V)$ is an isometry

i.e. $d_{int}(V, W) = d_{bot}(B(V), B(W))$

III Persistent Homology

1. Some symplectic geometry

$$H: M\times[0,1]\to\mathbb{R} \rightsquigarrow \iota_{X_H}\omega = -dH$$

$$\downarrow$$

ϕ_H^t. $\quad \phi = \phi_H^1$: Hamiltonian diffeomorphism

$\mathrm{Ham}(M,\omega)$

we normalize H_t s.t. $\int_M H_t \frac{\omega^n}{n!} = 0$

Thm (Banyaga) ① $\{\gamma_t\}_{t\in[0,1]}$ smooth path in $\mathrm{Ham}(M,\omega)$

then $\exists F: M\times[0,1]\to\mathbb{R}$ s.t. $\frac{d}{dt}(\gamma_t(x)) = X_{F_t}(\gamma_t(x))$

② (M,ω) closed then $\mathrm{Ham}(M,\omega)$ is a simple group

2. Hofer's geometry

$$\mathrm{Lie}(\mathrm{Ham}(M,\omega)) \longleftrightarrow \frac{d}{dt}\Big|_{t=0}(\gamma_t(x)) \overset{①}{\longleftrightarrow} X_{F(0,x)} \Leftrightarrow \underset{\mathfrak{g}}{\underset{\parallel}{C^\infty(M)/\mathbb{R}}}$$

norm on $\mathfrak{g}$ $\|F\| = \max F - \min F$

$\uparrow$

Hofer length: $\mathrm{length}(\gamma) = \int_0^1 \|F_t\| dt$ $\quad$ adjoint invariant

Hofer's metric on $\mathrm{Ham}(M,\omega)$: $d_{Hofer}(\phi,\psi) = \inf\limits_{\substack{\gamma \text{ connects} \\ \phi,\psi}} \mathrm{length}(\gamma)$

Hofer norm: $\|\phi\|_{Hofer} = d_{Hofer}(\phi, \mathrm{Id}_M)$

It's easy to check d_{Hofer} is a bi-variant pseudo-metric.

Thm. d_{Hofer} is non-degenerate. thus actually a metric.

(pf later) $\exists \phi \in Ham(M,\omega)$

$\phi(A) \cap A = \phi$

Def. For a displaceable subset $A \subset M$.

displacement energy $e(A) = \inf \{ \|\phi\|_{Hofer} \mid \begin{matrix} \phi \in Ham(M,\omega) \\ \phi(A) \cap A = \phi \end{matrix} \}$

(clearly $e(A) = e(\psi(A))$ for $\psi \in Ham(M,\omega)$)

Cor $e(A) > 0$ for any open subset $A \neq \phi$

(pf. Lem $\|[\phi, \psi]\|_{Hofer} \leq 2\|\psi\|_{Hofer}$

then take non-commuting $f, g \in Ham(M,\omega)$ supported in A

and $\theta \in Ham(M,\omega)$ with $\theta(A) \cap A = \phi$.

$\theta g^{-1} \theta^{-1}$ is supported in $\theta(A) \Rightarrow$ commutes with $f \rightarrow$ $\theta g \theta^{-1} f = f \theta g \theta^{-1}$, $f^{-1} \theta g \theta^{-1} = \theta g \theta^{-1} f^{-1}$

$\Rightarrow [f, [g, \theta]] = f(g\theta g^{-1}\theta^{-1}) f^{-1} (\theta g \theta^{-1} g^{-1}) = f g f^{-1} g^{-1} = [f, g]$

$\Rightarrow 4\|\theta\|_{Hofer} \geq \|[f,g]\|_{Hofer}$ $\Rightarrow e(A) \geq \frac{1}{4} \|[f,g]\|_{Hofer} > 0$)

3. Hamiltonian Persistence Modules

(1) C-Z Index

Def For a path $\Phi: [0,1] \to Sp(2n)$ $(\Phi(0) = Id)$, $t \in [0,1]$ is a <u>crossing</u> if $|\Phi(t) - Id| = 0$. Let $S(t) = \Omega \dot{\Phi}(t) \Phi(t)^{-1}$ ($\xrightarrow{\text{check}}$ symmetric)

then at a crossing t, $\Gamma(\Phi, t) \triangleq S(t)|_{\ker(\Phi(t) - Id)}$ gives a quadratic form. t is regular if $\Gamma(\Phi, t)$ non-degenerate.

If Φ only has regular crossings, then define

Conley-Zehnder index $\mu_{CZ}(\Phi) \triangleq \frac{1}{2} \operatorname{sign}(\Gamma(\Phi, 0)) + \sum_{\substack{t \in (0,1) \\ \text{crossing}}} \operatorname{sign}(\Gamma(\Phi, t))$

Ex. ① $\mathbb{C}^1$, $q + ip$ $\quad H(z) = \pi\alpha |z|^2$ $(\alpha \notin \mathbb{Z})$ $\Rightarrow X_H(q,p) = \begin{pmatrix} & 2\pi\alpha \\ -2\pi\alpha & \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$

$\Rightarrow \phi_H^t(z) = e^{-2\pi\alpha i t} z$

$\rightsquigarrow \Phi: [0,1] \to Sp(2)$

$$t \mapsto \begin{pmatrix} \cos(2\pi\alpha t) & \sin(2\pi\alpha t) \\ -\sin(2\pi\alpha t) & \cos(2\pi\alpha t) \end{pmatrix}$$

crossing $\Leftrightarrow t = \frac{k}{\alpha}$ $(k \in \mathbb{Z}^*)$ $\quad \Gamma(\Phi, t) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 2\pi\alpha \\ -2\pi\alpha & \end{pmatrix}$

$= 2\pi\alpha I_2$

$$\Rightarrow \mu_{CZ}(\Phi) = \begin{cases} -2|\lceil \alpha \rceil| - 1, & \alpha < 0 \\ 2\lfloor \alpha \rfloor + 1, & \alpha > 0 \end{cases}$$

② Let $\operatorname{Ind}(\Phi) = n - \mu_{CZ}(\Phi)$ and if Φ is generated by a sufficiently small quadratic H, then $\operatorname{Ind}(\Phi) = \underline{\operatorname{ind}}(H)$

Morse ind.

③ If Φ is a loop, then $\sum_{t \in [0,1)} \operatorname{sign}(\Gamma(\Phi, t))$ is the Maslov index

(2) Filtered Hamilton-Floer Theory

For symplectic mfld M with $\pi_2 M = 0$ and consider $\mathbb{Z}_2$-homology

$\mathcal{L}M = \{\text{smooth contractible loop}\}$ $\qquad A : \mathcal{L}M \to \mathbb{R}$

$T_x \mathcal{L}M \leftrightarrow$ tangent v.f. along x $\qquad x \mapsto -\int_D \omega$

D: disc spanning x.

$\downarrow$ metric

$\langle \xi, \eta \rangle = \int_0^1 \omega(\xi(t), J(t)\eta(t))\,dt$

$J(t)$ a loop of compatible AC structure.

Fix a Hamiltonian $H: \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ $\quad A_H \triangleq A + \int_0^1 H(t,x)$

$\downarrow$ action functional

Fact: $dA_H(\xi) = \int_0^1 dH(\xi) - \omega(\xi, \dot{x}(t)) = \int_0^1 \omega(\xi, X_H - \dot{x})$

$\downarrow$

Prop. $\mathrm{Crit}(A_H) \leftrightarrow$ contractible 1-period orbit of Ham. flow

$\overset{\cong}{P}$

$\phi = \varphi_H^1$

$x \in P$ called non-deg if $\phi_* : T_{x(0)}M \to T_{x(0)}M$ doesn't have eigenvalue 1

$\updownarrow$ in fact

non-deg as crit pt

then $(\varphi_H^t)_*$ lifts to $\Phi: [0,1] \to Sp(2n)$

$\Phi(0) = Id$

$\Phi(1)$ doesn't have eigenvalue 1

$\downarrow$

define $\mathrm{Ind}(x) \triangleq \mathrm{Ind}(\Phi)$ (well-defined!)

Gradient trajectory: $u(s,t): \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to M$

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0$$

Define as in Morse theory $CF_k(M,H) = \mathrm{span}_{\mathbb{Z}_2}\langle x\in P \mid \mathrm{Ind}(x)=k\rangle$

$$\partial_k : CF_k(M,H) \to CF_{k-1}(M,H)$$
$$x \mapsto \sum_{\mathrm{Ind}(y)=k-1} n(x,y)\, y$$
$$HF_k(M,H) = \ker\partial_k / \mathrm{Im}\,\partial_{k+1}$$

Or fitered ver $\quad CF_k^{\lambda}(M,H) = \mathrm{span}_{\mathbb{Z}_2}\langle x\in P \mid \mathrm{Ind}(x)=k,\ \mathcal{A}_H(x)<\lambda\rangle$

$$HF_k^{\lambda}(M,H) = \ker\partial_k / \mathrm{Im}\,\partial_{k+1}$$

$\lambda\le\eta$: $\iota_{\lambda,\eta} : HF_k^{\lambda}(M,H) \to HF_k^{\eta}(M,H)$ natural map

$$\iota_{\lambda,\theta} = \iota_{\eta,\theta}\circ\iota_{\lambda,\eta}$$

Prop (Schwarz) For normalized Hamitonian H. $HF_k^{\lambda}(H)$ only depends on $\phi=\phi_H^1$. then write $HF_k^{\lambda}(\phi) \triangleq HF_k^{\lambda}(H)$

Def. For (M,ω) with $\pi_2(M)=0$. $\phi=\phi_H^1$ as above then $\left\{\{HF_*^{\lambda}(\phi)\}_{\lambda\in\mathbb{R}},\ \{\iota_{\lambda,\eta}\}_{\lambda\le\eta}\right\}$ is the Hamitonian persistence module. denoted as $\mathbb{HF}_*(\phi)$

Barcode of $\mathbb{HF}_*(\phi)$ is $B_*(\phi)$ $\qquad B(\phi) \triangleq \bigcup_{*\in\mathbb{Z}} B_*(\phi)$

Ex ① H · C^∞-small autonomous Morse. then one can show that HF complex reduces to Morse complex.

$\Rightarrow B(\phi)$ is the barcode of filtered Morse homology

$A = \max_M H$ $B = \min_M H$. then $B(\phi)$ contains $[A, \infty)$, $[B, \infty)$

② $H = 0$: degenerate. let $H_i \to 0$ $B(Id_M) = \lim B(\phi_{H_i})$

then $B(Id_M) = ([0, \infty))^B$ B: total Betti number

Thm (Dynamical Stability) (M, ω) with $\pi_2(M) = 0$, then for any non-deg $\phi, \psi \in Ham(M, \omega)$ $d_{bot}(B(\phi), B(\psi)) \le d_{Hofer}(\phi, \psi)$

Cor. M as above. $\phi \ne Id_M \in Ham(M, \omega)$ then $\underline{d_{Hofer}(\phi, Id_M)} > 0$

non-degeneracy of d_{Hofer}

pf Take H as in ① $\phi = \phi_H^1$

$A = \max_M H > 0 \Rightarrow \begin{cases} [A, \infty) \in B(\phi) \\ [0, \infty) \in B(Id_M) \end{cases}$

$\Rightarrow 0 < A \le d_{bot}(B(\phi), B(Id_M)) \le d_{Hofer}(\phi, Id_M)$)

For H, G define $E_{H,G} = \int_0^1 \max_M (G-H)(t, \cdot) - \min_M (G-H)(t, \cdot) \, dt$

Thm. M, H, G as above, then $\forall \lambda \in \mathbb{R}$ $* \in \mathbb{Z}$ $\exists$ chain maps

$\phi_\lambda : CF_*^\lambda(H) \to CF_*^{\lambda + E_{H,G}}(G)$

$\psi_\lambda : CF_*^\lambda(G) \to CF_*^{\lambda + E_{H,G}}(H)$ s.t

$\psi_{\lambda + E_{H,G}} \circ \phi_\lambda \overset{\text{homotopic}}{\simeq} i_H : CF_*^\lambda(H) \hookrightarrow CF_*^{\lambda + 2E_{H,G}}(H)$

$\phi_{\lambda + E_{H,G}} \circ \psi_\lambda \simeq i_G : CF_*^\lambda(G) \to CF_*^{\lambda + 2E_{H,G}}(G)$

Pf of Thm $\phi = \phi_H^1$ $\psi = \phi_G^1$ $\Rightarrow \exists$ $\Phi_\lambda : HF_*^\lambda(H) \to HF_*^{\lambda+\varepsilon_{H,G}}(G)$

$\Psi_\lambda : HF_*^\lambda(G) \to HF_*^{\lambda+\varepsilon_{H,G}}(H)$

s.t.

$$\begin{tikzcd} HF_*^\lambda(H) \arrow[r, "\Phi_\lambda"] \arrow[rr, bend right, "\iota"'] & HF_*^{\lambda+\varepsilon_{H,G}}(G) \arrow[r, "\Psi_{\lambda+\varepsilon_{H,G}}"] & HF_*^{\lambda+2\varepsilon_{H,G}}(H) \end{tikzcd}$$

$$\begin{tikzcd} HF_*^\lambda(G) \arrow[r, "\Psi_\lambda"] \arrow[rr, bend right, "\iota"'] & HF_*^{\lambda+\varepsilon_{H,G}}(H) \arrow[r, "\Phi_{\lambda+\varepsilon_{H,G}}"] & HF_*^{\lambda+2\varepsilon_{H,G}}(G) \end{tikzcd}$$

$\Rightarrow \mathbb{HF}_*(\phi)$. $\mathbb{HF}_*(\psi)$ are $\varepsilon_{H,G}$-interleaved

$\Rightarrow d_{bot}(\mathcal{B}_*(\phi), \mathcal{B}_*(\psi)) = d_{int}(\mathbb{HF}_*(\phi), \mathbb{HF}_*(\psi)) \leq \varepsilon_{H,G}$ ✓

(3) Non-contractible version.

M above fix a non-trivial class $\alpha \in \pi_0(\mathcal{L}M)$

$p: \mathcal{L}M \to \pi_0(\mathcal{L}M)$ natural projection $\qquad \mathcal{L}_\alpha M = p^{-1}(\alpha)$

If α satisfies <u>symplectically atoroidal</u> condition:

$\forall$ loop in $\mathcal{L}_\alpha M$ which is a topological torus $\rho: \mathbb{T}^2 \to M$

$$\int_{\mathbb{T}^2} \rho^* \omega = \int_{\mathbb{T}^2} \rho^* \underset{\substack{\uparrow \\ c_1(M)}}{c_1} = 0$$

Fix $x_\alpha \in \mathcal{L}_\alpha M$ $\Rightarrow \mathcal{A}_H : \mathcal{L}_\alpha M \to \mathbb{R}$

$$x \mapsto -\int_{\bar{x}} \omega + \int_0^1 H(t, x(t))\, dt$$

$\bar{x}$: cylinder connecting x to x_α

Again $\mathrm{Crit}(\mathcal{A}_H) \Longleftrightarrow x$ 1-periodic orbit of Ham. flow with $[x]=\alpha$

$\mathcal{P}_\alpha(H)$

and $CF_k^\lambda(M,H)_\alpha = \mathrm{Span}_{\mathbb{Z}_2}\{x \in \mathcal{P}_\alpha(H) \mid \mathrm{Ind}(x)=k,\ \mathcal{A}_H(x)<\lambda\}$

$HF_k^\lambda(H)_\alpha = \ker \partial_k / \mathrm{Im}\, \partial_{k+1}$

$\mathbb{HF}_*(H)_\alpha = \left(\{HF_*^\lambda(H)_\alpha\}_{\lambda\in\mathbb{R}},\ \{\iota_{\lambda,\eta}: HF_*^\lambda(H)_\alpha \to HF_*^\eta(H)_\alpha\}_{\lambda\le\eta}\right)$

$\downarrow$ barcode

$B(H)_\alpha$

Fact: $B(H)_\alpha$ consists only of finite length bars

The dynamical stability theorem also extends to this case.

4. Symplectic Persistence Modules

(1) Liouville mfld: (M,ω,X) X generates X^t, s.t.

① $\omega = d\lambda$, $\lambda = \iota_X\omega$

② $\exists$ closed hypersurface $P\subset M$ s.t. $X \pitchfork P$ and bounds an open set U with $\bar{U}$ cpt, $M = U \sqcup \bigcup_{t\ge 0} X^t(P)$

A symplectomorphism ϕ of (M,ω,X) is called exact, if $\phi^*\lambda - \lambda = dF$ for some F.

$M = M_{X \cdot P} \sqcup \mathrm{Core}_P M \qquad M_X P = \bigcup_{t\in\mathbb{R}} X^t(P) \quad \mathrm{Core}_P M = \bigcap_{t<0} X^t(U)$

$\forall m \in M_{X \cdot P} \Leftrightarrow (x,u) \in P \times \mathbb{R}_+ \quad m = X^{\ln u}(x)$

$P = \{u=1\} \quad \underset{I}{U} : \{P<1\} \quad \mathrm{Core}_M(P) = \{u=0\}$

called star-shape

(2) Symplectic persistence module

Def Downward directed Poset $(I, \leq)$ $(\forall i, j \ \exists k \leq i, k \leq j)$

An inverse system of vector space over $\mathbb{Z}_2$:

$\forall i \in I$. A_i : vector space over $\mathbb{Z}_2$

i, j $\sigma_{ij} : A_i \to A_j$ $\mathbb{Z}_2$-linear with $\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}$, $\sigma_{ii} = \mathrm{Id}$

inverse limit $\varprojlim_{i\in I} A = \left\{ \{x_i\}_{i\in I} \in \prod_{i\in I} A_i \,\middle|\, i \leq j \Rightarrow \sigma_{ij}(x_i) = x_j \right\}$

Then for (M, ω, X, U) as above. $\mathcal{H}(U)$: autonomous function on M that is compactly supported in U. $H \leq G \Leftrightarrow H(x,u) \leq G(x,u) \ \forall (x,u)$

$\to (\mathcal{H}(U), \leq)$ downward directed Poset

Fact $\exists$ "monotone homotopy" $\{H_s\}_{0\leq s\leq 1}$ $H_0 = H$ $H_1 = G$ $\partial_s H_s \leq 0$

$\xrightarrow{\text{induce}} \sigma_{H,G} \quad HF_*^{(a,\infty)}(H) \to HF_*^{(a,\infty)}(G)$

Def (M, ω, X, U) then $\forall a > 0$ the filtered symplectic homology of U: $SH_*^{(a,\infty)}(U) = \varprojlim_{H\in\mathcal{H}(U)} HF_*^{(a,\infty)}(H)$

Let $SH_*^{\ln a}(U) \triangleq SH_*^{(a,\infty)}(U)$

$SH_*(U) = \left\{ \{SH_*^{\ln a}(U)\}_{a>0}, \{\theta_{s,t} : SH_*^{\ln a}(U) \to SH_*^{\ln b}(U)\}_{a<b} \right\} \xrightarrow{\text{barcode}} B_*(U)$

symplectic persistence module

Ex ① $cU \overset{\Delta}{=} \varphi_X^{\ln c}(U)$ then $\exists r_c : SH_*^{t+\ln c}(cU) \simeq SH_*^t(U)$

$\to$ uniform shift of $B_*(U)$ by $\ln c$

② $E(1, N, \cdots N) \subseteq \mathbb{R}^{2n} = \mathbb{C}^n$ star-shaped

$\{(z_1, \cdots z_n) \in \mathbb{C}^n \mid \pi(\frac{|z_1|^2}{1} + \frac{|z_2|^2}{2} + \cdots \frac{|z_n|^2}{n}) < 1\}$

Fact: $SH_*^{(a,\infty)}(E(1, N, \cdots N)) = \begin{cases} \mathbb{Z}_2 & * = -2|\lceil -a \rceil| - 2(n-1)|\lceil \frac{-a}{N} \rceil| \\ 0 & \text{other} \end{cases}$

$\Rightarrow \mathbb{B}H_0(E(1, N, \cdots N)) = \mathbb{Z}_2(-\infty, 0)$

(3) Symplectic Banach-Mazur distance

S^{2n} all non-degenerate star-shaped domain

$\forall \phi \in Symp_{ex}(M)$ $\quad \phi(c) \overset{\Delta}{=} X^{\ln c} \circ \phi \circ X^{-\ln c} \in Symp_{ex}(M)$

For $U, V \in S^{2n}$ a Liouville morphism $U \xrightarrow{\Phi} V$ is a compact supported exact symplectomorphism with $\varphi(\bar{U}) \subset V$

Def. $U, V \in S^{2n}$. $C \geq 1$ is (U, V)-admissible if $\exists \phi, \psi \in Symp_{ex}(M)$ s.t. $\frac{1}{C}U \xrightarrow{\Phi} V \xrightarrow{\Psi} CU$ with $\psi \circ \phi \in Symp_{ex}^{\circ}(M)$

then the symplectic Banach-Mazur distance is

$$d_{SBM}(U, V) = \inf\{\ln C \mid C \text{ is } (U, V)\text{-admissible}\}$$

$\downarrow$

a pseudo-metric. don't know if it's a metric on $S^{2n}/Symp_{ex}(M)$

Thm (Topological Stability Theorem) $U, V \in S^{2n} \Rightarrow d_{bot}(B_*(U), B_*(V)) \le d_{SBM}(U,V)$

Ex. $* = 0$ $\quad B_0(E(1,8)) = (-\infty, 0)$ $\quad B_0(E(2,4)) = (-\infty, \ln 2)$

so $d_{SBM}(E(1,8), E(2,4)) \ge \ln 2$

More generally, $d_{SBM}(E(r, rN, \cdots rN), B^{2n}(R)) \ge |\ln r - \ln R|$

Now we prove the thm.

Thm (Functorial Property) (M, ω, X) $U, V \in S^{2n}$

(1) $\forall$ Liouville morphism $\phi: U \to V$ induces $\mathbb{Z}_2$-linear map

$f_\phi^a : SH_*^{(a,\infty)}(V) \to SH_*^{(a,\infty)}(U)$ and the commutative diagram

$$\begin{tikzcd}
SH_*^{(a,\infty)}(V) \arrow[r, "f_\phi^a"] \arrow[d, "\theta_{a,b}^V"'] & SH_*^{(a,\infty)}(U) \arrow[d, "\theta_{a,b}^U"] \\
SH_*^{(b,\infty)}(V) \arrow[r, "f_\phi^b"] & SH_*^{(b,\infty)}(U)
\end{tikzcd}$$

and $f_{\psi\circ\phi}^a = f_\phi^a \circ f_\psi^a$

(2) Write $i^a = f_{Id}^a$ $\Rightarrow$

$$\begin{tikzcd}
SH_*^{(a,\infty)}(U) \arrow[dr, "\theta_{a,ca}"'] \arrow[r, phantom, "\circlearrowleft"] & & SH_*^{(ca,\infty)}(cU) \arrow[ll, "r_c"', "\cong"] \arrow[dl, "i^{ca}"] \\
& SH_*^{(ca,\infty)}(U) &
\end{tikzcd}$$

$$\begin{tikzcd}
SH_*^{(a,\infty)}(U) & & SH_*^{(ca,\infty)}(cU) \arrow[ll, "r_c"', "\cong"] \\
& SH_*^{(a,\infty)}(cU) \arrow[ul, "i^a"] \arrow[ur, "\theta_{a,ca}"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

(3) $\bar{U} \subset V$. ϕ: Liouville morphism from U to V. if $\phi \in Symp_{ex}^0(M)$

then $f_\phi = i = f_{Id}$

pf of stability thm: $\forall \varepsilon > 0$, $\exists C > 1$ s.t.

$\frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU$ with $\psi\circ\phi \in \mathrm{Symp}^0_{ex}(M)$ and $\ln C \le d_{SBM}(U,V)+\varepsilon$

$$\overset{(1)(3)}{\Rightarrow} \begin{tikzcd} SH_*^{(a,\infty)}(CU) \arrow[r, "f_\psi^a"] \arrow[rr, bend right, "f^a_{\psi\circ\phi}"'] & SH_*^{(a,\infty)}(V) \arrow[r, "f_\phi^a"] & SH_*^{(a,\infty)}(\frac{U}{C}) \\ SH_*^{(a-\ln C,\infty)}(U) \arrow[u, phantom, "\cong r_C"] & i^a_{CU,\frac{U}{C}} & SH_*^{(a+\ln C,\infty)}(U) \arrow[u, phantom, "\cong r_C"] \end{tikzcd}$$

$\overset{(2)}{\Rightarrow}$ $SH_*(U)$, $SH_*(V)$ are $\ln C$-interleaved

$\Rightarrow d_{bot}(B_*(U), B_*(V)) = d_{int}(SH_*(U), SH_*(V)) \le \ln C \le d_{SBM}(U,V)+\varepsilon$

$\varepsilon \to 0$ ✓

(4) Application: Non-squeezing

Thm. For $R' \ge R$ and $\exists$ a Liouville morphism $B^{2n}(r) \to E(R, R', \cdots R') \Rightarrow R \ge r$

pf. WLOG let $r=1$, take a Liouville morphism ϕ $\quad B_1 \overset{\Delta}{=} B^{2n}(1)$

Assume $\mathrm{supp}\,\phi \subseteq \underbrace{B^{2n}(R_0)}_{B_2}$ for R_0 large $\Rightarrow \phi(B_1) \subseteq E \subseteq \phi(B_2)$

$$\Rightarrow \begin{tikzcd} SH_*^{(a,\infty)}(\phi(B_2)) \arrow[r] \arrow[d, "\cong"] & SH_*^{(a,\infty)}(E) \arrow[r, "i^a"] & SH_*^{(a,\infty)}(\phi(B_1)) \arrow[d, "\cong"] \\ SH_*^{(a,\infty)}(B_2) \arrow[rr, "i^a_{B_2,B_1}"] \arrow[dr, "\cong"'] & & SH_*^{(a,\infty)}(B_1) \\ & SH_*^{(\frac{a}{R_0},\infty)}(B_1) \arrow[ur, "\theta_{\frac{a}{R_0},a}"'] & \end{tikzcd}$$

We know $SH_0(E) = \mathbb{Z}_2(-\infty, \ln R)$ $\qquad SH_0(B_1) = \mathbb{Z}_2(-\infty, 0)$

for $a < R_0$. $\theta_{\frac{a}{R_0},a} \neq 0 \Rightarrow i^a_{B_2,B_1} \neq 0$

$\Rightarrow i: \mathbb{Z}_2(-\infty, \ln R) \to \mathbb{Z}_2(-\infty, 0)$ nonzero

$\Rightarrow \ln R \ge 0$. $R \ge 1$ ✓

IV J-hol Curves

1. Def

(M,J) almost complex (Σ, j) Riemann surface

$u: \Sigma \to M$ J-hol curve if $J \circ du = du \circ j$ (written $\bar\partial_J(u) = 0$)

$$\bar\partial_J(u) = \tfrac{1}{2}(du + J \circ du \circ j)$$

$\bar\partial_J u \in \Omega^{0,1}(\Sigma, u^*TM)$

$\mathcal{E}$

$\downarrow$

$\mathcal{B} = C^\infty(\Sigma, M)$ fiber $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$

$u \mapsto (u, \bar\partial_J(u))$ section zero set $\Leftrightarrow$ J-hol

In local coordinate $\{U_\alpha, \phi_\alpha\}$ on Σ $\Rightarrow d\phi_\alpha \circ j = i\, d\phi_\alpha$ ($z = s + it$)

u J-hol $\Leftrightarrow u_\alpha = u \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to M$ J-hol

$$\bar\partial_J(u_\alpha) = \tfrac{1}{2}(\partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha)ds + \tfrac{1}{2}(\partial_t u_\alpha - J(u_\alpha)\partial_s u_\alpha)dt$$

so u_α J-hol $\Leftrightarrow \partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha = 0$ $\leftarrow$ nonlinear C-R eq

(M,ω) almost symplectic (ω may not be closed)

$J \in \mathcal{J}_\tau(M)$ a ω-tame AC structure

The energy of $u: \Sigma \to M$ is defined as

$$E(u) = \tfrac{1}{2}\int_\Sigma |du|_J^2 \, dvol_\Sigma$$

here $L = du(z): T_z\Sigma \to T_{u(z)}M$ $\quad |L|_J \triangleq |\zeta|^{-1}\sqrt{|L(\zeta)|_J^2 + |L(j_z\zeta)|_J^2}$ independent of ζ

Lemma (Energy identity) ω: nondegenerate 2-form

if J is ω-tame then $\forall$ J-hol curve $u: \Sigma \to M$ satisfies

$$E(u) = \int_\Sigma u^*\omega$$

If J is ω-compatible then $\forall u: \Sigma \to M$ satisfies

$$E(u) = \int_\Sigma |\bar\partial_J(u)|_J^2 \, dVol_\Sigma + \int_\Sigma u^*\omega$$

pf WLOG Σ is an open subset of $\mathbb{C}$

$$\frac{1}{2}|du|_J^2 dVol_\Sigma = \frac{1}{2}(|\partial_s u|^2 + |\partial_t u|_J^2) ds \wedge dt$$

$$= \frac{1}{2}|\partial_s u + J\partial_t u|_J^2 \, ds \wedge dt - \langle \partial_s u, J\partial_t u\rangle_J ds \wedge dt$$

$$= |\bar\partial_J(u)|_J^2 \, dVol_\Sigma + \frac{1}{2}(\omega(\partial_s u, \partial_t u) + \omega(J\partial_s u, J\partial_t u)) \, ds \wedge dt$$

2. Unique Continuation

Unique continuation problem is obvious when J is integrable

If M is only almost complex?

Assume $u: B_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\} \to \mathbb{R}^{2n}$ (since the question is local)

$$\partial_s u + J(u)\partial_t u = 0 \qquad (*)$$

where $J: \mathbb{R}^{2n} \to GL(2n, \mathbb{R})$ is C^1 $J^2 = -Id$

Rmk. $\forall W^{1,p}$ $(p>2)$ solution of $(*)$ is a weak solution of

$$\Delta u = (\partial_t J(u))\partial_s u - (\partial_s J(u))\partial_t u$$

bootstrap $\Rightarrow$ if $J \in C^\ell$ then $u \in W^{\ell+1,p} \overset{\ell=1}{\Rightarrow} u \in C^1$

An integrable function $w : B_\varepsilon \to \mathbb{C}^n$ vanishes to ∞-ord at 0

if $\forall k \in \mathbb{N}$ $\int_{|z|\le r} |w(z)| = O(r^k)$

Thm (Unique Continuation) $u, v \in C^1(B_\varepsilon, \mathbb{R}^{2n})$, $J \in C^1$ satisfies $(*)$

if $u-v$ vanishes to ∞-ord at 0, then $u \equiv v$

Cor. Σ connected Riem surf. J C^1 AC structure on M

if J-hol curve $u, v : \Sigma \to M$ agrees on a point z to ∞-ord $\Rightarrow u \equiv v$

(pf. connectedness)

pf of thm: $w = u - v \in C^1$ and J, ∇J are bounded, so $w \in W^{2,p}$ ($\forall p < \infty$) and vanishes to ∞-ord ①

$$|\Delta w(z)| \le C(|w| + |\partial_s w| + |\partial_t w|) \quad ②$$

the theorem of Aronszajn claims that ①+② $\Rightarrow w \equiv 0$)

Thm (Carleman Similarity Principle) $p > 2$. $C \in L^p(B_\varepsilon, \mathrm{End}(\mathbb{R}^{2n}))$

$J \in W^{1,p}(B_\varepsilon, GL(2n, \mathbb{R}))$ $J^2 = -Id$. then if $u \in W^{1,p}(B_\varepsilon, \mathbb{R}^{2n})$

is a solution of $\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0$, $u(0) = 0$ (Δ)

then $\exists \delta \in (0, \varepsilon)$, $\Phi \in W^{1,p}(B_\delta, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}^{2n}))$, $G : B_\delta \xrightarrow{\text{holomorphic}} \mathbb{C}^n$

s.t. $\Phi(z)$ is invertible and $\begin{cases} u(z) = \Phi(z) G(z) \\ G(0) = 0 \\ \Phi(z)^{-1} J(z) \Phi(z) = i \end{cases}$ $(\forall z \in B_\delta)$

(this can be used to show unique continuation

$w = u - v$ satisfies (Δ) and use connectedness)

3 Critical Points

crit pt isolated

Lemma. Σ cpt Riem. surface without bdry. J: C^1 AC structure

u: C^1 nonstant J-hol curve then $X = u^{-1}\{u(z) \mid z \in \Sigma,\ du(z) = 0\}$

is finite. Moreover, $\forall x \in M$ $u^{-1}(x)$ is finite.

pf. Again let $\Sigma = \Omega \subset \mathbb{C}$. $M = \mathbb{R}^{2n}$ then u fits (0) with $C = 0$

by Carleman's theorem. $\forall p \in M$. $u^{-1}(p)$ is finite.

take derivative $\Rightarrow$ $\partial_s v(z) + J(z)\partial_t v(z) + (\partial_s J(z)) J(z) v(z) = 0$ $\quad (v = \partial_s u)$

again zero set of v is isolated $\Rightarrow$ finite critical points $\Rightarrow |X| < \infty$

Lemma (good coordinate) $\ell \geq 2$. J: C^ℓ AC structure on M.

$\Omega \subset \mathbb{C}$ nbhd of 0 $\quad u: \Omega \to M$ J-hol curve $\quad du(0) \neq 0$

then $\exists\, C^{\ell-1}$ chart $\psi: U \to \mathbb{C}^n$ near $u(0)$ s.t.

$$\psi(u(z)) = (z, 0, \cdots, 0) \qquad d\psi(u(z))\, J(u(z)) = J_0\, d\psi(u(z))$$

Now we show that the accumulation point of the intersection points of two J-hol curves must be critical w.r.t. both curves.

Lemma (local ver) J: C^2 AC on M $\Omega\subset\mathbb{C}$ nbhd of 0

$u,v: \Omega\to M$ J-hol curves with $u(0)=v(0)$ $du(0)\neq 0$

if $\exists z_\nu\to 0$ $\zeta_\nu\to 0$ $(\zeta_\nu\neq 0)$ s.t. $u(z_\nu)=v(\zeta_\nu)$

then $\exists$ holomorphic $\phi: B_\varepsilon(0)\to\Omega$ s.t. $\phi(0)=0$. $v=u\circ\phi$

pf. Use the coordinate above. assume $J:\mathbb{C}^n\to GL(2n,\mathbb{R})$

$u(z)=(z,0)$ $v=(v_1,\tilde{v})$. then one can show that

$\hat{C}:\Omega\to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^{n-1})$. $\hat{C}(z)\tilde{\xi}=\tilde{\pi}\left(\left(\int_0^1 \frac{d}{ds}\Big|_{s=0} J(v_1(z), T\tilde{v}(z)+s\tilde{\xi})\,dT\right)\partial_t v(z)\right)$

is continuous with $\partial_s\tilde{v}+i\partial_t\tilde{v}+\hat{C}\tilde{v}=0$

Again by Carleman's thm. $\exists\delta>0$. $\tilde{\Phi}\in W^{1,p}(B_\delta, GL(n-1,\mathbb{C}))$

and holomorphic function $\tilde{\sigma}: B_\delta\to\mathbb{C}^{n-1}$. s.t.

$$\tilde{v}(z)=\tilde{\Phi}(z)\tilde{\sigma}(z). \quad \tilde{\sigma}(0)=0$$

$$\tilde{v}(\zeta_\nu)=0 \Rightarrow \tilde{\sigma}(\zeta_\nu)=0 \ (\nu \text{ large}) \Rightarrow \tilde{\sigma}=0 \Rightarrow \tilde{v}=0.$$

Thm (Global) J: C^2 AC on M Σ_0 Σ_1: cpt Riem surface

without bdry. $u_i:\Sigma_i\to M$ J-hol curves

Assume $u_0(\Sigma_0)\neq u_1(\Sigma_1)$ and u_0 non-constant. then

$u_0^{-1}(u_1(\Sigma_1))$ is at most countable and can only

accumulate at the critical point of u_0

4. Somewhere Injective Curve

(Σ, j) cpt Riem. surface $\qquad (M, J)$ AC

$u: \Sigma \to M$ J-hol curve is called multiply covered if

$\exists (\Sigma', j') \xrightarrow{u'} (M, J)$ as above with $\phi: \Sigma \to \Sigma'$

where ϕ is a holomorphic branched covering. $\quad u = u' \circ \phi$

otherwise u is called simple

A simple J-hol curve is called somewhere injective

if $\exists z \in \Sigma$ s.t $du(z) \neq 0 \quad u^{-1}(u(z)) = \{z\}$

Then z is an injective point.

$Z(u) \triangleq \{ z \in \Sigma \mid du(z) = 0 \ \ |u^{-1}(u(z))| > 1 \}$

Prop $J: C^2$ AC. Σ cpt without boundary $\quad u: \Sigma \to M$ simple

then u is somewhere injective

And $Z(u)$ is at most countable and can only accumulate at crit pt of u

(pf. The idea is by showing that any $u: \Sigma \to M$ can be decomposed into $u = u' \circ \phi \quad u': \Sigma' \to M$ somewhere injective

Then if u is simple, there must be $\deg(\phi) = 1$

$\Rightarrow u$ is also somewhere injective. $\qquad)$

Cor. ① $J: C^2$ AC Σ_0, Σ_1 cpt connected without boundary

$u_j: \Sigma_j \to M$ simple with $u_0(\Sigma_0) = u_1(\Sigma_1)$ then

$\exists$ holomorphic diffeomorphism s.t. $u_1 = u_0 \circ \phi$

② $J, \Sigma_0 \cdots \Sigma_N$ as above $u_0 \cdots u_N$ J-hol u_0 simple

with $u_0(\Sigma_0) \neq u_j(\Sigma_j)$ ($\forall j > 0$). then $\forall z_0 \in \Sigma_0$ and $u_0 \ni z_0$

$\exists$ an open annulas $\overset{\subset u_0}{A_0}$ at z_0 s.t. $u_0: A_0 \to M$ is embedding

with $u_0(A_0) \cap u_j(\Sigma_j) = \phi$ ($j > 0$)

$$\underset{\shortparallel}{\mathrm{Crit}(u_i)} \cup \{z \in \Sigma_i \mid |u_i^{-1}(u_i(z))| > 1\}$$

pf. ① $Z_i \subset u_i$ noninjective pts $\Rightarrow Z_i = u_i^{-1}(u_i(Z_i))$

then Z_i is at most countable and can accumulate

only at finite subset of $\mathrm{crit}(u_i)$. Since $u_0(\Sigma_0) = u_1(\Sigma_1)$

$\exists!$ bijection $\phi: \Sigma_1 \setminus u_1^{-1}(u_0(Z_0) \cup u_1(Z_1)) \to \Sigma_0 \setminus u_0^{-1}(u_0(Z_0) \cup u_1(Z_1))$

s.t. $u_1 = u_0 \circ \phi$. Since $du_0(\phi(z_1))$ $du_1(z_1)$ nonzero and cplx

linear in the domin of $\phi \Rightarrow \phi$ holomorphic

Then since $u_1^{-1}(u_0(Z_0) \cup u_1(Z_1))$ at most countable $\cdots$

then ϕ extends to a holomorphic diffeomorphism $\phi: \Sigma_1 \to \Sigma_0$

② $Z_j = u_0^{-1}(u_j(\Sigma_j))$ Z_0 noninjective pts

then Z_i ($0 \le i \le N$) can only accumulate at $\mathrm{Crit}(u_0)$.

Choose $A_0 \subset \Sigma_0 \setminus (Z_0 \cup \cdots \cup Z_N)$.

V. Moduli Space and Transversality

1. Moduli spaces of simple curve

(M, ω, J) cpt symplectic $(\Sigma, j_\Sigma, dVol_\Sigma)$ cpt Riem surface

(J ω-tamed) $(\mathcal{M}(A;J) = \mathcal{M}(A, \mathbb{CP}^1; J)$ $\mathcal{M}^*(A, J) = \mathcal{M}^*(A, \mathbb{CP}^1; J))$

$\forall A \in H_2(M; \mathbb{Z})$ $\mathcal{M}(A, \Sigma; J) \triangleq \{u \in C^\infty(\Sigma, M) \mid J \circ du = du \circ j_\Sigma, \ [u] = A\}$

$\mathcal{M}^*(A, \Sigma; J) = \{u \in \mathcal{M}(A, \Sigma, J) \mid u \text{ is simple}\}$

Let $\mathcal{B} \subset C^\infty(\Sigma, M)$: all smooth $u: \Sigma \to M$ representing $A \in H^2(M)$

$T_u\mathcal{B} = \Omega^0(\Sigma, u^*TM)$: all v.f. $\xi(z) \in T_{u(z)}M$ along u

Bundle $\mathcal{E} \to \mathcal{B}$. fiber $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$

$S: \mathcal{B} \to \mathcal{E}$ section $\Rightarrow \mathcal{M}(A, \Sigma; J) = \text{zero}(S)$

$u \mapsto (u, \bar\partial_J(u))$

we want $\mathcal{M}^*(A, \Sigma; J)$ finite-dim mfld

$\Uparrow$

$S \pitchfork$ zero section $\Leftarrow \operatorname{Im} dS(u) + T_u\mathcal{B} = T_{(u,0)}\mathcal{E}$

$D_u \triangleq D_{J,u} \triangleq DS(u) : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$

(vertical differential) $\overset{\shortparallel}{\pi \circ dS(u)}$ where $\pi: T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$

$\partial_s u + J(u)\partial_t u = 0$

derivative $\Rightarrow$ in direction ξ $D_u\xi = \bar\partial_J \xi - \frac{1}{2}(J\partial_\xi J)(u)\partial_J(u)$

where $\partial_J(u) = \frac{1}{2}(du - J \circ du \circ j)$

Generally, if u is not J-hol. then it's a problem to split $T_{(u,\bar\partial_J(u))}\mathcal{E}$: depends on connection!

Let $\tilde\nabla_v X = \nabla_v X - \frac{1}{2}J(\nabla_v J)X$: preserves J $\Rightarrow$ fiber is transport -invariant

Given $\xi \in \Omega^0(\Sigma, u^*TM)$ $\Phi_u(\xi): u^*TM \to \exp_u(\xi)^*TM$

defined by parallel transport w.r.t. $\tilde\nabla$ along geodesic $s \mapsto \exp_{u(z)}(s\xi(z))$ (w.r.t. ∇)

$$\mathcal{F}_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$$

$$\xi \mapsto \Phi_u(\xi)^{-1}\bar\partial_J(\exp_u(\xi))$$

Prop $\forall u: \Sigma \to M$ $\quad D_u: \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$

$$\xi \mapsto d\mathcal{F}_u(0)\xi$$

then $D_u\xi = \frac{1}{2}(\nabla\xi + J(u)\nabla\xi\circ j_\Sigma) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J(u)$

pf. path : $\mathbb{R} \to C^\infty(\Sigma, M)$ $\quad \Rightarrow \Phi_u(\lambda\xi)\mathcal{F}_u(\lambda\xi) = \bar\partial_J(u_\lambda)$

$\lambda \mapsto u_\lambda = \exp_u(\lambda\xi)$

since $\Phi_u(\lambda\xi)$ is given by transport along the geodesic $\lambda \mapsto u_\lambda(z)$

$$D_u\xi = \frac{d}{d\lambda}\mathcal{F}_u(\lambda\xi)\Big|_{\lambda=0} = \frac{d}{d\lambda}\left(\Phi_u(\lambda\xi)^{-1}\bar\partial_J(u_\lambda)\right)\Big|_{\lambda=0} = \tilde\nabla_\lambda \bar\partial_J(u_\lambda)\Big|_{\lambda=0}$$

$$= \frac{1}{2}(\tilde\nabla_\lambda du_\lambda + J(u_\lambda)\tilde\nabla_\lambda du_\lambda \circ j_\Sigma)\Big|_{\lambda=0}$$

$$= \frac{1}{2}(\nabla_\lambda du_\lambda + J(u_\lambda)\nabla_\lambda du_\lambda \circ j_\Sigma)\Big|_{\lambda=0} - \frac{1}{4}($$

∇ torsion-free $\quad J(u_\lambda)(\nabla_{\partial_\lambda u_\lambda}J)(u_\lambda)du_\lambda - (\nabla_{\partial_\lambda u_\lambda}J)(u_\lambda)du_\lambda \circ j_\Sigma)\Big|_{\lambda=0}$

$$\mathcal{L} = \frac{1}{2}(\nabla\xi + J(u)\nabla\xi \circ j_\Sigma) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J(u)$$

$\nabla_\lambda\partial_t\gamma = \nabla_t\partial_\lambda\gamma$ especially $\gamma(\lambda,t) = u_\lambda(z(t))$

If u is not smooth but only $W^{k,p}$ then

$$\mathcal{F}_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$$

and D_u is a Fredholm operator

Outline: ① R-R thm $\Rightarrow$ $\operatorname{ind} D_u = n(2-2g) + 2c_1(u^*TM)$

② Let $\mathcal{J}$ be a large space of AC on M with C^∞-topology

$\mathcal{J}^l$: corresponding space with C^l AC

universal moduli space $\mathcal{M}^*(A, \Sigma; \mathcal{J}^l) = \{(u, J) \mid J \in \mathcal{J}^l, u \in \mathcal{M}^*(A, \Sigma; J)\}$

$\pi: \mathcal{M}^*(A, \Sigma; \mathcal{J}^l) \to \mathcal{J}^l$ is Fredholm since $d\pi = D_u$

$(u, J) \mapsto J$

③ IFT $\to$ $\mathcal{M}^*(A, \Sigma; J^l)$ a finite-dim mfld whose tangent space at u $= \ker D_u$ when J is a regular value

Sard-Smale Thm: regular value is residual for l large

④ Taubes: still hold in smooth case

Formal statement:

Def. Fix cpt Riem surface $(\Sigma, j_\Sigma, dVol_\Sigma)$ and $A \in H_2(M; \mathbb{Z})$ then an AC J on M is said to be regular if D_u is onto for any $u \in \mathcal{M}^*(A, \Sigma; J)$. $\mathcal{J}_{reg}(A, \Sigma) = \{J \in \mathcal{J} \mid J \text{ is regular}\}$

Especially write $\mathcal{J}_{reg}(A) = \mathcal{J}_{reg}(A, S^2)$

Thm 1. Let $\mathcal{J} = \mathcal{J}(M,\omega)$ or $\mathcal{J}_\tau(M,\omega)$

(1) $J \in \mathcal{J}_{reg}(A,\Sigma)$ then $\mathcal{M}^*(A,\Sigma;J)$ is a smooth oriented mfld of dim $n(2-2g)+2c_1(A)$

(2) $\mathcal{J}_{reg}(A,\Sigma)$ is residual in $\mathcal{J}$

For a homotopy $[0,1] \to \mathcal{J}$, $\lambda \mapsto J_\lambda$ of AC. define

$$\mathcal{W}^*(A,\Sigma,\{J_\lambda\}_\lambda) = \{(\lambda,u) \mid 0 \le \lambda \le 1,\ u \in \mathcal{M}^*(A,\Sigma;J_\lambda)\}$$

Def. Let $J_0, J_1 \in \mathcal{J}_{reg}(A,\Sigma)$, a homotopy from J_0 to J_1 is regular if $\Omega^{0,1}(\Sigma, u^*TM) = \operatorname{Im} D_{J_\lambda,u} + \mathbb{R} v_\lambda$ $(\forall(\lambda,u) \in \mathcal{W}^*(A,\Sigma;\{J_\lambda\}_\lambda))$, where $v_\lambda = (\partial_\lambda J_\lambda)\, du \circ j_\Sigma$

$\mathcal{J}_{reg}(A,\Sigma;J_0,J_1) = \{$regular homotopy from $J_0 \to J_1\}$

Thm 2 Let $\mathcal{J} = \mathcal{J}(M,\omega)$ or $\mathcal{J}_\tau(M,\omega)$ $J_0, J_1 \in \mathcal{J}_{reg}(A,\Sigma)$

(1) If $\{J_\lambda\}_\lambda \in \mathcal{J}_{reg}(A,\Sigma,J_0,J_1)$ then $\mathcal{W}^*(A,\Sigma;\{J_\lambda\}_\lambda)$ is a smooth oriented mfld with $\partial\mathcal{W}^*(A,\Sigma;\{J_\lambda\}_\lambda) = \mathcal{M}^*(A,\Sigma;J_0) \cup \mathcal{M}^*(A,\Sigma;J_1)$

(2) $\mathcal{J}_{reg}(A,\Sigma;J_0,J_1)$ is residual in the space of homotopies from J_0 to J_1.

2 Elliptic Regularity

$\mathcal{J}^{\ell} = \mathcal{J}^{\ell}(M,\omega)$: C^{ℓ} ω-tame AC

$p>2,\ k\geq 1$ $\quad B^{k,p} = \{u \in W^{k,p}(\Sigma, M) \mid [u] = A\}$

$$T_u W^{k,p}(\Sigma, M) = W^{k,p}(\Sigma, u^*TM)$$

$W^{k,p}$, $B^{k,p}$: smooth seperable Banach mfld

Prop J AC C^{ℓ}, $u: \Sigma \to M$ J-hol, $W^{1,p}$ $(p>2)$

then $u \in W^{\ell+1,p} \Rightarrow u \in C^{\ell}$. So if J is $C^{\infty} \Rightarrow u$ is C^{∞}

Rmk. So $W^{k,p}$ J-hol independent of k if $k \leq \ell+1$

If we want to well-define $D_u \Rightarrow k \leq \ell$ (since there's a derivative to J)

Prop. $\ell \geq 1$. $p>2$ $\quad J \in \mathcal{J}^{\ell}_{\tau}$ $\quad u \in W^{\ell,p}(\Sigma, M)$ $\quad 1 \leq k \leq \ell$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

then (1) $D_u: W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$

$\qquad D_u^*: W^{k,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM) \to W^{k-1,p}(\Sigma, u^*TM)$

are Fredholm with $\operatorname{ind} D_u = -\operatorname{ind} D_u^* = n(2-2g) + 2c_1(u^*TM)$

(2) If $\eta \in L^q(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$ s.t. $\int_\Sigma \langle \eta, D_u \xi \rangle \, dVol_\Sigma = 0$ $(\forall \xi \in W^{k,p}(\Sigma, u^*TM))$

then $\eta \in W^{\ell,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$. $D_u^* \eta = 0$

(3) If $\xi \in L^q(\Sigma, u^*TM)$ s.t. $\int_\Sigma \langle \xi, D_u^* \eta \rangle \, dVol_\Sigma = 0$ $(\forall \eta \in W^{k,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM))$

the $\xi \in W^{\ell,p}(\Sigma, u^*TM)$. $D_u \xi = 0$

3. Transversality

Recall $\mathcal{M}^*(A,\Sigma;\mathcal{J}^l) = \{(u,J) \mid J \in \mathcal{J}^l,\ u \in \mathcal{M}^*(A,\Sigma,J)\}$

$\qquad \cap\!| \quad \mathcal{B}^{k,p} \times \mathcal{J}^l \quad (\forall 1 \le k \le l,\ p > 2)$

$\mathcal{J}^l$: seperable smooth Banach mfld

$T_J \mathcal{J}^l = C^l$-sections Y of bundle $\underbrace{\mathrm{End}(TM, J, \omega)}$

fiber : $\{Y : YJ = -JY,\ \omega(Yv,w) + \omega(v,Yw) = 0\}$

Prop For $A \in H_2(M;\mathbb{Z})$, $\underset{\in \mathbb{Z}}{l \ge 2}$, $\underset{\in \mathbb{R}}{p > 2}$, $1 \le k \le l$, then $\mathcal{M}^*(A,\Sigma;\mathcal{J}^l)$ is a seperable C^{l-k}-Banach submfld of $\mathcal{B}^{k,p} \times \mathcal{J}^l$.

pf Bundle $\mathcal{E}^{k-1,p} \to \mathcal{B}^{k,p} \times \mathcal{J}^l$ with fiber $\mathcal{E}^{k-1,p}_{(u,J)} = W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$

$J\ C^l \Rightarrow \tilde{\nabla}$ transport $C^{l-1} \Rightarrow$ transition $C^{l-1} \xRightarrow[l-k \text{ times}]{\text{differentiate}} W^{k-1,p}$

$\Rightarrow \mathcal{E}^{k-1,p}$ is a C^{l-k} Banach mfld

Consider a $\underset{C^{l-k}}{\text{section}}$: $\mathcal{F} : \mathcal{B}^{k,p} \times \mathcal{J}^l \to \mathcal{E}^{k-1,p}$

$(u,J) \mapsto \bar{\partial}_J(u)$

$D\mathcal{F}(u,J) : W^{k,p}(\Sigma, u^*TM) \times C^l(M, \mathrm{End}(TM,J,\omega)) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$

$(\xi, Y) \mapsto D_u \xi + \frac{1}{2} Y(u) \circ du \circ j$

D_u Fredholm $\Rightarrow \mathrm{Im}(D\mathcal{F})$ is closed $\xRightarrow[\text{density}]{\text{can prove}}$ $D\mathcal{F}$ surjective

$\Downarrow$ (has right inverse)

$\overset{\mathrm{IFT}}{\Rightarrow} \mathcal{M}^*(A,\Sigma;\mathcal{J}^l)$ is a C^{l-k} submfld.

4 A Regularity Criterion

We give some examples of regularity without proof.

For $\Sigma = S^2 = \mathbb{C}P^1$. every holomophic bundle E of $\mathbb{C}P^1$ (by Grothendieck)

$\underset{\text{equivalent}}{\overset{\text{holomophic}}{\equiv}}$ $L_1 \oplus \cdots \oplus L_n$ L_i line bundle → summands of E

$C_1(E) = \sum_i C_1(L_i)$

Prop. J is integrable and $u: \mathbb{C}P^1 \to M$ J-hol. If every summand of u^*TM has Chern number $C_1 \geq -1$. then D_u is onto

Prop': J AC on M^4 $u: \mathbb{C}P^1 \to M$ immersed J-hol. then D_u is onto $\Leftrightarrow$ $C_1(u^*TM) \geq -1$

Cor. ① J AC on M^4. C an embedded J-hol sphere with self intersection number $C \cdot C = p$. then J is regular $\Leftrightarrow$ $p \geq -1$

② $\tilde{M} = S^2 \times (M, \omega)$ $\tilde{A} \in H_2(\tilde{M}; \mathbb{Z})$ presented by the spheres $S^2 \times \{pt\}$

then $\forall J \in \mathcal{J}(M, \omega)$ $\tilde{J} = i \times J$ is regular for $\tilde{A}$

5. Compactness

Since $\forall$mfld V is cobordant to ϕ via the non-cpt cobordism $V\times[0,1)$. Thm 2 is meaningless unless there is some kind of cptness.

Recall the energy defined as $E(u)=\frac{1}{2}\int_\Sigma |du|_J^2\, dA$
$= \int_\Sigma u^*\omega$ (if u J-hol, J: ω-tame)

Thm. $u_\nu : \overset{\text{might noncpt}}{\Sigma} \to M$ a sequence of J-hol curves s.t.
$\sup_\nu \|du_\nu\|_{L^\infty(K)} < \infty$ ($\forall K\subset\Sigma$ cpt). then $\exists$ subsequence u_{ν_i} which converges uniformly compactly with all order derivatives
(pf. use elliptic bootstrapping)

Thm (Removal of Singularities) J-smooth ω-tame AC $\leftrightarrow g_J$ on $\overset{\text{cpt}}{M}$
if $u: B\setminus\{0\}\to M$ J-hol with $E(u)<\infty$. then u extends smoothly to B.

$\mathbb{C}P^1 = S^2 = \mathbb{C}\cup\{\infty\}$ $G \triangleq PSL(2,\mathbb{C})$

$u\in\mathcal{M}(A,J) \overset{\text{regarded as}}{\Longrightarrow}$ J-hol $u:\mathbb{C}\to M$ s.t. $u(\frac{1}{z}): \mathbb{C}\setminus\{0\}\to M$
$\downarrow$ extends
$\mathbb{C}\to M$

$\{u_\nu:\mathbb{C}\to M\}$ converges on $\mathbb{C}\cup\{\infty\}$ $\Leftrightarrow$ $\{u_\nu(z)\}$ $\{u_\nu(\frac{1}{z})\}$ converges uniformly cptly on $\mathbb{C}$

Thm If there is no spherical class $B\in H_2(M)$ s.t. $0<\omega(B)<\omega(A)$ then $\mathcal{M}(A,J)/G$ is cpt

The class A is called indecomposable if it doesn't decompose as $A = A_1 + \cdots + A_k$ with A_i spherical and $\omega(A_i) > 0$.

Thm A is indecomposable. then for J compatible. $\mathcal{M}(A,J)/G$ cpt.
(*)

$G \circlearrowright \mathcal{M}(A,J) \times S^2 \rightsquigarrow \mathcal{M}(A,J) \times_G S^2$

$\phi \cdot (u, z) = (u \circ \phi^{-1}, \phi(z))$

evaluation map $e_{A,J} : \mathcal{M}(A,J) \times_G S^2 \to M$

$(u, z) \longmapsto u(z)$

Thm. A is indecomposable. $J_1, J_2 \in \mathcal{J}_{reg} \Rightarrow e_{A,J_1}, e_{A,J_2}$ cptly cobordant

"Gromov cptness" (implies Thm(*)): M cpt. $J_\nu \in \mathcal{J}_\tau(M, \omega)$

$J_\nu \xrightarrow{C^\infty} J$ then for $u_\nu : \mathbb{CP}^1 \to M$ J_ν-hol with $\sup_\nu E(u_\nu) < \infty$

$\exists$ subsequence $u_{\nu_i} \xrightarrow{\text{weakly}} u : \mathbb{CP}^1 \to M$ J-hol.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad\quad\quad\quad (u^1, \cdots, u^N)$

here weakly convergence means:

(i) $\forall j \leq N$. $\exists \phi_\nu^j : \mathbb{CP}^1 \to \mathbb{CP}^1$ fractional linear transform

$X^j \subset \mathbb{CP}^1$ finite s.t $u_\nu \circ \phi_\nu^j \to u^j$ uniformly cptly in all order on $\mathbb{CP}^1 - X^j$

(ii) $\exists$ orientation-preserving diffeo $f_\nu : \mathbb{CP}^1 \to \mathbb{CP}^1$

s.t $u_\nu \circ f_\nu \xrightarrow{C^0} v : \mathbb{CP}^1 \to M$