LINKED TWISTED MAPS AND HOFER DISTANCE

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Abstract. We show that, on a large class of closed symplectic manifolds, the Hofer distance from a time-dependent Hamiltonian diffeomorphism to an autonomous Hamiltonian diffeomorphism can be arbitrarily large. This generalize the previous results of [PS16, Zha19], and the main tool is the persistence module theory [PRSZ20] and the construction of linked twisted map[Wan25].

Contents

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1. Introduction

1.1. **Background.** Let (M,ω) be a closed symplectic manifold. Let $\text{Diff}(M)$ be the group of diffeomorphisms on M, with the natural group structure defined from map composition. There are many interesting subgroups of this big group and in this paper we will investigate some of them.

Firstly, the set of symplectic form-preserving diffeomorphisms on (M,ω) forms a subgroup of $\text{Diff}(M)$, denoted by

$$\text{Symp}(M,\omega)=\{\phi\in\text{Diff}(M)\mid\phi^*\omega=\omega\}.$$

There are some elements of this group, called the Hamiltonian diffeomorphism that are of great interest. For any smooth function $H:S^1\times M\to\mathbb{R}$, one can determine a unique vector field X_t by

$$\iota_{X_t}\omega=-dH_t,$$

where $H_t:M\to\mathbb{R}$ is defined by $H_t(x)=H(t,x)$. The time-one map ϕ_H^1 of the flow generated by X_t is called a Hamiltonian diffeomorphism generated by H. One can easily check that the set of Hamiltonian diffeomorphisms is a subgroup of $\text{Symp}(M,\omega)$, and we denote it by $\text{Ham}(M,\omega)$.

There is a celebrated metric on the group $\text{Ham}(M,\omega)$, called the Hofer metric, firstly introduced by Hofer [Hof90]. For any element $\phi\in\text{Ham}(M,\omega)$, define the *Hofer norm* as

$$\|\phi\|_H:=\inf_H\int_0^1\big(\max_M H_t-\min_M H_t\big)dt,$$

where the infimum is taken over all Hamiltonian H with $\phi=\phi_H^1$. This norm induces a metric by

$$d_H(\phi,\psi)=\|\phi^{-1}\circ\psi\|_H,\forall\phi,\psi\in\text{Ham}(M,\omega).$$

Remark 1.1. The non-degeneracy of the above metric is highly non-trivial. It is first proved by [Hof90] for linear symplectic space, and extended to rational symplectic manifolds by [Pol93]. Finally the general case was settled by Lalonde-McDuff[LM95].

The global(or large-scale geometry) of the Hofer metric on Hamiltonian diffeomorphism group has been of great interest and we give some of the problems that are intensively studied.

(1) It is conjectured that the diameter of $\text{Ham}(M,\omega)$ is infinite with respect to the Hofer metric for any closed symplectic manifold (M,ω). This conjecture

is also referred to as the Hofer diameter conjecture [MS17, Question 20]. The work of [McD10] has covered a large class of symplectic manifolds.

(2) Taking a step further, one might ask if there exists a quasi-isometric embedding from some unbounded group to the Hamiltonian diffeomorphism group. We usually consider the case where the source is the standard Euclidean space and the existence of quasi-isometric embedding from Euclidean space of any dimension is also called the existence of infinite dimensional *quasi-flat*. See [Ush13, Ush14, Sun24, CGHS24] for some effort toward this direction.

(3) The problem of embedding free groups into the asymptotic cone of $\mathrm{Ham}(M,\omega)$ has also been considered. [AGKK+19, Cho24] solves this problem for the free group of two generators and surface of genus $g \geq 2$, and [Wan25] confirmed this phenomenon on some higher dimensional manifolds.

In this paper, we also focus on the Hofer geometry of $\mathrm{Ham}(M,\omega)$ from a global perspective, to be precise, we pay our attention on the following two subsets of $\mathrm{Ham}(M,\omega)$.

The first subset is $\mathrm{Aut}(M,\omega) \subseteq \mathrm{Ham}(M,\omega)$, which consists of the elements that can generated by time-independent(as called autonomous) Hamiltonian function. We would call the elements in this subset autonomous Hamiltonian diffeomorphisms.

The second subset is $\mathrm{Ham}_k(M,\omega)$, defined by

$$\mathrm{Ham}_k(M,\omega) = \{\phi \in \mathrm{Ham}(M,\omega) \mid \exists \psi \in \mathrm{Ham}(M,\omega), \phi = \psi^k\}$$

for any integer k.

We observe that there is $\mathrm{Aut}(M,\omega) \subseteq \mathrm{Ham}_k(M,\omega)$ for any $k \geq 1$. For any $\phi \in \mathrm{Aut}(M,\omega)$ generated by a time-independent function H, let ψ be the Hamiltonian diffeomorphism generated by $F(t,x) = \frac{1}{k}H(x)$, then we have $\phi = \psi^k$.

We consider how far can an element in $\mathrm{Ham}(M,\omega)$ be from the above two subsets. More specifically, we give the following definitions.

Definition 1.2. For any closed symplectic manifold (M,ω) and a prime number p, define

$$\mathrm{aut}(M,\omega) = \sup_{\varphi \in \mathrm{Ham}(M,\omega)} d_H(\phi, \mathrm{Aut}(M,\omega))$$

and

$$\mathrm{pow}_p(M,\omega) = \sup_{\varphi \in \mathrm{Ham}(M,\omega)} d_H(\phi, \mathrm{Ham}_p(M,\omega)).$$

Polterovich-Shelukhin proposed the following conjecture.

Conjecture 1.3. *For any closed symplectic manifold* (M, ω), *there is*

$$\operatorname{aut}(M, \omega) = +\infty$$

1.2. **Main results.** The main task of this paper is to confirm the above conjecture for a broad class of manifolds. First we give the following definition.

Definition 1.4. A closed symplectic manifold (M, ω) is said to fit condition (*), if it satisfies

1. (M, ω) is symplectically aspherical.
2. There are two embedded Lagrangian tori T_1, T_2 in M, intersecting transversally at one point.
3. The natural homomorphism $\mathbb{Z}^n * \mathbb{Z}^n \simeq \pi_1(T_1 \cup T_2) \to \pi_1(M)$ is monomorphic.
4. There are homotopically non-trivial curves $a \subseteq T_1$ and $b \subseteq T_2$ so that M is α-atoroidal for any $\alpha \in \langle a, b \rangle$, that is, for any smooth map $\rho : \mathbb{T}^2 = S^1 \times S^1 \to M$ with $\rho(S^1 \times \{t\}) \in \alpha$ for any $t \in S^1$, one has
$$\int_{\mathbb{T}^2} \rho_1^* = \int_{\mathbb{T}^2} \rho^* \omega = 0,$$
where c_1 is the first Chern class of (M, ω).

Our main theorem is

Theorem 1.5. *For any closed symplectic manifold* (M_0, ω_0) *satisfying condition (*) and any close symplectic manifold* (M_1, ω_1), *let* $(M, \omega) = (M_0 \times M_1, \omega_0 \oplus \omega_1)$, *one has*

$$\operatorname{pow}_p(M, \omega) = +\infty$$

for some prime number p.

Since $\operatorname{Aut}(M, \omega) \subseteq \operatorname{Ham}_p(M, \omega)$, the following corollary is immediate.

Corollary 1.6. *For any closed symplectic manifold* (M, ω), *if* (M, ω) *satisfies one of the followings:*

(1) (M, ω) *is a closed oriented surface of genus 1, equipped with the standard area form.*
(2) (M, ω) *is the annulus* $S^1 \times [0, 1]$ *with standard symplectic form.*
(3) There is a closed symplectic manifold (M_0, ω_0) *satisfying condition (*) and some closed symplectic manifold* (M_1, ω_1), *so that* $(M, \omega) = (M_0 \times M_1, \omega_0 \oplus \omega_1)$.

$$\mathrm{aut}(M, \omega) = +\infty.$$

Proof. The statements when (M, ω) being genus-1 surface and the annulus are shown to be true in [Kha22] and [MB21] respectively. The product case follows Theorem 1.5. □

Remark 1.7. We note that both the closed oriented surface of genus 1 and the annulus does not satisfy condition (*). We only take the surface of genus 1 as example, that is the standard torus $\mathbb{T}^2$.

For any two embedded Lagrangian tori T_1, T_2 in M intersecting transversally at one point, let $a = [T_1], b = [T_2]$, then a, b generate $\pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$. Since $\pi_1(T_1) * \pi_1(T_2) \simeq \mathbb{Z} * \mathbb{Z}$ and there exists no injective homomorphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$, the third requirement of condition (*) is not satisfied.

Remark 1.8. In [MB21, Kha22], the authors did not show that the Hofer distance to Ham_p can be arbitrarily large. In fact, their arguments rely heavily on the two-dimensional nature and do not involve careful analysis of persistence modules.

1.3. **Examples.**

Now we present some examples where our main theorem applies.

Example 1.9. Let Σ_g be the closed oriented surface of genus g, and $g \geq 2$. The symplectically aspherical condition is trivial. Besides, it is well-known that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle.$$

Let T_1, T_2 be the embedded generating curves representing a_1, b_1, then they are embedded Lagrangian tori satisfying (2) from condition (*).

There is a small canonical neighborhood N of T_1, T_2 homeomorphic to an one-punctured torus, so

$$\pi_1(N) \simeq F_2 = \langle a_1, b_1 \rangle.$$

Then (3) from condition (*) holds.

Since any continuous map $\rho : \mathbb{T}^2 \to \Sigma_{g\geq 2}$ has vanishing mapping degree, the atoroidal condition is satisfied. So $\Sigma_{g\geq 2}$ fits condition (*).

So our main theorem covers the result of [PS16, Zha19] and [Cho24] which proved Conjecture 1.3 for $\Sigma_{g\geq 4} \times M$(for any closed M) and $\Sigma_{g=2,3} \times M$(for any closed symplectic aspherical M), respectively.

Example 1.10. Let S_1, S_2 be closed surfaces of genus ≥ 2. Suppose a_0 and b_0 generate a free product in $\pi_1(S_1)$. Let $\psi(a_0)$ be a reducible symplectomorphism on S_2, fixing the curve c that is pseudo-Anosov in $S_2 - \{c\}$, similarly $\psi(b_0) \in \mathrm{Symp}(S_2)$ fixes d and is pseudo-Anosov in $S_2 - \{d\}$.

By [Wan25, Lemma 2.1], there exists $N \in \mathbb{N}$, such that for any $|k| > N$, and $\psi : \pi_1(S_1) \to \mathrm{Symp}(S_2)$ where $\psi(a_0) = \psi_0^k(a_0), \psi(b_0) = \psi_0^k(b_0)$, we have that $M = S_1 \ltimes_\psi S_2$ satisfies condition (*).

Example 1.11. Let $M_1 = S_1 \ltimes_{\psi_1} S_2$ be a surface bundle constructed as in Example 1.10, and and $M_2 = S_2 \ltimes_{\psi_2} S_3$ is another surface bundle with ψ_2 has a fixed point in S_3.

If M_2 has a section of Euler class zero, then by [Wan25, Proposition 2.3], their symplectic sum M satisfies condition (*), with Lagrangian tori T_1 and T_2 given by those in M_1.

1.4. **Outline of the proof.** Before we give a brief outline of the proof of Theorem 1.5, we review the previous methods on tackling Conjecture 1.3.

The argument in [PS16](dealing with the product with symplectically aspherical manifolds) relies on the persistent homology theory[ZC05]. They defined a numerical measurement related a Hamiltonian diffeomorphism ϕ by considering the spectral spread of the $\mathbb{Z}_p$-action on the Hamiltonian Floer persistence module of ϕ^p. They studied the dynamic and geometry of the eggbeater model [AGKK+19] to yield the final result.

However, since the version of persistent homology theory developed by [ZC05] only works for monotone symplectic manifold, where the construction Floer chain complex does not involve Novikov theory, to extend the argument of [PS16] to more general manifolds, a persistent homology theory cooperating with Floer-Novikov theory must be involved. This kind of persistent homology theory is realized in [UZ16], then Zhang [Zha19] worked on a p-cyclic version of this theory to drop the symplectically aspherical condition.

Wang [Wan25] constructed the linked twisted map and somehow upgraded the method of [PS16] to a high dimensional nature, since the eggbeater map only serves in the surface setting.

So to prove Theorem 1.5, where we work on the high-dimensional setting and the symplectically aspherical condition on the product manifold is dropped, we need to put the linked twisted map into the framework of p-cyclic persistent homology theory. Now we give a more refined discussion of this.

For a closed symplectic manifold (M, ω) with a fixed regular almost complex structure J, the Floer chain complex $\mathrm{CF}_k(M, H)$ with grading k is a finite dimensional vector space over the Novikov field Λ, along with a filtration ℓ induced from the action.

Since $(\mathrm{CF}_*(M, H), \partial, \ell)$ is a Floer-Novikov type complex in the sense of [UZ16], where ∂ is the Floer boundary operator, the singular value decomposition gives rise to to invariants of the complex, called the *verbose(concise) barcode*.

In order to incorporate with the geometry of $\mathbb{Z}_p$-action on the loop space, Zhang[Zha19] defined a filtered isomorphism T which giving a $\mathbb{Z}_p$-action on the degree k Floer persistence module of ϕ^p and consider the following mapping-cone complex as follow. Fix a homotopy class α and ξ_p a primitive p-th root of the unity, we consider $\mathrm{Cone}_*(H)$ to be the self-mapping cone of the Floer chain complex of $H^{(p)}$ in the class α with respect to $T - \xi_p \cdot \mathrm{id}$.

The following result is crucial in our proof, giving a lower bound of $\mathrm{pow}_p(M, \omega)$ using quantity in p-cyclic persistent homology, which is more computable.

Theorem 1.12. [Zha19, Theorem 1.28] *For any closed symplectic manifold (M, ω), suppose that there exists a Hamiltonian diffeomorphism $\phi = \phi_H$ such that for some $k \in \mathbb{Z}$, $p \nmid m_k$ where m_k is the multiplicity of degree-k concise barcode of $\mathrm{Cone}_*(H)$. Then we have*

$$\mathrm{pow}_p(M, \omega) \geq \frac{1}{24p} \beta_{m_k}(\phi_H).$$

So we only need to seek for the followings.

(i) Find a family of $\phi_N = \phi^1_{H_N} \in \mathrm{Ham}(M, \omega)$ such that $\beta_{m_k}(\phi_N) \to \infty$ as $N \to \infty$;
(ii) Control the non-divisibility of the multiplicity of the concise barcode of $\mathrm{Cone}_*(H_N)$ by prime number p.

The appropriate model that we will work on is the linked twisted map constructed in [Wan25]. It escapes from the two-dimensional nature of eggbeater maps, while preserving the good dynamical properties of them.

More specifically, let $\tau(N, (ab)^p)$ be the linked twisted map, for some primitive loop class γ_N, there are exactly 2^{2p} p-tuples of generators of the corresponding Floer chain complex in class γ_N. Here each tuple $\mathcal{O}_{\alpha,N}$(α is the index of the tuple taking value in $\{+,-\}^{2p}$) consists of p fixed points with same action and Conley-Zehnder index.

Firstly we have that the difference of actions of two different tuples is controlled. This will lead to the unboundedness of boundary depth.

Theorem 1.13. *(Theorem 4.4) There exists constant $c>0, C\in\mathbb{R}$, satisfying the following.*

For N large enough, there exist $\alpha_{i,N},\beta_{i,N}(i=1,\dots,p)$ with $\frac{|\alpha_i|}{N},\frac{|\beta_i|}{N}\in[\frac{\varepsilon}{4},\frac{\varepsilon}{3}]$ so that $\gamma_N=\alpha_{1,N}*\beta_{1,N}*\cdots*\alpha_{p,N}*\beta_{p,N}$ is primitive, and for any $\alpha\neq\alpha'\in\{+,-\}^{2p}$, there holds

$$|\mathcal{A}(\mathcal{O}_{\alpha,N})-\mathcal{A}(\mathcal{O}_{\alpha',N})|\geq cN-C.$$

And the Conley-Zehnder index of the fixed points are rather computable, so it turns out that the divisibility can be controlled when p is large.

Theorem 1.14. *Denote by m_1 the multiplicity of degree-1 concise barcode of $Cone_M(\tau_N)_*$, the self-mapping cone of $\mathrm{CF}_*(M_0\times M_1,\tau_N^{(p)}\times id)_{\gamma_N\times\{pt\}}$. If*

$$p\nmid(qb_{r_p+p-1}(M_1)+2qb_{r_p-1}(M_1)+qb_{r_p-p-1}(M_1)), \tag{1.1}$$

where $r_p=2-n+\frac{1}{2}p(1-n)$ and

$$qb_k(M_1)=\sum_{2N|k_1-k}b_{k_1}(M_1)$$

and N is the minimal Chern number of (M_1,ω_1), then $p\nmid m_1$. In particular, if

$$p>2\sum_{i=0}^{\dim(M_1)}b_i(M_1),$$

then (1.1) is satisfied.

So the linked twisted map will serve for our purpose and combining the above results would yield Theorem 1.5.

This paper is organized as follow.

- In Section 2 we give preliminaries on p-cyclic persistent homology theory, using a language that is more in accord with Hamiltonian Floer theory.
- In Section 3, we review the basic aspects of Hamiltonian Floer theory and put them into the algebraic framework built in Section 2. As application we prove Theorem 1.12.

- In Section 4 we construct linked twisted map on manifolds satisfying condition (*), and give computation on the actions and Conley-Zehnder indices of its (capped) fixed points. Based on the analysis of fixed points and actions we prove Theorem 1.13.
- In Section 5 we complete the proof, including the proof of Theorem 1.14 and thus Theorem 1.5.

Acknowledgments.

2. p-CYCLIC PERSISTENT HOMOLOGY THEORY

2.1. **Non-Archimedean normed vector space.** Recall the following definition of the valuation on a field.

Definition 2.1. A *valuation* ν on a field $\mathbb{K}$ is a function $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ such that

(1) $\nu(x) = \infty$ if and only if $x = 0$;
(2) For any $x, y \in \mathbb{K}, \nu(xy) = \nu(x) + \nu(y)$;
(3) For any $x, y \in \mathbb{K}, \nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ with equality when $\nu(x) = \nu(y)$.

We fix a ground field $\mathbb{K}$ and an additive subgroup $\Gamma \leq \mathbb{R}$, and throughout this article, we always assume that

(1) $\mathbb{K}$ has characteristic zero and contains all the p-th roots of unity;
(2) For all primitive p-th root of unity ξ_p, there exists no solution of $x^p = \xi_p^q$ unless $p \mid q$.

The associated *Novikov field* is defined as

$$\Lambda = \Lambda^{\mathbb{K},\Gamma} = \big\{ \sum_{g \in \Gamma} a_g T^g \mid a_g \in \mathbb{K}, \#\{g \mid a_g \neq 0, g < C\} < \infty, \forall C \in \mathbb{R} \big\},$$

where T is a formal variable. Let

$$\nu(\sum_{g \in \Gamma} a_g T^g) = \min\{g \mid a_g \neq 0\},$$

then ν gives a valuation on the Novikov ring Λ.

Definition 2.2. A *non-Archimedean normed vector space* over Λ is a pair (C, ℓ) where C is a vector space over Λ endowed with a *filtration*

$$\ell : C \to \mathbb{R} \cup \{-\infty\}$$

satisfying the following axioms:

(1) $\ell(x) = -\infty$ if and only if $x = 0$;
(2) For any $\lambda \in \Lambda$ and $x \in C$, one has $\ell(\lambda x) = \ell(x) - \nu(\lambda)$;
(3) For any $x, y \in C, \ell(x+y) \leq \max\{\ell(x), \ell(y)\}$.

We need the following notation of orthogonality, which is similar to the standard one in Euclidean space.

Definition 2.3. Let (C, ℓ) be a non-Archimedean normed vector space over a Novikov field Λ.

- Two subspaces V and W of C are said to be *orthogonal* if for all $v \in V$ and $w \in W$, we have
$$\ell(v+w) = \max\{\ell(v), \ell(w)\}.$$
- A finite ordered collection $(w_1, \ldots, w_r)$ of elements of C is said to be *orthogonal* if, for all $\lambda_1, \ldots, \lambda_r \in \Lambda$, we have
$$\ell\left(\sum_{i=1}^{r} \lambda_i w_i\right) = \max_{1 \leq i \leq r} \ell(\lambda_i w_i). \tag{4}$$

The following little lemma will be used later.

Lemma 2.4. *Given a set of orthogonal elements $\{v_1, \ldots, v_n\}$ over Λ and any strictly lower filtration perturbation for each v_i, i.e. $v_i + w_i$ where $\ell(w_i) < \ell(v_i)$, for $1 \leq i \leq n$, then*
$$\{v_1 + w_1, \ldots, v_n + w_n\}$$
are also orthogonal over $\Lambda^{\mathbb{K},\Gamma}$.

Proof. For any $\lambda_1, \ldots, \lambda_n \in \Lambda$,by [UZ16, Proposition 2.3],
$$\begin{aligned}\ell\big(\lambda_1(v_1+w_1) + \cdots + \lambda_n(v_n+w_n)\big) &= \ell(\lambda_1 v_1 + \cdots + \lambda_n v_n) \\ &= \max_{1 \leq i \leq n}\{\ell(\lambda_i v_i)\} \\ &= \max_{1 \leq i \leq n}\{\ell(\lambda_i(v_i + w_i))\}.\end{aligned}$$
So $\{v_1 + w_1, \ldots, v_n + w_n\}$ are also orthogonal. □

The main object that we study is the Floer-type complex.

Definition 2.5. [UZ16, Definition 4.1] A *Floer-type complex* $(C_*, \partial_C, \ell_C)$ over a Novikov field $\Lambda = \Lambda^{\mathbb{K},\Gamma}$ is a chain complex $(C_* = \bigoplus_{k\in\mathbb{Z}} C_k, \partial_C)$ over Λ together with a function $\ell_C : C_* \to \mathbb{R}\cup\{-\infty\}$ so that each $(C_k, \ell_C|_{C_k})$ is an orthogonalizable Λ-space, and for each $x \in C_k$, there is $\ell_C(\partial_C x) \leq \ell_C(x)$.

For a linear map $T : (V, \ell_1) \to (W, \ell_2)$, we can choose a nice orthogonal bases of both source space and target space so that the structure of map T is easy under the bases.

Proposition 2.6. [UZ16, Theorem 3.4] *Let (C, ℓ_C) and (D, ℓ_D) be orthogonalizable Λ-spaces and let $T : C \to D$ be a linear map with rank r. Then there exists a* ***singular value decomposition(SVD)*** *of T, that is, a choice of orthogonal ordered bases $(y_1, \dots, y_n)$ for C and $(x_1, \dots, x_m)$ for D such that:*

(i) $(y_{r+1}, \dots, y_n)$ *is an orthogonal ordered basis for* $\ker T$*;*
(ii) $(x_1, \dots, x_r)$ *is an orthogonal ordered basis for* $\operatorname{Im} T$*;*
(iii) $Ay_i = x_i$ *for* $i \in \{1, \dots, r\}$*;*
(iv) $\ell_C(y_1) - \ell_D(x_1) \geq \cdots \geq \ell_C(y_r) - \ell_D(x_r)$*.*

We can the above result to the boundary operator of a Floer-type complex. To be precise, let (C_*, ∂, ℓ) be a Floer-type complex, for any degree k, consider the restriction $\partial_{k+1} : C_{k+1} \to \ker(\partial_k)$, there is a singular value decomposition given by $((y_1, \dots, y_n), (x_1, \dots, x_m))$. The degree k *verbose barcode* of (C_*, ∂, ℓ) consists of multiset of elements of $(\mathbb{R}/\Gamma) \times [0, \infty]$ of the form

- $(\ell(x_i) \mod \Gamma, \ell(y_i) - \ell(x_i))$ for $i = 1, \dots, r$;
- $(\ell(x_i) \mod \Gamma, \infty)$ for $i = r+1, \dots, m$.

And the degree k *concise barcode* is the submultiset of the verbose barcode, consisting of the elements with positive second entry.

In this article we will work with the following algebraic structure called mapping cone. To be specific, let (C_*, ∂_*, ℓ) be a Floer-type complex and T be a filtration-preserving map on it, we define a chain complex $(\mathrm{Cone}_C(T)_*, \partial_{co})$ by

$$\mathrm{Cone}_C(T)_k := C_k \oplus C_{k-1}$$

and the boundary map defined by

$$\partial_{co} = \begin{pmatrix} \partial & -T \\ 0 & -\partial \end{pmatrix}.$$

And there is a filtration ℓ_{co} on $\mathrm{Cone}_C(T)_*$ by

$$\ell_{co}(x_1, x_2) = \max\{\ell(x_1), \ell(x_2)\}.$$

The filtered chain complex is called the *self mapping-cone* with respect to T.

We note the following lemma, which will explain that our later constructed mapping cone under the Hamiltonian Floer setting is independent of the choice of the almost complex structure.

Lemma 2.7. [Zha19, Lemma 4.1] *For two filtration preserving chain maps Φ and Ψ on a Floer-type complex (C_*, ∂_*, ℓ), if Φ and Ψ are filtered homotopic, then the associated self-mapping cones $\mathrm{Cone}_C(\Phi)_*$ and $\mathrm{Cone}_C(\Psi)_*$ are filtered isomorphic to each other.*

For each linear map F on C_*, we can define its associated double map by

$$\mathcal{D}_F(x_1, x_2) = (Fx_1, Fx_2), \forall (x_1, x_2) \in \mathrm{Cone}_C(T)_*.$$

2.2. **p-cyclic singular value decomposition.** In this section we always consider a self-mapping cone $\mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id}) := (\mathrm{Cone}_C(T - \xi_p \cdot \mathrm{id})_*, \partial_{co}, \ell_{co})$ of (C_*, ∂, ℓ), with respect to some linear map $T - \xi_p \cdot \mathrm{id}$, satisfying the following conditions, where ξ_p is a primitive p-th root of unity:

1. The homology of the mapping cone $\mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id})$ vanishes.
2. T is a strictly lower perturbation of R_p which is commutative with ∂, where R_p is a $\mathbb{Z}_p$-action on (C_*, ∂, ℓ).
3. There exists a strictly lower perturbation S of R_{p^2} which is commutative with ∂ so that $\mathcal{D}_S^p = \mathcal{D}_T$, and $\mathcal{D}_{R_p^2}$ is a $\mathbb{Z}_{p^2}$-action on $\mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id})$.
4. There is a positive constant $\hbar > 0$ so that

$$\ell_{co}(\mathcal{D}_T x - \mathcal{D}_{R_p} x) \leq \ell(x) - \hbar, \quad \ell_{co}(\mathcal{D}_S x - \mathcal{D}_{R_{p^2}} x) \leq \ell(x) - \hbar$$

holds for any $x \in \mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id})$.

We first note that the above conditions ensure the invertibility of $\mathcal{D}_T$ and $\mathcal{D}_S$.

Proposition 2.8. *Both $\mathcal{D}_T$ and $\mathcal{D}_S$ are invertible.*

Proof. One can write

$$\mathcal{D}_T^p = 1 - Q_T \qquad \text{and} \qquad \mathcal{D}_S^{p^2} = 1 - Q_S,$$

since $\mathcal{D}_{R_p}^p = \mathcal{D}_{R_{p^2}}^{p^2} = \mathrm{id}$, where Q_T and Q_S strictly lower filtration by at least $\hbar$. Then $\ell_{co}(Q_T^k(x))$ diverges to $-\infty$ for any $x \in \mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id})$.

So the operator

$$B_T = \mathrm{id} + Q_T + Q_T^2 + \dots$$

is well defined and it is the inverse of $\mathrm{id} - Q_T$. Moreover, it is direct to check that B_T commutes with $\mathcal{D}_T$. So let $B'_T = (\mathcal{D}_T)^{p-1} \circ B_T$, there is

$$\mathcal{D}_T B'_T = (\mathcal{D}_T)^p B_T = (\mathrm{id} - Q_T)(\mathrm{id} - Q_T)^{-1} = \mathrm{id} = (\mathcal{D}_T)^{p-1} B_T \mathcal{D}_T = B'_T \mathcal{D}_T.$$

So B'_T is the inverse of $\mathcal{D}_T$. The construction with respect to S is almost identical, so we omit it here. □

By an almost purely algebraic argument [Zha19, Lemma 5.3, Corollary 5.4], one has

Corollary 2.9. *There are strictly lower filtration perturbation T', S' of T and S, so that $\mathcal{D}_{T'}^p = id$ and $\mathcal{D}_{S'}^p = \mathcal{D}_{T'}$. Moreover, both $\mathcal{D}_{T'}$ and $\mathcal{D}_{S'}$ commute with ∂_{co}.*

Now since $[\mathcal{D}_{T'}, \partial_{co}] = 0$, $\ker(\partial_{co})$ is a $\mathcal{D}_{T'}$-invariant subspace of $\mathrm{Cone}_k(T - \xi_p \cdot \mathrm{id})$. Then one can construct a $\mathcal{D}_{T'}$-invariant orthogonal complement of $\ker(\partial_{co,k+1}) = \mathrm{Im}(\partial_{co,k+2})$ in $\mathrm{Cone}_k(T - \xi_p \cdot \mathrm{id})$, where the equality comes from our vanishing condition on the homology of self-mapping cone.

We write the complement as W. Then because $\mathcal{D}_{T'}^p = \mathrm{id}$, the eigenvalues of $\mathcal{D}_{T'}$ lie in the set $\{\xi_p, \dots, \xi_p^{p-1}, 1\}$, which is contained in the ground field $\mathbb{K}$ by our assumption. By a simple algebraic argument, this set is also contained in the Novikov ring $\Lambda^{\mathbb{K},\Gamma}$, so $\mathcal{D}_{T'}$ is diagonalizable.

Consider the eigenspace decomposition

$$W = F_0 \oplus F_1 \cdots \oplus F_{p-1} \text{ and } \mathrm{Im}(\partial_{co}) = G_0 \oplus G_1 \cdots \oplus G_{p-1},$$

where F_i is the eigenspace of $\mathcal{D}_{T'}$ with respect to eigenvalue ξ_p^i, similarly for G_i. And the decomposition is orthogonal in the sense that $\{F_i\}_{i=0}^{p-1}$ are mutually orthogonal to each other and so do $\{G_i\}_{i=0}^{p-1}$.

By a sophisticated algebraic analysis, there is

Proposition 2.10. [Zha19, Proposition 5.24] *Let $K_0 = G_0^+ \oplus F_0$, where G_0^+ is the 1-eigenspace from the decomposition of $Im(\partial_{co})$ as above, then $\partial_{co}|_{K_0} : K_0 \to G_0$ only contributes 0-length bars.*

On the other hand, since $\ker(\partial_{co,k+1}) = \mathrm{Im}(\partial_{co,k+2})$, the decomposition of G_i's gives rise to a SVD of $\partial_{co}|_{\ker \partial_{co}}$. Besides, since there is a SVD on each of the $\partial_{co}|_{F_i}$ by [UZ16, Theorem 3.5], the SVD on $\partial_{co}|_W$ is obtained then. So we have conducted singular value decomposition of $\partial_{co} : \mathrm{Cone}_*(T - \xi_p \cdot \mathrm{id}) \to \mathrm{Im}(\partial_{co})$.

By inductively working more carefully on $\partial_{co}|_{F_i} : F_i \to G_i$ piece by piece, one can obtain a SVD

$$\partial_{co}|_{\oplus F_i} : \bigoplus_{i=1}^{p-1} F_i \to \bigoplus_{i=1}^{p-1} G_i$$

in the form of p-tuple, which is compatible with $\mathcal{D}_{S'}$ in the sense of [Zha19].

3. Hamiltonian Floer theory

3.1. **Basic aspects.** In this section we review the basic aspects of Hamiltonian Floer theory. Consider a closed symplectic manifold (M, ω) and a Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ on it. Let ϕ_H^t be the flow generated by H and $\mathrm{Fix}(H)$ be the set of fixed points of $\phi = \phi_H^1$. Any element in $\mathrm{Fix}(H)$ can be also regarded as a 1-periodic orbit γ of X_{H_t}. An element $\gamma \in \mathrm{Fix}(H)$ is said to be *non-degenerate* if the linearization of its Poincar é return map does not have 1 as its eigenvalue. A Hamiltonian H is non-degenerate if all the elements in $\mathrm{Fix}(H)$ are non-generate.

For any homotopy class $\alpha \in \pi_1(M)$, write the loop space in the class α on M as $\mathcal{L}_\alpha(M)$ and the universal cover is $\tilde{\mathcal{L}}_\alpha(M)$. When $\alpha = [pt]$ we omit it from the notations.

A *capping* u of $\gamma \in \mathrm{Fix}(H)$ is a smooth map $u : \mathbb{D} \to M$ with

$$u(e^{2\pi i t}) = \gamma(t), \forall t \in [0, 1].$$

Two cappings u, u' of the same orbit γ are called equivalent if and only if

$$\omega(u) = \omega(u'), c_1(u) = c_1(u'),$$

where $c_1 = c_1(TM, \omega)$ is the Chern class of the triple (M, ω, J), which is independent of the choice of compatible almost structure J. And the set of equivalence classes is denoted $\widetilde{\mathrm{Fix}}(H)$.

The action of an element $[\gamma, u] \in \widetilde{\mathrm{Fix}}(H)$ is defined as

$$\mathcal{A}_H([\gamma, u]) = \int_S^1 H(t, \gamma(t))dt + \int_{\mathbb{D}} u^*\omega$$

and the degree is

$$\deg([\gamma, u]) = n + \mathrm{CZ}([\gamma, u]),$$

where $\mathrm{CZ}([\gamma, u])$ is the Conley-Zehnder index defined as in [CZ83].

Let Γ be defined as

$$\Gamma = \{\int_{S^2} u^*\omega \mid u : S^2 \to M, c_1(u) = 0\},$$

then $\Gamma \leq \mathbb{R}$ is an additive subgroup and we can define the associated Novikov ring $\Lambda = \Lambda^{\mathbb{K},\Gamma}$, where $\mathbb{K}$ is a fixed ground field.

For a S^1-parametrized almost complex structure $J = (J_t)_t$ in the sense of [HS95], define the Floer chain complex in class α as

$$\mathrm{CF}_k(H, J)_\alpha = \{x = \sum a_{[\gamma,u]}[\gamma, u] \mid a_{[\gamma,u]} \in \mathbb{K}, [\gamma, u] \in \mathcal{L}_\alpha(M) CZ([\gamma, u]) = k,$$
$$\#\{[\gamma, u] \mid a_{[\gamma,u]} \neq 0, \mathcal{A}_H([\gamma, u]) > C\} < \infty (\forall C \in \mathbb{R})\}.$$

The Novikov field Λ acts on $\mathrm{CF}_k(H, J)_\alpha$ by

$$t^g \cdot [\gamma, u] = [\gamma, u\#v],$$

where v is a sphere with symplectic area g. Then it is direct to check that $\mathrm{CF}_k(H, J)_\alpha$ is a finite dimensional vector space over the Novikov field Λ, and the dimension is equal to the number of $\gamma \in \mathcal{L}_\alpha(X)$ so that there is a capping u with $CZ([\gamma, u]) = k$.

Define a function ℓ_H on $\mathrm{CF}_k(H, J)_\alpha$ by

$$\ell_H(\sum a_{[\gamma,u]}[\gamma, u]) = \max\{\mathcal{A}_H([\gamma, u]) \mid a_{[\gamma,u]} \neq 0\},$$

then one can see that $\mathrm{CF}_k(H, J)_\alpha$ is a non-Archimedean normed vector space with filtration ℓ_H, in the sense of Definition 2.2.

The boundary operator $(\partial_{H,J})_k : \mathrm{CF}_k(H, J) \to \mathrm{CF}_{k-1}(H, J)$ is defined by

$$(\partial_{H,J})_k([\gamma_-, u_-]) = \sum a_{[\gamma_+,u_+]} \cdot [\gamma_+, u_+],$$

where $a_{[\gamma_+,u_+]}$ is given by counting the moduli space(modulo the $\mathbb{R}$-action) of the solutions of the following differential equation,

$$\frac{\partial u}{\partial s} + J_t(u(s, t))\big(\frac{\partial u}{\partial t} - X_{H_t}(u(s, t))\big) = 0,$$

satisfying that

$$u(s, \cdot) \to \gamma_\pm(\cdot), s \to \pm\infty \text{ and} [\gamma_+, u_+] = [\gamma_+, u_-\#u].$$

For semipositive manifolds, the above counting, thus the boundary operator is well-defined by [HS95]. And [Par16] adapted virtual techniques to cover the general case.

Besides, by the Gromov compactness theorem, there always exists $\hbar > 0$ so that

$$\ell_H(\partial_{H,J}x) \leq \ell_H(x) - \hbar, \forall x \in \mathrm{CF}_*(H, J). \tag{3.1}$$

Moreover, by the standard Floer continuation argument(see [Sal99, AD14]), the homology of the complex $(\mathrm{CF}_*(H, J), \partial_{H,J})$ does not depend on the choice of the pair (H, J), and we will denote this homology by $\mathrm{HF}_*(M, \omega)$. By choosing a C^2-small non-degenerate autonomous function, there is

Theorem 3.1. *For any $k \in \mathbb{Z}$, there is*

$$\mathrm{HF}_k(M, \omega) = \bigoplus_{j \equiv k \mod 2N} H_j(M; \mathbb{K}) \otimes \Lambda^{\mathbb{K},\Gamma},$$

where N is the minimal Chern number of (M, ω).

The above discussion works in the setting of $\mathrm{CF}_k(H, J)_\alpha$ in a parallel way [Ush13], where α is a non-trivial homotopy class. The only difference is that when α is non-trivial, if we consider C^2-small autonomous Hamiltonian function, there is no generators of the Floer chain complex in the class α. So the invariance of Floer homology gives $\mathrm{HF}_*(M, \omega)_\alpha = 0$.

3.2. **Construction of the operators and numerical measurements.** Now we construct the operators that satisfy the conditions on Section 2.2. The idea is to perturb the rotation operator to make sure that we work on the same chain complex. In this section, we fix a primitive free non-trivial homotopy class α.

Proposition 3.2. [Zha19, Proposition 1.15] *For any closed symplectic manifold (M, ω) and $\phi = \psi^p \in \mathrm{Ham}_p(M, \omega)$, where p is a prime number. Let H be the Hamiltonian generating ϕ and $R_p : \mathrm{CF}_k(H^{(p)}, J_t)_\alpha \to \mathrm{CF}_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha$ be induced from the rotation operator*

$$R_p(x(t)) = x(t + \frac{1}{p}),$$

where $H(t, x) = pF(pt, x)$ and F generates ψ. Also let

$$R_{p^2} : \mathrm{CF}_k(H^{(p)}, J_t)_\alpha \to \mathrm{CF}_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha$$

be the chain map defined as in [PS16, Theorem 4.22]. *Then there exists continuation maps*

$$C : \mathrm{CF}_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \to \mathrm{CF}_k(H^{(p)}, J_t)_\alpha$$

and

$$C' : \mathrm{CF}_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha \to \mathrm{CF}_k(H^{(p)}, J_t)_\alpha,$$

so that $T = S^p$ *for* $T = C \circ R_p, S = C' \circ R_{p^2}$.

So by the above proposition and (3.1), the Floer-type complex $\mathrm{CF}_*(H^{(p)}, J)_\alpha$ and the operators T, S, R_p, R_{p^2} satisfy the conditions in Section 2.2, where the commutativity condition can be checked through a direct diagram chasing. Moreover, by Lemma 2.7, the self-mapping cone $\mathrm{Cone}_{\mathrm{CF}_*(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathrm{id})_*$ is independent of the choice of the continuation map in Proposition 3.2, up to a filtered isomorphism.

Recall that in Section 2.2, we proved the existence of a compatible SVD of

$$\partial_{co}|_{\oplus F_i} : \bigoplus_{i=1}^{p-1} F_i \to \bigoplus_{i=1}^{p-1} G_i$$

in the p-tuple form. Besides, by Proposition 2.10, the restriction $\partial_{co}|_{K_0} : K_0 = G_0^+ \oplus F_0 \to G_0$ only contributes 0-length bars. So for any degree $k \in \mathbb{Z}$, each bar in the degree k concise barcode of the boundary map

$$(\partial_{co})_{k+1} : \mathrm{Cone}_{\mathrm{CF}_*(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathrm{id})_{k+1} \to \mathrm{Im}(\partial_{co})_{k+1}$$

has a p-divisible multiplicity.

This motivate us to come up with the following definition.

Definition 3.3. Let $\{\beta_i(\phi_H, \xi_p, k)\}_i$ be the collection of lengths of bars in the degree k concise barcode of $\mathrm{Cone}_{\mathrm{CF}_*(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathrm{id})_*$. Order then as

$$\beta_1(\phi_H, \xi_p, k) \geq \cdots \geq \beta_{m_k}(\phi_H, \xi_p, k) > 0,$$

then m_k is the multiplicity of the degere k concise barcode. Define

$$\mathfrak{o}(\phi_H, \xi_p, k) := \max_{s \in \mathbb{N}} (\beta_{sp+1}(\phi_H, \xi_p, k) - \beta_{sp+p}(\phi_H, \xi_p, k)),$$

where $\beta_i(\phi_H, \xi_p, k) = 0$ if $i > m_k$. And the *divisbility sensitive invariant* of ϕ_H is

$$\mathfrak{o}(\phi) = \max_{\xi_p} \sup_{k \in \mathbb{Z}} \mathfrak{o}(\phi_H, \xi_p, k),$$

where the maximum is taken over all the primitive p-th roots of unity.

Note that as explained in [Zha19, Remark 1.21], the above invariant $\mathfrak{o}(\phi)$ is independent of the choice of Hamiltonian function H.

By definition, we have that

$$\mathfrak{o}(\phi) = 0, \forall \phi \in \mathrm{Ham}_p(M, \omega). \tag{3.2}$$

Moreover, if $p \nmid m_k$ for some k, let s_0 be the largest multiple of p that is smaller than m_k, then

$$\beta_{s_0 p+1}(\phi) \neq 0, \beta_{(s_0+1)p}(\phi) = 0.$$

So

$$\mathfrak{o}(\phi) \geq \mathfrak{o}(\phi)_k \geq \beta_{s_0 p+1}(\phi) - \beta_{(s_0+1)p}(\phi) = \beta_{s_0 p+1}(\phi) \geq \beta_{m_k}(\phi). \tag{3.3}$$

3.3. **Lipschitz type results.** The main purpose of this section is to connect the above defined invariants to the Hofer distance. For simplicity we write

$$\mathrm{Cone}(H) = \left(\mathrm{Cone}_{\mathrm{CF}_*(H^{(p)}, J_t)_\alpha}(T - \xi_p \cdot \mathrm{id})_*, \partial_{co,H}\right)$$

throughout this section. Firstly we introduce some algebraic notions.

Definition 3.4. (1) For a $\Lambda^{\mathbb{K},\Gamma}$-linear map

$$F : (V, \ell_1) \to (W, \ell_2)$$

between two orthogonalizable $\Lambda^{\mathbb{K},\Gamma}$-spaces, F is called a *δ-morphism* if there exists a $\delta \geq 0$ such that for any $v \in V$,

$$\ell_2(Fv) \leq \ell_1(v) + \delta.$$

(2) Given two Floer-type complexes $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$, suppose that chain map $\Phi : C_* \to D_*$ is a δ_+-morphism and $\Psi : C_* \to D_*$ is a δ_--morphism. Then we say that Φ and Ψ are $(\delta_+, \delta_-, \lambda)$-*homotopic* where $\lambda \in \mathbb{R}_{\geq 0}$ if there exists a degree-1 $\lambda(\delta_+ + \delta_-)$-morphism $K : C_* \to D_{*+1}$ such that

$$\Phi - \Psi = K \circ \partial_C + \partial_D \circ K,$$

where K is then called a $(\delta_+, \delta_-, \lambda)$-homotopy.

We address the following.

Theorem 3.5. [Zha19, Proposition 7.6] *Let $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ be two Floer-type complexes with filtration-preserving chain maps W, W' respectively. If there exists $\delta_+, \delta_- > 0$ so that*

- *there exist chain maps $F_{C,D} : C \to D$ and $F_{D,C} : D \to C$ so that $F_{D,C} \circ F_{C,D}$ provides a (δ_+, δ_-)-quasiequivalence in the sense of* [UZ16]*;*

- *$W' \circ F_{C,D}$ and $F_{C,D} \circ S$ are $(\delta_+, \delta_+, \frac{1}{2})$-homotopic;*
- *$W \circ F_{D,C}$ and $F_{D,C} \circ S'$ are $(\delta_-, \delta_-, \frac{1}{2})$-homotopic.*

Then there exist $\Delta_+, \Delta_- > 0$ with $\Delta_+ + \Delta_- \leq 6(\delta_+ + \delta_-)$ so that $Cone_C(W)$ and $Cone_D(W')$ are (Δ_+, Δ_-)-quasiequivalent.

Now we consider any two Hamiltonian diffeomorphisms $\phi, \psi \in \mathrm{Ham}(M, \omega)$ generated by H and G respectively. Let $W = T^H - \xi_p \cdot \mathrm{id}, W' = T^G - \xi_p \cdot \mathrm{id}$, then they are filtration-preserving chain maps on $\mathrm{CF}_*(H^{(p)}, J)_\alpha$ and $\mathrm{CF}_*(G^{(p)}, J_t)_\alpha$ respectively. Let

$$F_{H,G} : \mathrm{CF}_k(H^{(p)}, J_t)_\alpha \to \mathrm{CF}_k(G^{(p)}, J_t)_\alpha \text{ and } F_{G,H} : \mathrm{CF}_k(G^{(p)}, J_t)_\alpha \to \mathrm{CF}_k(H^{(p)}, J_t)_\alpha$$

be the Floer continuation map.

Define

$$\delta_+ = p \int_0^1 \max_M (H - G) dt, \delta_- = p \int_0^1 - \min_M (H - G) dt,$$

then it is standard that $F_{G,H} \circ F_{H,G}$ is a (δ_+, δ_-)-quasiequivalence. Moreover, by [Ush11, Section 2], $W' \circ F_{H,G}$ and $F_{H,G} \circ W$ are $(\delta_+, \delta_+, \frac{1}{2})$-homotopic, and $F_{G,H} \circ W'$ and $W \circ F_{G,H}$ are $(\delta_-, \delta_-, \frac{1}{2})$-homotopic. So all the conditions in Theorem 3.5 are met, hence there is a (Δ_+, Δ_-)-quasiequivalence between $\mathrm{Cone}(H)$ and $\mathrm{Cone}(G)$.

Let d_Q be the quasiequivalence distance, then

$$d_Q(\mathrm{Cone}(H), \mathrm{Cone}(G)) \leq \frac{\Delta_+ + \Delta_-}{2} \leq 3(\delta_+ + \delta_-) = 3p\|G - H\|_H.$$

On the other hand, by [UZ16, Corollary 8.8], there is

$$\begin{aligned} |\beta_i(\phi) - \beta_i(\psi)| &\leq 4 d_Q(\mathrm{Cone}(H), \mathrm{Cone}(G)) \\ &\leq 12p\|G - H\|_H. \end{aligned}$$

Then for any degree k and primitive p-th root of unity ξ_p, if $\mathfrak{o}(\phi, \xi_p, k)$ is realized by $s_0 \in \mathbb{N}$, there is

$$\begin{aligned} \mathfrak{o}(\phi, \xi_p, k) - \mathfrak{o}(\psi, \xi_p) &\leq \beta_{s_0 p+1}(\phi) - \beta_{s_0 p+p}(\phi) - \beta_{s_0 p+1}(\psi) + \beta_{s_0 p+p}(\psi) \\ &= \big(\beta_{s_0 p+1}(\phi) - \beta_{s_0 p+1}(\psi)\big) + \big(\beta_{s_0 p+p}(\psi) - \beta_{s_0 p+p}(\phi)\big) \\ &\leq 24p\|H - G\|_H. \end{aligned}$$

By taking supremum over degree k and root of unity ξ_p, consideing symmetry, we have that

$$|\mathfrak{o}(\phi) - \mathfrak{o}(\psi)| \leq 24p\|H - G\|_H. \tag{3.4}$$

Now we can give a proof of Theorem 1.12.

Proof of Theorem 1.12. For any $\varepsilon > 0$, there exists some $\psi \in \mathrm{Ham}_p(M,\omega)$ so that

$$d_H(\phi_H, \mathrm{Ham}_p(M,\omega)) + \varepsilon \geq d_H(\phi_H, \psi).$$

Since by (3.3), there is $\mathfrak{o}(\psi) = 0$, so

$$\mathfrak{o}(\phi_H) = |\mathfrak{o}(\phi_H) - \mathfrak{o}(\psi)| \leq 24p d_H(\phi_H, \psi),$$

where the last inequality comes from (3.4).

By (3.2) and condition $p \nmid m_k$, there is $\mathfrak{o}(\phi_H) \geq \beta_{m_k}(\phi_H)$, then

$$d_H(\phi_H, \mathrm{Ham}_p(M,\omega)) + \varepsilon \geq d_H(\phi_H, \psi) \geq \frac{1}{24p}\mathfrak{o}(\phi_H) \geq \frac{1}{24p}\beta_{m_k}(\phi_H).$$

By letting $\varepsilon \to 0$, the result follows. □

4. Geometry of linked twisted map

4.1. **Geometric construction.** We review the geometric construction of linked twisted map by [Wan25]. Throughout this section, let (M_0, ω_0) be a closed symplectic manifold satisfying condition (*) and T_1, T_2 be the embedded Lagrangian tori. By Weinstein's theorem, there exists a tubular neighborhood of T_1 and T_2 which is symplectomorphic to the plumbing of the cotangent bundles of two tori $T^*_\varepsilon\mathbb{T}^n \cup_\psi T^*_\varepsilon\mathbb{T}^n$, where $T^*_\varepsilon\mathbb{T}^n \subseteq T^*\mathbb{T}^n$ is the subset of cotangent vectors with lengths $\leq \varepsilon$, and the plumbing is defined as follow.

Definition 4.1. Given two manifolds L_1 and L_2 with points $p_i \in L_i$ and chosen Riemannian metric g_i on $L_i (i = 1, 2)$, consider neighborhoods U_i of p_i. Assume that there exist local isometries

$$\psi_i : U_i \to B(2) \subseteq \mathbb{R}^n$$

such that $\psi_i(p_i) = 0$ and $\psi_i(U_i) = B(1)$. These induce symplectomorphisms $T^*\psi_i$ from the unit disk bundles D^*U_i to $D^n \times D^n \subseteq D^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$.

Let $J : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be the standard almost-complex linear symplectomorphism, then we can define symplectomorphism from D^*U_1 to D^*U_2 by

$$\psi = (T^*\psi_2)^{-1} \circ J \circ T^*\psi_1.$$

The resulting plumbing space $T^*L_1 \#_\psi T^*L_2$ is defined as the completion of the plumbing domain

$$P_\psi(L_1, L_2) := D^*L_1 \cup_\psi D^*L_2 / \{x \sim \psi(x), \forall x \in D^*U_1\}.$$

Let τ^t be a time change of the geodesic flow on T^*T^n, given by $\tau^t(v,x) = (v, x + t\rho(|v|)v)$, where

$$\rho(r) = \begin{cases} 1, & r \le \varepsilon/2, \\ \dfrac{2(\varepsilon - r)}{\epsilon}, & \varepsilon/2 < r \le \varepsilon, \\ 0, & r > \varepsilon, \end{cases}$$

and ϵ sufficiently small. For small positive δ, let

$$\tau^t_\delta(v,x) = (v, x + t\rho_\delta(|v|)v)$$

be the δ-smoothing of τ^t, where $\rho_\delta(r)$ is a δ-smoothing of ρ.

The Hamiltonian path τ^t_δ is generated by a δ-smoothing of the Hamiltonian function $H(v,x) = h(|v|)$, where

$$h(r) = \begin{cases} \dfrac{r^2}{2} - \dfrac{7}{24}\varepsilon^2, & r \le \varepsilon/2, \\ r^2 - \dfrac{2r^3}{3\varepsilon} - \dfrac{\varepsilon^2}{3}, & \varepsilon/2 < r \le \varepsilon, \\ 0, & r > \varepsilon. \end{cases}$$

Then for any word w in the free group $\mathbb{F}_2 = \langle a, b \rangle$ and $N \in \mathbb{N}$, if

$$w = a^{l_1} b^{l_2} \dots a^{l_{2m-1}} b^{l_{2m}},$$

define

$$\tau(N, w) = \tau^{l_1 N}_{1, \frac{1}{N^2}} \circ \tau^{l_2 N}_{2, \frac{1}{N^2}} \circ \cdots \circ \tau^{l_{2m-1} N}_{1, \frac{1}{N^2}} \circ \tau^{l_{2m} N}_{2, \frac{1}{N^2}}$$

where $\tau^t_{1,\delta}, \tau^t_{2,\delta}$ are two copies of τ^t_δ on the two copies of $T^*\mathbb{T}^n$. Then $\tau(N, w)$ extends to a Hamiltonian diffeomorphism on M_0.

4.2. **Action and index.** In this section we focus on the word $w = (ab)^p$, so

$$l_1 = l_2 = \cdots = l_{2m-1} = l_{2m} = 1.$$

For any fixed point (v_0, x_0) of $\tau(N, (ab)^p)$, with corresponding orbit in a homotopy class $\gamma = \alpha_1 * \beta_1 * \cdots * \alpha_p * \beta_p \in \pi_1(M, *)$(the base point is the plumbing point), where $\alpha_i \in \pi_1(T_1)$ and $\beta_i \in \pi_1(T_2)$, and they can be also regarded as elements in $\mathbb{Z}^n$.

If $0 \ne \alpha_i, \beta_i$, define intermediate points as

$$(v_i, x_i) = \tau^N_{1, \frac{1}{N^2}} \circ \tau^N_{2, \frac{1}{N^2}} \circ \cdots \circ \tau^N_{1, \frac{1}{N^2}} \circ \tau^N_{2, \frac{1}{N^2}}(v_0, x_0),$$

then by our construction, we have

$$x_{i+1} = x_i + N\rho_{\frac{1}{N^2}}(|v_i|)v_i - \alpha_i,$$
$$v_{i+1} = v_i - N\rho_{\frac{1}{N^2}}(|x_{i+1}|)x_{i+1} - \beta_i,$$
$$x_p = x_0, \qquad v_p = v_0.$$

For any $N > \frac{10}{\varepsilon}$, choose $\alpha_i \in \langle a\rangle, \beta_i \in \langle b\rangle$ with $|\alpha_i|, |\beta_i| \in [\frac{N\varepsilon}{4}, \frac{N\varepsilon}{3}]$. Then a direct computation yields that $x_i \in B_i^- \cup B_i^+$, where

$$B_i^* = B(r_i^* \cdot \frac{\beta_{i-1}}{|\beta_{i-1}|}, \frac{1}{N}), * \in \{+,-\},$$

and r_i^-, r_i^+ are roots of equation

$$r\rho(|r|) = -\frac{|\beta_{i-1}|}{N}$$

with $|r_i^-| \in (\frac{\varepsilon}{4}, \frac{\varepsilon}{3})$ and $|r_i^+| \in (\frac{\varepsilon}{2}, \varepsilon)$.

Similarly, $v_i \in C_i^- \cup C_i^+$, where

$$C_i^* = B(s_i^* \cdot \frac{\alpha_{i-1}}{|\alpha_{i-1}|}, \frac{1}{N}), * \in \{+,-\},$$

and s_i^-, s_i^+ are roots of equation

$$s\rho(|s|) = \frac{|\alpha_{i-1}|}{N}$$

with $|s_i^-| \in (\frac{\varepsilon}{4}, \frac{\varepsilon}{3})$ and $|s_i^+| \in (\frac{\varepsilon}{2}, \varepsilon)$.

In fact, for N large enough, there are exactly 2^{2p} fixed points in the base-pointed class γ as bove, each of which can be characterized by a set of $2p$ signs.

Proposition 4.2. *For N sufficiently large and α_i, β_i be as above, then for any choice of p signs $\sigma_i, \xi_i \in \{+,-\}$, there is exactly one fixed point which orbit is in the base-pointed class γ, with intermediate points $x_i \in B_i^{\sigma_i}, v_i \in C_i^{\xi_i}$ given as above.*

Then by the proof of [Wan25, Proposition 4.1] and above discussion, if class $\gamma_N = \alpha_{1,N} * \beta_{1,N} * \cdots * \alpha_{p,N} * \beta_{p,N}$ has no symmetry and

$$\frac{|\alpha_i|}{N}, \frac{|\beta_i|}{N} \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{3}],$$

then the class is primitive, and there are exactly 2^{2p} p-tuples

$$\mathcal{O}_{\alpha,N} = \{z_{\alpha,N}, \tau(N, ab)z_{\alpha,N}, \ldots, \tau(N, (ab)^{p-1})z_{\alpha,N}\}$$

of non-degenerate primitive fixed points in class γ_N, with point $z_{\alpha,N}$ is a fixed point as given in Proposition 4.2. We might index α using $\{+,-\}^{2p}$.

All the fixed points in a same tuple have the same action and Conley-Zehnder index, and we use the notation $\mathcal{A}(\mathcal{O}_{\alpha,N})$ to denote the common action of the fixed points in the tuple $\mathcal{O}_{\alpha,N}$.

We give the calculation results on the actions and indices of the fixed points.

Proposition 4.3. [Wan25, Proposition 4.3,Proposition 5.1] *Let (v_0, x_0) be a fixed point of the linked twist map $\tau(N,(ab)^p)$ as in Proposition 4.2, and (v_i, x_i) be the intermediate points.*

(1) Suppose $x_i \in B_i^{\sigma_i}$ and $v_i \in C_i^{\xi_i}$ for $\sigma_i, \xi_i \in \{+,-\}$. Then the Conley–Zehnder index of (v_0, x_0) is

$$\begin{aligned}\mathrm{CZ}(v_0,x_0) &= n - \frac{1}{2}\sum_{i=1}^{p}\Big((\sigma_i + 1 - n) + (\xi_i + 1 - n)\Big)\\ &= n - \frac{1}{2}p(1-n) - \frac{1}{2}\sum_{i=1}^{p}(\sigma_i + \xi_i).\end{aligned}$$

(2) Suppose N is large enough with $x_i \in B_i^{\sigma_i}, v_i \in C_i^{\xi_i}$ for $\sigma_i, \xi_i \in \{+.-\}$, then

$$\mathcal{A}(v_0,x_0) = \sum_{i=0}^{p-1}(\mathcal{A}_{2i} + \mathcal{A}_{2i+1}),$$

where

$$\mathcal{A}_{2i} = N\big(h(r_i^{\sigma_i}) + r_i^{\sigma_i}\frac{|\beta_i|}{N}\big) + O(1) \tag{4.1}$$

and

$$\mathcal{A}_{2i+1} = N\big(h(s_i^{\xi_i}) + s_i^{\xi_i}\frac{|\alpha_i|}{N}\big) + O(1). \tag{4.2}$$

Now for any degree $k \in [n - \frac{1}{2}p(1-n) - p, n - \frac{1}{2}p(1-n) + p] \cap \mathbb{Z}$, if there are x many $+1$ and y many -1 in the choice of $\sigma_i, \xi_i (i = 1, \ldots, p)$, then

$$x - y = 2n - p(1-n) - 2k, x + y = 2p.$$

So there are $\binom{2p}{n-k-\frac{1}{2}p(1-n)+p}$ choices of the places of $+1$, then

$$\dim \mathrm{CF}_k(\tau_N^{(p)}, J_t)_{\gamma_N} = \binom{2p}{n-k-\frac{1}{2}p(1-n)+p}.$$

Now we prove the following main result of this section.

Theorem 4.4. *There exists constant $c > 0, C \in \mathbb{R}$, satisfying the following.*

For N large enough, there exist $\alpha_{i,N}, \beta_{i,N} (i = 1, \dots, p)$ with $\frac{|\alpha_i|}{N}, \frac{|\beta_i|}{N} \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{3}]$ so that $\gamma_N = \alpha_{1,N} * \beta_{1,N} * \cdots * \alpha_{p,N} * \beta_{p,N}$ is primitive, and for any $\alpha \neq \alpha' \in \{+,-\}^{2p}$, there holds

$$|\mathcal{A}(\mathcal{O}_{\alpha,N}) - \mathcal{A}(\mathcal{O}_{\alpha',N})| \geq cN - C.$$

Proof. Write $I = [\frac{\varepsilon}{4}, \frac{\varepsilon}{3}]$. For $u \in I$, the two solutions $r^{\pm}(u)$ of the equation $r\rho(|r|) = -u$ are negative, with $|r^-| \leq \frac{\varepsilon}{2}$ and $|r^+| \in (\frac{\varepsilon}{2}, \varepsilon)$.

Since $\rho(|r|) = 1$ when $|r| \leq \frac{\varepsilon}{2}$, one has $r^-(u) = u$. Since $\rho(|r|) = \frac{2(\varepsilon - |r|)}{\varepsilon}$, one can compute that

$$r^+(u) = -\frac{\varepsilon + \sqrt{\varepsilon(\varepsilon - 2u)}}{2}.$$

Define

$$F^{\pm}(u) = h(|r^{\pm}(u)|) + ur^{\pm}(u)$$

and

$$\Delta F(u) = F^+(u) - F^-(u),$$

then a direct computation yields

$$\Delta F(u) = \frac{(\varepsilon - 2u)\big(4\sqrt{\varepsilon(\varepsilon - 2u)} + 3(\varepsilon - 2u)\big)}{24}.$$

Clearly ΔF is a continuous and non-constant function on I.

Similarly define

$$G^{\pm}(v) = h(s^{\pm}(v)) + vs^{\pm}(v)$$

for $v \in I$ and $s^{\pm}$ are the solutions of $s\rho(|s|) = v$. Then

$$\Delta G(u) = G^+(v) - G^-(v) = \frac{\varepsilon^2}{8} + \frac{\varepsilon v}{2} - \frac{3v^2}{2} + \frac{\varepsilon + 4v}{6}\sqrt{\varepsilon(\varepsilon - 2v)},$$

which is also continuous and non-constant on I.

Since both $\Delta F(I)$ and $\Delta G(I)$ are uncountable, there are $u_1, \dots, u_p, v_1 \dots, v_p \in I$ so that the set

$$\mathcal{W} = \{\Delta F(u_1), \dots, \Delta F(u_p), \Delta G(v_1), \dots, \Delta G(v_p)\}$$

is $\mathbb{Q}$-linear independent. Without loss of generality we also assume that they satisfy the no symmetry condition.

Then for $\alpha = (\sigma_1, \xi_1, \dots, \sigma_p, \xi_p) \in \{+,-\}^{2p}$, define

$$\Lambda_\alpha = \sum_{i=1}^{p} \big(F^{\sigma_i}(u_i) + G^{\xi_i}(v_i)\big).$$

If there exists $\alpha \neq \alpha'$ with $\Lambda_\alpha = \Lambda_{\alpha'}$, then there are $c_i, d_i \in \{-1, 0, 1\}, \forall i = 1, \dots, p$ so that

$$\sum_{i=1}^{p} c_i \Delta F(u_i) + \sum_{i=1}^{p} d_i \Delta G(v_i) = 0,$$

which contradicts to the fact that $\mathcal{W}$ is $\mathbb{Q}$-linear independent. So the Λ_α's are pairwise different.

Now let $r_N^{\pm}, s_N^{\pm}, F_N^{\pm}, G_N^{\pm}, \Delta F_N, \Delta G_N$ and $\Lambda_{\alpha,N}$ be defined as the same way as above, with ρ and h substituted by ρ_N and h_N, the $\frac{1}{N^2}$-smoothing of ρ and h. Then since both ρ_N and h_N converges uniformly to their un-smoother version when $N \to +\infty$, by the implicit function theorem, there is also the convergence

$$\Lambda_{\alpha,N} \to \Lambda_N, N \to \infty, \forall \alpha \in \{+,-\}^{2p}.$$

Let

$$\eta_0 = \min_{\alpha \neq \alpha'} |\Lambda_\alpha - \Lambda_{\alpha'}| > 0,$$

then there exists N_0 for any $N > N_0$ and $\alpha \in \{+,-\}^{2p}$, there is $|\Lambda_\alpha - \Lambda_{\alpha,N}| \leq \frac{\eta_0}{4}$, so

$$\min_{\alpha \neq \alpha'} |\Lambda_{\alpha,N} - \Lambda_{\alpha',N}| > \frac{\eta_0}{2} > 0, \forall N > N_0.$$

Choose rational approximation $u_i(N) = \frac{n_i(N)}{N} \to u_i, v_i(N) = \frac{m_i(N)}{N} \to v_i$, where $m_i(N), n_i(N) \in \mathbb{Z}$ and these integers satisfy the no symmetry condition, then let $\beta_{i,N} = n_i(N)b, \alpha_{i,N} = m_i(N)a$, the class

$$\gamma_N = \alpha_{1,N} * \beta_{1,N} * \cdots * \alpha_{p,N} * \beta_{p,N}$$

is primitive.

Let $\Lambda'_{\alpha,N}$ be defined as $\Lambda_{\alpha,N}$, with u_i, v_i replaced by $u_i(N), v_i(N)$, then $\Lambda'_{\alpha,N} - \Lambda_{\alpha,N} \to 0$ when $N \to \infty$. So again

$$\min_{\alpha \neq \alpha'} |\Lambda'_{\alpha,N} - \Lambda_{\alpha,N}| > 0, \forall \alpha \in \{+,-\}^{2p} \text{ and } N \text{ sufficiently large.} \tag{4.3}$$

By Proposition 4.3, with respect to the primitive class γ_N, there is

$$\mathcal{A}(\mathcal{O}_{\alpha,N}) = N\Lambda'_{\alpha,N} + O(1),$$

then the result follows from (4.3). ∎

5. Completing the proof

5.1. Proof of Theorem 1.14. Now we prove Theorem 1.14.

Let $\tau_N = \tau(N, ab)$ and write

$$\mathrm{Cone}_{M_0}(\tau_N)_* = \mathrm{Cone}_{\mathrm{CF}_*(\tau_N^{(p)}, J_t)_{\gamma_N}}(T - \xi_p \cdot \mathrm{id})_*,$$

where γ_N is as in Theorem 4.4 and the self-mapping cone and T is defined as in Section 3.The boundary operator is denoted by $(\partial_{co})_*$. Now for any closed symplectic manifold (M_1, ω_1) and a S^1-family of almost complex structure $(J'_t)_{t\in S^1}$ on it so that $(J_t \oplus J'_t)_{t\in S^1}$ is a regular family of almost complex structure on $(M, \omega) = (M_0 \times M_1, \omega_0 \oplus \omega_1)$. We denote the self-mapping cone of

$$\mathrm{CF}_*(\tau_N^{(p)} \times \mathrm{id}, J_t \oplus J'_t)_{\gamma_N \times \{pt\}} = \mathrm{CF}_*(\tau_N^{(p)}, J_t)_{\gamma_N} \otimes \mathrm{CF}_*(\mathrm{id}, J'_t)_{\{pt\}}$$

as $\mathrm{Cone}_M(\tau_N)_*$.

Let m_k be the multiplicity of bars in the degree k concise barcode $\mathrm{Cone}_M(\tau_N)_*$. Since

$$\begin{aligned}\mathrm{Cone}_M(\tau_N)_k &\simeq \bigoplus_m \mathrm{Cone}_{M_0}(\tau_N)_m \otimes \mathrm{CF}_{k-m}(\mathrm{id}_{M_1}, J'_t)\\ &\simeq \bigoplus_m \mathrm{Cone}_{M_0}(\tau_N)_m \otimes (\oplus_s H_{m-k+2Ns}(M_1)),\end{aligned}$$

the total multiplicity of degree 1 concise barcode of $\mathrm{Cone}_M(\tau_N)_*$ is

$$\begin{aligned}m_1 &= \sum_{k=n-\frac{1}{2}p(1-n)-p}^{k=n-\frac{1}{2}p(1-n)+p} \binom{2p}{n-k-\frac{1}{2}p(1-n)+p} \cdot qb_{1-k}(M_1)\\ &= \sum_{l=-p+1}^{p+1} \binom{2p}{p+l-1} \cdot qb_{r_p-l}(M_1)\\ &= qbr_p + p - 1(M_1) + qb_{r_p-p-1}(M_1) + \binom{2p}{p} \cdot qb_{r_p-1}(M_1)\\ &\quad + \sum_{l\notin\{1,1-p,1+p\}} \binom{2p}{p+l-1} \cdot qb_{r_p-l}(M_1).\end{aligned}$$

where $r_p = 2 - n + \frac{1}{2}p(1-n)$ and we recall that

$$qb_k(M_1) = \sum_{2N|k_1-k} b_{k_1}(M_1).$$

Since

$$p \mid \binom{2p}{p+l-1} \text{ for } l \notin \{1, 1-p, 1+p\}$$

and

$$p \mid \binom{2p}{p} - 2,$$

Theorem 1.14 is immediate.

5.2. **Proof of Theorem 1.5.** Fix

$$p > 2 \sum_{i=0}^{2\dim(M_1)} b_i(M_1),$$

then by Theorem 1.14, there is $p \nmid m_1$, where m_k is the multiplicity of bars in the degree k concise barcode $\mathrm{Cone}_M(\tau_N)_*$.

Let $\beta_i(\tau_N^{(p)} \times \mathrm{id})$ be the length of the i-th bar in the degree 1 concise barcode of $\mathrm{Cone}_M(\tau_N)_*$, then by Theorem 1.12, we have

$$\mathrm{pow}_p(M, \omega) \geq \frac{1}{24p} \beta_{m_1}(\tau_N^{(p)} \times \mathrm{id}). \tag{5.1}$$

On the other hand, due to [Zha19, Proposition 9.2], the smallest length of bar of $\mathrm{Cone}_M(\tau_N)_*$ equals to the one of $\mathrm{Cone}_{M_0}(\tau_N)_*$. So by Theorem 4.4, there is

$$\beta_i(\tau_N^{(p)} \times \mathrm{id}) \to +\infty, N \to \infty$$

for any $i \leq m_1$.

Taking $i = m_1$ and $N \to +\infty$, we have $\mathrm{pow}_p(M, \omega) = +\infty$ by (5.1).

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