BARCODE ENTROPY AND EMBEDDED CONTACT HOMOLOGY

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ABSTRACT. Adapting the theory of persistence modules and barcodes, we can define the barcode entropy associated to embedded contact homology of a three dimensional contact manifold. We show that this barcode entropy is less or equal than the topological entropy of the Reeb flow.

CONTENTS

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1. Introduction

1.1. **Background.** The algebraic object persistence module, firstly introduced in [ZC05], has been playing a central role in since it was brought to the study of symplectic geometry by Polterovich-Shelukin[PS16]. After [UZ16] developed a quite complete theory of persistence module in the Floer-Novikov setting, many symplectic geometers have been using persistence modules arising from various Floer homologies to define geometric invariants and study the dynamics and topology of symplectic manifolds[Zha19, LRSV21, KS21, Sug21, She22a, BX23, AL25]. For contact manifolds, similar approach has also been taken under the setting of Liouville fillings, using symplectic homologies [SZ21, She22b]. In the present article, we use the persistence module constructed from embedded contact homology, introduced by Hutchings[Hut10], to study the topological entropy of the Reeb flow on a three dimensional contact manifold. Note that any type of symplectic filling of the contact manifold is not required in our paper.

Now we give a brief introduction to the history and motivation of the study of the topological and barcode entropy of some diffeomorphisms.

There has been a longstanding question asking if on a contact structure, every Reeb flow has positive topological entropy. Though the general answer to this question is still open, some examples have been discovered. In [FS06, MS11], the authors showed that every Reeb flow of the standard contact structure on the unit cotangent bundle of an energy hyperbolic manifold has a positive topological entropy. They studied some certain exponential rate of the filtered wrapped Floer homology to provide a lower bound to the topological entropy. More precisely, they proved that if the positivity of this growth rate holds for some Reeb flow, then it holds for all Reeb flows. Similar arguments toward the positivity of topological entropy of Reeb flows have been made by [ACH19, Mei18], which concern contact homology and Rabinowitz Floer homology respectively. More related works appear in [Alv16a, Alv16b, Dah20, AP22].

In general, the main method toward the positivity of topological entropy involves some certain quantities concerning the exponential growth rate of some Floer homologies. In virtue of the above observation, Çineli, Ginzburg, and Gürel [CGG24] introduced a Floer-theoretic entropy, called barcode entropy, using persistence homology theory and Lagrangian Floer theory. And they showed that the barcode entropy is bounded above by the topological entropy of Hamiltonian diffeomorphisms.

Following [CGG24], [CGG22, FLS26] defined barcode entropy of geodesic flows on unit-tangent bundles of Riemannian manifolds and Reeb flows of Liouville-fillable contact manifolds respectively. The construction about geodesic flows utilized the Morse homology of the energy functional on the free loop spaces, and the work on Reeb flows uses symplectic homology of the Liouville filling. They both established a lower bound of topological entropy of the corresponding flows through the barcode entropy. Similar constructions concerning wrapped Floer homology and relative symplectic cohomology can also be found in [Fer24, Ahn26].

Remark 1.1. Though the construction in [FLS26] relies on a Liouville filling of the contact manifold, it is also shown that their definition of barcode entropy does not depend on the choice of the fillings with vanishing first Chern classes. It is also expected that the vanishing condition of first Chern class can be removed.

Remark 1.2. A lower bound of barcode entropy provided by the topological entropy of hyperbolic sets has also been established in [CGG24, CGGM25]. The main tool is a certain type of crossing energy lemma.

In this paper, we define a barcode entropy for Reeb flows using embedded contact homology, generalizing the construction of [CGG24, CGG22, FLS26].

1.2. **Main results.** Let (Y, α) be a closed oriented and non-degenerate contact 3-manifold, the action filtration of embedded contact homology of (Y, α) gives rise to a persistence module $V(Y, \alpha) = \{V_s(Y, \alpha)\}_s = \{\mathrm{ECH}^s_*(Y, \alpha)\}_s$. The structure theorem(Theorem 2.5), a fundamental result in persistence module theory, associates a barcode $B(Y, \alpha)$ to the persistence module above. A barcode is a multiset of intervals with each interval called a bar in it. Let $V^T(Y, \alpha)$ and $B^T(Y, \alpha)$ be the T-truncation of the persistence module and barcode respectively and we write $b^T_\varepsilon(Y, \alpha)$ be the number of the bars with length greater than ε in $B^T(Y, \alpha)$. Then we can define the **ε-ECH barcode entropy** of (Y, α) to be

$$h^{\mathrm{ECH}}_\varepsilon(Y, \alpha) := \limsup_{T\to\infty} \frac{1}{T} \log\Big(b^T_\epsilon\big(Y, \alpha\big)\Big).$$

And the **ECH barcode entropy** of (Y, α) is

$$h^{\mathrm{ECH}}(Y, \alpha) := \lim_{\varepsilon\searrow 0} h^{\mathrm{ECH}}_\varepsilon(Y, \alpha).$$

Our main result is the following, which is analogue of [CGG24, Theorem A] and [CGG22, Theorem A].

Theorem 1.3. *Let (Y, α) be a closed oriented and non-degenerate contact 3-manifold. If the Reeb flow ϕ_α of (Y, α) is an Anosov flow, then the ECH-barcode entropy of (Y, α) is less than or equal to the topological entropy of the flow ϕ_α.*

We give a sketch of the proof. By doing some algebra, estimating the number of bars reduces to estimating the number of generators of the ECH complex, that is, simple Reeb orbits with multiplicities. We use the method of generating functions in combinatorics to show that the exponential growth rate of the ECH generators is equal to the growth rate of simple Reeb orbits with respect to period. The Anosov condition ensures us to bound the exponential growth rate by topological entropy. Note that the topological entropy of Reeb flows Anosov 3-contact manifold must be positive[Alv16b]. The proof detail will be presented in Section 5.

Remark 1.4. In the setting of [CGG24, CGG22], the generators of chain complex can be regarded as fixed points of certain Hamiltonian diffeomorphisms, thus the intersections of the diagonal submanifold and certain manifold. Then using the concept of *Lagrangian tomograph* developed from [Arn90a, Arn90b], with the Crofton inequality [APF98, APF07], the number of intersections is related to volume of the graph. Finally, Yomdin Theorem [Yom87], one can bound the exponential growth rate of volume of the graphs by topological entropy and conclude the main result.

However, it is difficult to adapt the above strategy in the Reeb orbit setting. One may try to translate the Reeb orbits into Reeb chords between certain Legendrains, then construct the tomograph in the Legendrain setting. This route might lead to the exponential growth rate estimation of Reeb orbits on general contact manifolds.

1.3. **Further discussions.** We close this section by describing some possible further directions.

As pointed out in Remark 1.2, in [CGG24, CGGM25], a lower bound of barcode entropy is given by the topological entropy of the flow restricted hyperbolic invariant set K. In our setting, it is natural to ask if a similar result holds. It is expected that establishing a crossing energy theorem for Reeb flows might lead to a positive answer to this question. This will be investigated in future research. Note that since on contact 3-manifold, there is

$$h_{\text{top}}(\phi_\alpha) = \sup_K h_{\text{top}}\big((\phi_\alpha)|_K\big)$$

by [LY12, LS19], where ϕ_α is the Reeb flow, this would lead to

$$h^{\mathrm{ECH}}(Y,\alpha) = h_{\mathrm{top}}(\phi_\alpha).$$

Another possible direction is to construct barcode entropy using different kinds of Floer homologies. A good candidate might be the periodic Floer homology(PFH), a cousin of ECH [HS05, LT12].

Acknowledgment.

2. Algebraic preliminaries

In this section we provide basic definitions and results in the theory of persistence module. Our conventions follow [PRSZ20] and one can find more details in [BL15, UZ16, PSS17], etc.

2.1. Persistence modules and barcodes.

First we lay out the relevant definitions about persistence module.

Definition 2.1. ([PRSZ20, Definition 1.1]) Let $\mathbb{F}$ be a field.

(1) A **persistence module** (V, π) consists of a finite dimensional $\mathbb{F}$-vector space V_t associated to each $t \in \mathbb{R}$ with homomorphisms $\pi_{s,t}\colon V_s \to V_t$ whenever $s \le t$ satisfying the functoriality properties that $\pi_{s,s} = I_{V_s}$, the identity map on module V_s, and $\pi_{s,u} = \pi_{t,u} \circ \pi_{s,t}$. We require that
   (a) For all but a finite number of points $t \in \mathbb{R}$ there exists a neighborhood U of t, such that $\pi_{s,r}$ is an isomorphism for any $s < r$ in U.
   (b) There is some $s_- \in \mathbb{R}$ so that $V_s = 0$ for any $s \le s_-$.
   (c) For any $t \in \mathbb{R}$ and any $s \le t$ sufficiently close to t, the map $\pi_{s,t}$ is an isomorphism.

   We omit the symbol π from the notation if it causes no problem of ambiguity.
(2) Let V be a persistence module and $\delta \in \mathbb{R}$. The δ-shift of V is the persistence module $V[\delta]$ with $V[\delta]^s = V^{s+\delta}$ and $\pi[\delta]^{s,t} = \pi_{s+\delta,t+\delta}$.
(3) Let V and W be two persistence modules. A morphism from V to W is a collection of linear maps $\boldsymbol{f} = (f^s : V^s \to W^s)_{s\in\mathbb{R}}$ such that for all $s \le t$ the

following diagram commutes.

$$\begin{array}{ccc} V^s & \xrightarrow{f^s} & W^s \\ {\scriptstyle \pi^V_{s,t}}\downarrow & & \downarrow{\scriptstyle \pi^W_{s,t}} \\ V^t & \xrightarrow[f^t]{} & W^t \end{array}$$

A morphism $A : V \to V'$ is an **isomorphism** if there is a morphism $B : V' \to V$ such that $A \circ B$ and $B \circ A$ are the identity morphisms on the corresponding persistence module.

(4) The **boundary depth** of V, denoted by $\beta(V)$, is the infimum of $\beta > 0$ such that for all $s \in \mathbb{R}$, $x \in V^s$, if $\pi_{s,t}(x) = 0$ for some $t > s$, then $\pi_{s,s+\beta}(x) = 0$.

Remark 2.2. There are many different definitions of persistence modules. The differences mainly take place in the condition imposed on the module.

(1) In some texts, for example [CB15, FLS26], the vector spaces V_s are not required to be finite dimensional in the definition of persistence module. However, since the finite dimensional condition is necessary in the structure theorem and all the persistence modules that we concern in this paper are all finite dimensional in a natural way, we omit the condition in the definition.
(2) The condition(a) in the definition is called *finite type condition* in [FLS26]. Since it enables us to count the number of intervals and this condition is met by the persistence module that we define through embedded contact homology, we put this condition in our definition.

There are many examples of persistence modules that appear in applications.

Example 2.3. (1) Let X be a closed manifold (i.e. a smooth compact manifold without boundary) and let $f : X \to \mathbb{R}$ be a Morse function. Fix $0 \le k \in \mathbb{Z}$ and put

$$V_t = H_k(\{f < t\}; \mathbb{F}).$$

Consider the natural inclusion $\{f < s\} \xhookrightarrow[i_{s,t}]{} \{f < t\}$ for $s \le t$. It induces the map $\pi_{s,t} := (i_{s,t})_* : V_s \to V_t$ in homology, and one can verify that we get a persistence module.

(2) Every non-empty interval $I \subset \mathbb{R}$ defines a persistence module $\mathbb{F}I$ as follows:

$$\mathbb{F}I_t := \begin{cases} \mathbb{F} \text{ if } t \in I, \\ 0 \text{ otherwise.} \end{cases} \qquad \pi_{s,t} := \begin{cases} id_{\mathbb{F}} \text{ if } s,t \in I, \\ 0 \text{ otherwise.} \end{cases}$$

More generally,let $\{I_i | i \in \text{ an index set } J\}$ be a collection of intervals $I_i \subset \mathbb{R}$. Let $\{(\pi_i)_{s,t}\}_{s \leq t}$ be the collection of linear maps for the interval persistence module $\mathbb{F}I_i$ for all $i \in J$. Then, this collection defines a persistence module $\bigoplus_{i \in J} \mathbb{F}I_i$ as follows:

$$\left(\bigoplus_{i \in J} \mathbb{F}I_i \right)_t := \bigoplus_{i \in J} (\mathbb{F}I_i)_t \, , \pi_{s,t} := \bigoplus_{i \in J} (\pi_i)_{s,t} \, .$$

(3) For a persistence module (V, π) and $\delta \in \mathbb{R}$, define a persistence module $(V[\delta], \pi[\delta])$ by taking $(V[\delta])_t = V_{t+\delta}$ and $(\pi[\delta])_{s,t} = \pi_{s+\delta,t+\delta}$. This new persistence module is called a *δ-shift* of V. For $\delta > 0$, the map $\Phi^\delta : (V, \pi) \to (V[\delta], \pi[\delta])$ defined by $\Phi^\delta_t = \pi_{t,t+\delta}$ is a morphism of persistence modules (it will be referred to as *δ-shift morphism*). Also, if we have a morphism $F : V \to W$ between two persistence modules, let us denote by $F[\delta] : V[\delta] \to W[\delta]$ the corresponding morphism between their δ-shifts.

(4) Let (V, π) be a persistence module, and let T be a real number. Then, the **truncation of $\boldsymbol{V}$ at $\boldsymbol{T}$**, denoted by $\boldsymbol{V^T}$, is a persistence module

$$V^T := \left(\{V^T_t\}_{t \in \mathbb{R}}, \{\pi^T_{a,b}\}_{a \leq b \in \mathbb{R}} \right),$$

defined as follows:

$$V^T_t = \begin{cases} V_t \text{ if } t < T \\ 0 \text{ otherwise} \end{cases} \quad , \pi^T_{a,b} = \begin{cases} \pi_{a,b} \text{ if } b < T \\ 0 \text{ otherwise} \end{cases} \quad .$$

In fact, any persistence module can be decomposed into the above defined interval modules. This is the structure theorem(also known as normal form theorem), which is fundamental in the persistence theory. To better describe the result of the theorem, we firstly introduce the definition of barcode.

Definition 2.4. A **barcode** $\mathcal{B}$ is a finite multiset of intervals, i.e. it is a finite collection $\{(I_i, m_i)\}$ of intervals I_i with given multiplicities $m_i \in \mathbb{N}$. The intervals in a barcode will be sometimes called **bars**.

Theorem 2.5 (Structure Theorem)**.** *Let (V, π) be a persistence module. Then there exists a finite collection $\{(I_i, m_i)\}_{i=1}^N$ of intervals I_i with their multiplicities m_i, where $I_i = (a_i, b_i]$ or $I_i = (a_i, \infty)$, $m_i \in \mathbb{N}$, $I_i \neq I_j$ for $i \neq j$, such that*

$$V = \bigoplus_{i=1}^{N} \mathbb{F}(I_i)^{m_i} \ .$$

By equality here we mean that they are isomorphic as persistence modules.
Moreover, this data is unique up to permutations, i.e., to any persistence module there corresponds a unique barcode $\mathcal{B}(V)$, which consists of the intervals I_i with multiplicity m_i. This barcode will be called the barcode of V.

Proof. See [Gab72, CB15] and [PRSZ20]. □

Remark 2.6. The definition of our barcode is different from the one in [FLS26] since we imposed the semicontinuity condition on the persistence module, which restricts the type of intervals appearing in the decomposition.

2.2. **Distance between persistence modules.** One can define a distance on isomorphism classes of persistence modules with the same vector space at $+\infty$.

Definition 2.7. ([PRSZ20, Definition 1.3.1]) Given a $\delta > 0$, we say that two persistence modules (V, π) and (W, θ) are *δ-interleaved* if there exist two morphisms $F : V \to W[\delta]$ and $G : W \to V[\delta]$, such that the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[rr, bend right, "\varphi_V^{2\delta}"'] & W[\delta] \arrow[r, "G[\delta]"] & V[2\delta] \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "G"] \arrow[rr, bend right, "\varphi_W^{2\delta}"'] & V[\delta] \arrow[r, "F[\delta]"] & W[2\delta] \end{tikzcd} ,$$

where $\varphi_V^{2\delta}$ and $\varphi_W^{2\delta}$ are the shift morphisms. We will also refer to such a pair of morphisms F and G as *δ-interleaving morphisms.*

For two persistence modules (V, π) and (W, θ), define the interleaving distance between them to be

$$d_{int}(V, W) = \inf \{\delta > 0 \mid (V, \pi) \text{ and } (W, \theta) \text{ are } \delta\text{-interleaved}\}.$$

Now we move on to show that this distance actually defines a metric, i.e. it is non-degenerate. The structure theorem connects persistence module and barcode, in fact one can define a distance on the space of barcodes to upgrade this connection to the metric level.

Given an interval $I = (a, b]$, denote by $I^{-\delta} = (a - \delta, b + \delta]$ the interval obtained from I by expanding by δ on both sides. Let $\mathcal{B}$ be a barcode. For $\varepsilon > 0$, denote by $\mathcal{B}_\varepsilon$ the set of all bars from $\mathcal{B}$ of length greater than ε. (That is, by considering $\mathcal{B}_\varepsilon$ we neglect "short bars".)

A *matching* between two finite multi-sets X, Y is a bijection $\mu : X' \to Y'$, where $X' \subset X$, $Y' \subset Y$. In this case, $X' = \mathrm{coim}\mu$, $Y' = \mathrm{im}\mu$, and we say that elements of X' and Y' are *matched*. If an element appears in the multi-set several times, we treat its different copies separately, e.g. it could happen that only part of its copies are matched.

Definition 2.8. A *δ-matching* between two barcodes $\mathcal{B}$ and $\mathcal{C}$ is a matching $\mu : \mathcal{B} \to \mathcal{C}$, such that:

(1) $\mathcal{B}_{2\delta} \subset \mathrm{coim}\mu$,
(2) $\mathcal{C}_{2\delta} \subset \mathrm{im}\mu$,
(3) If $\mu(I) = J$, then $I \subset J^{-\delta}$, $J \subset I^{-\delta}$.

Definition 2.9. The *bottleneck distance*, $d_{bot}(\mathcal{B}, \mathcal{C})$, between two barcodes $\mathcal{B}, \mathcal{C}$ is defined to be the infimum over all δ for which there is a δ-matching between $\mathcal{B}$ and $\mathcal{C}$.

The following isometry theorem shows that this is the desired metric on the space of barcodes that corresponds to the interleaving distance of persistence modules.

Theorem 2.10 (Isometry Theorem)**.** *The map $V \mapsto \mathcal{B}(V)$ is an isometry, i.e. for any two persistence modules V, W, we have $d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W))$.*

Proof. See [BL15, PRSZ20]. □

It is straightforward to check that $d_{bot}(\mathcal{B}, \mathcal{C}) = 0$ if and only if $\mathcal{B} = \mathcal{C}$, so the isometry theorem gives

Corollary 2.11. *$d_{int}(V, W) = 0$ if and only if V and W are isomorphic.*

Remark 2.12. If one drops the semicontinuity condition(c) in the definition of persistence module, then the interleaving distance is only a pseudometric on the isomorphism class of persistence modules. For example for a ground field $\mathbb{K}$, $\mathbb{K}[0, 1]$ and $\mathbb{K}(0, 1)$ have vanishing interleaving distance, but they are not isomorphic.

2.3. **Singular value decomposition.** In this subsection, we consider filtered non-Archimedean normed vector space as a model of persistence module. Almost every persistence module constructed from various Floer theories arises this way. Throughout the secion we fix a field $\mathbb{F}$ and only consider the vector spaces over $\mathbb{F}$. For more details the reader can refer to [UZ16].

Definition 2.13.

(1) A **valuation** ν on $\mathbb{F}$ is a function $\nu : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ such that
  - (V1) $\nu(a) = \infty$ if and only if $a = 0$,
  - (V2) for any $a, b \in \mathbb{F}$, $\nu(ab) = \nu(a) + \nu(b)$, and
  - (V3) for any $a, b \in \mathbb{F}$, $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

(2) A valuation ν is **trivial** if $\nu(a) = 0$ for all nonzero $a \in \mathbb{F}$.

(3) Let $\mathbb{F}$ be a field equipped with a valuation ν. A **non-Archimedean normed vector space** over $\mathbb{F}$ is a pair $(V, \ell : V \to \mathbb{R} \cup \{-\infty\})$ of a $\mathbb{F}$-vector space V and a function ℓ on V such that
  - (F1) $\ell(x) = \infty$ if and only if $x = 0$,
  - (F2) for any $a \in \mathbb{F}$ and $x \in V$, $\ell(ax) = \ell(x) - \nu(a)$, and
  - (F3) for any $x, y \in V$, $\ell(x + y) \leq \max\{\ell(x), \ell(y)\}$.

Now, we define the notion of *orthogonality* on a non-Archimedean vector space.

Definition 2.14. Let (V, ℓ) be a non-Archimedean vector space over $\mathbb{F}$.

(1) A finite ordered collection $(v_1, \ldots, v_r)$ of elements of V is said to be **orthogonal** if, for all $a_1, \ldots, a_r \in \mathbb{F}$, the following holds:
$$\ell\left(\sum_{i=1}^{r} a_i v_i\right) = \max\left\{\ell(a_i v_i) | i = 1, \ldots, r\right\}.$$

(2) An **orthogonalizable space** (V, ℓ) is a finite-dimensional non-Archimedean normed vector space (V, ℓ) over $\mathbb{F}$ such that there exists an orthogonal basis for V.

We will use Lemmas 2.15 and 2.16 in later sections.

Lemma 2.15. *Let (C, ℓ_C) and (D, ℓ_D) be orthogonalizable $\mathbb{F}$-spaces and let $A : C \to D$ be a linear map with rank r. Then, C (resp. D) admits an orthogonal basis $\{v_1, \ldots, v_n\}$ (resp. $\{y_1, \ldots, y_m\}$) such that*

(1) $\{v_{r+1}, \ldots, v_n\}$ is an orthogonal basis for $\ker A$,

(2) $\{w_1, \ldots, w_r\}$ *is an orthogonal basis for* $\mathrm{Im}A$*,*
(3) $Av_i = w_i$ *for all* $i = 1, \ldots, r$*, and*
(4) $\ell_C(v_1) - \ell_D(w_1) \leq \cdots \leq \ell_C(v_r) - \ell_D(w_r)$*.*

Proof. See the proof of [UZ16, Theorem 3.4]. We omit the proof. □

Lemma 2.16. *Let* (V, ℓ) *be a finite-dimensional non-Archimedean normed vector space with two different orthogonal bases*

$$\{v_1, \ldots, v_n\} \text{ and } \{w_1, \ldots, w_n\}.$$

If $\ell(v_1) \leq \ell(v_2) \leq \cdots \leq \ell(v_n)$ *and* $\ell(w_1) \leq \ell(w_2) \leq \cdots \leq \ell(w_n)$*, then*

$$\ell(v_i) = \ell(w_i),$$

for all $i = 1, \ldots, n$*. In particular, there is a one-to-one map between two bases preserving the non-Archimedean norm.*

Proof. Let $V_{<r}$ be a subspace of V defined as follows:

$$V_{<r} := \{v \in V | \ell(v) < r\}.$$

Since V is finite dimensional, there is a unique finite sequence $(r_0 < r_1 < \cdots < r_k)$ of real numbers such that

- $0 = V_{<r_0} \subsetneq V_{<r_1} \subsetneq \cdots \subsetneq V_{<r_k} = V$, and
- for any $r \in [r_i, r_{i+1})$, $V_{<r_i} = V_{<r}$.

We note that the sequence $(r_0, \ldots, r_k)$ is an invariant of (V, ℓ), not depending on the choice of bases.

Since the sequence determines $\ell(v_i)$ and $\ell(w_i)$, Lemma 2.16 holds. □

Let us assume that we have a finite-dimensional, orthogonalizable non-Archimedean normed vector space (V, ℓ) equipped with a linear map $\partial : V \to V$ satisfying that

- $\partial \circ \partial = 0$, and
- for all $v \in V$, $\ell(\partial v) \leq \ell(v)$.

The first condition means that ∂ can be seen as a differential map. In other words, the *homology* vector space of (V, ∂), denoted by $H(V, \partial)$, is defined as

$$H(V, \partial) := \ker(\partial)/\mathrm{im}(\partial).$$

Moreover, thanks to the non-Archimedean vector space structure on V and the second condition on ∂, $H(V, \partial)$ admits a persistence module structure.

Let V^a be the subspace of V consisting of all vectors v such that $\ell(v) < a$. Then, the second condition on ∂ implies that the restriction of ∂ on V^a can be seen as a linear map from V^a to itself. Thus, the following H^a is well-defined:

$$H^{<a} := \ker(\partial|_{V^a})/\mathrm{im}(\partial|_{V^a}).$$

Moreover, for any $a \leq b$, there exists an inclusion map $\pi_{a,b} : V^a \to V^b$. It is easy to observe that $\pi_{a,b}$ is a chain map and induces a linear map from H^a to H^b. Let $\pi_{a,b}$ denote the induced map on H^a again. Now, the following is a persistence module in the sense of [FLS26](that is, the finite type and semicontinuity condition is not guaranteed):

$$H(V) := \left(\{H^a\}_{a\in\mathbb{R}}, \{\pi_{a,b}\}_{a\leq b\in\mathbb{R}}\right).$$

3. Quantitative embedded contact homology

3.1. **Reeb and Anosov dynamics.** Let Y be an oriented closed 3-manifold and α be a contact form supporting $\xi = \ker(\alpha)$. The contact form determines the Reeb vector field R_α by the characterization

$$d\alpha(R_\alpha, \cdot) = 0, \alpha(R_\alpha) = 1.$$

A closed orbit of the Reeb flow is called a Reeb orbit, and we say a Reeb orbit is *non-degenerate* if its linearzied return map does not have 1 as its eigenvalue. A contact form α is said to be non-degenerate if all of its Reeb orbits are non-degenerate.

A non-degenerate Reeb orbit is *hyperbolic* if its linearzied return map has real eigenvalues, otherwise it is *elliptic*.

In this paper we mainly focus on the case where the Reeb flow is Anosov. Anosov flows are important in the theory of dynamical systems and the first systematic study of them are conducted in [Ano67]. We firstly give the definition here.

Definition 3.1. Let Y be a smooth 3-manifold and X be a smooth vector field on Y without singularities. We denote by ϕ_X the flow generated by X. We say that the flow ϕ_X is *Anosov* when there exist line fields E_S and E_U in Y, and real numbers $\mu > 0$ and $C > 0$ such that:

(1) at every point $y \in Y$ we have $T_yY = \mathbb{R}X(y) \oplus E_S(y) \oplus E_U(y)$, where $\mathbb{R}X(y)$ is the 1-dimensional subspace of T_yY generated by $X(y)$,
(2) the line fields E_S and E_U are left invariant by the flow ϕ_X,
(3) $|D\phi^t_X(y)v| \geq Ae^{\mu t}|v|$ for every $y \in Y$, $v \in E_U(y)$ and $t \geq 0$,

(4) $|D\phi_X^t(y)v| \leq Ae^{-\mu t}|v|$ for every $y \in Y$, $v \in E_S(y)$ and $t \geq 0$.

Here $|\cdot|$ is a norm induced by a fixed chosen Riemannian metric on Y.

An Anosov flow on a 3-manifold is called *transversely orientable* when the line bundles E_S and E_U are trivial.

There are many dynamical properties of Anosov flows. Firstly there are many results on the positivity of the topological entropy of Anosov flow on a 3-contact manifold.

Theorem 3.2. *(1)* [Alv16b] *Let (M, ξ) be a compact contact 3-manifold and assume that there exists a contact form α_0 on (M, ξ) such that its Reeb flow is Anosov. Then every Reeb flow on (M, ξ) has positive topological entropy.*

(2) [ACH19] *Let (M, ξ) be a closed cooriented three-dimensional contact manifold which admits a supporting open book decomposition whose binding is connected and whose monodromy is isotopic to a pseudo-Anosov homeomorphism with fractional Dehn twist coefficient $\frac{n}{k}$. If $k \geq 5$ then every Reeb vector field for ξ has positive topological entropy.*

Besides, Anosov flows are usually accompanied by some types of exponential growth. For example, for symplectically fillable Anosov contact manifolds in any dimension, Macarini-Paternain [MP12] established the exponential growth of S^1-equivariant symplectic homology. Vaugon[Vau] showed for Anosov contact 3-manifolds with a transversely orientable Anosov Reeb flow, the cylindrical contact homology has exponential growth property.

On top of the above results, the following one on the exponential growth rate of simple periodic orbits is crucial in our proof of main theorem.

Theorem 3.3. [PP83] *Let ϕ^t be an Anosov flow and $\mathcal{P}$ be the set of simple closed orbits of ϕ^t. For each $\gamma \in \mathcal{P}$, let T_γ be the primitive period of γ. Write*

$$\mathcal{P}(\lambda) = \{\gamma \in \mathcal{P} \mid T_\gamma \leq \lambda\}.$$

Then

$$F(\lambda) = |\mathcal{P}(\lambda)| \sim \frac{e^{h\lambda}}{h\lambda}, \lambda \to +\infty. \tag{3.1}$$

Here $h = h_{top}$ is the topological entropy of the Anosov flow.

3.2. **Definition of ECH.** Now we give a brief review of embedded contact homology. For more details one can refer to [HT07, Hut09, Hut11].

Now we begin to define the chain complex of embedded contact homology. We use coefficient $\mathbb{Z}_2$ for simplicity. Firstly we describe the chain vector space.

Let $ECC(Y, \alpha)$ be the vector space over $\mathbb{Z}_2$ generated by finite sets of pairs $\theta = \{(\theta_i, m_i)\}$, where the θ_i's are different embedded Reeb orbits and m_i are positive integers satisfying:

whenever θ_i is a hyperbolic Reeb orbit, there is $m_i = 1$.

Every such an θ is called an ECH generator. One can also use a formal notation $\theta = \prod_i \theta_i^{m_i}$ to represent an ECH generator.

Remark 3.4. The necessity of the condition restraining the multiplicity of hyperbolic orbits is from the theory of generating functions. It is explained in detail in [Hut14, Section 2.7].

Then we need to define the differential. Just as in almost all Floer theories, the differential is given by counting certain pseudoholomorphic curves or cylinders, so an almost complex structure should be chosen first.

Let $\mathcal{J}(Y, \alpha)$ denote the set of all almost complex structures on $\mathbb{R} \times Y$ satisfying the following conditions:

- J is preserved by the natural action of $\mathbb{R}$ on $\mathbb{R} \times Y$.
- J preserves ξ. Moreover $g_J(v, w) := d\alpha(v, Jw)$ is a positive-definite symmetric bilinear form on ξ.
- $J(\partial_s) = R_\alpha$, where s denotes the coordinate on $\mathbb{R}$.

Such almost complex structure is said to be *symplectization-admissible*. The space of symplectization-admissible almost complex structures is contractible. To define the differential we give the following definition of "J-holomorphic curve" in the ECH setting.

Definition 3.5. [HT13, Definition 1.1] Given a symplectization-admissible J, and given orbit sets $\theta = \{(\theta_i, m_i)\}$ and $\theta' = \{(\theta'_j, m'_j)\}$, define a *$J$-holomorphic curve from θ to θ'* to be a J-holomorphic curve in $\mathbb{R} \times Y$ (whose domain is a possibly disconnected punctured compact Riemann surface) with positive ends at covers of θ_i with total multiplicity m_i, negative ends at covers of θ'_j with total multiplicity m'_j, and no other ends.

Here a *positive end* of a holomorphic curve at a (not necessarily embedded) Reeb orbit γ is an end which is asymptotic to the cylinder $\mathbb{R} \times \gamma$ as the $\mathbb{R}$ coordinate $s \to +\infty$. A *negative end* is defined analogously with $s \to -\infty$.

Let $\mathcal{M}^J(\theta, \theta')$ denote the moduli space of J-holomorphic curves from θ to θ', where two such curves are considered equivalent if they represent the same current in $\mathbb{R} \times Y$, up to translation of the $\mathbb{R}$ coordinate.

For two ECH generators θ, θ', the differential coefficient $\langle \partial\theta, \theta' \rangle$ is the mod-2 count of J-holomorphic curves in $\mathcal{M}^J(\theta, \theta')$ with ECH index equal to 1. The definition and discussion of ECH index can be found in [Hut02, Hut09], we omit them here for the sake of brevity. For generic symplectization-admissible almost complex structure, one can show that $\partial^2 = 0$. A comprehensive proof can be found in [Hut14, Section 5].

Then we denote the chain complex by $ECC(Y, \alpha, J) = (ECC(Y, \alpha), \partial)$. The resulting homology, called the embedded contact homology, is denoted by $ECH(Y, \alpha)$.

Remark 3.6. The reason that we drop the variable J in the homology is that ECH does not depend on the choice of the almost complex structure J. The explanation is as follow.

By Taubes [Tau10a, Tau10b, Tau10c, Tau10d, Tau10e], ECH is isomorphic to a version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [KM07]. More specifically is that there is a canonical isomorphism of relatively graded $\mathbb{Z}_2$-modules

$$ECH_*(Y, \alpha, \Gamma; J) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}). \tag{3.2}$$

Here $ECH_*(Y, \alpha, \Gamma; J)$ is the homology of the subcomplex $(ECC(Y, \alpha, \Gamma), \partial)$ generated by ECH generators $\theta = \{(\theta_i, m_i)\}$ so that $\sum_i m_i[\theta_i] = \Gamma \in H^1(Y; \mathbb{Z})$. And $\widehat{HM}^*$ denotes Seiberg-Witten Floer cohomology with $\mathbb{Z}_2$ coefficients, and $\mathfrak{s}_{\xi,\Gamma}$ denotes the spin-c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$ on Y, where $\mathfrak{s}_\xi$ denotes the spin-c structure determined by oriented 2-plane field ξ. This isomorphism is an analogue of the fact that the GromovTaubes invariant of a closed symplectic 4-manifold is equal to the corresponding Seiberg-Witten invariant.

Then the invariance about J on ECH follows directly from the invariance property of $\widehat{HM}^*$.

By the above isomorphism, we have further that the embedded contact homology is independent on the contact form α, so we might also use the notation $ECH_*(Y, \xi)$ later.

3.3. **Filtration structure and ECH spectral invariants.** If $\theta = \{(\theta_i, m_i)\}$ is an orbit set, its *symplectic action* is defined by

$$\mathcal{A}(\theta) := \sum_i m_i \int_{\theta_i} \alpha.$$

Then for any $\sum_i v_i \theta^{(i)}$ in $ECC(Y, \alpha, \Gamma)$, where $\theta^{(i)}$'s are ECH generators, we define

$$\mathcal{A}(\sum_i v_i \theta^{(i)}) = \max_i \mathcal{A}(v_i \theta^{(i)}).$$

We write $ECC^L_*(Y, \alpha, \Gamma; J) \subseteq ECC_*(Y, \alpha, \Gamma; J)$ as the subset of elements with action less than L.

Proposition 3.7. *If J is symplectization-admissible, then $(ECC^L_*(Y, \alpha, \Gamma; J), \partial)$ is a subcomplex of $(ECC_*(Y, \alpha, \Gamma; J), \partial)$. And we define $ECH^L_*(Y, \alpha, \Gamma; J)$ to be the homology of this subcomplex.*

Proof. It suffices to show that if $\theta = \{(\theta_i, m_i)\}$ and $\theta' = \{(\theta'_j, m'_j)\}$ are ECH generators with $\mathcal{M}^J(\theta, \theta') \neq \emptyset$, then $\mathcal{A}(\theta) \geq \mathcal{A}(\theta')$. Since J is symplectization-admissible, the restriction of $d\theta$ to any J-holomorphic curve in $\mathbb{R} \times Y$ is pointwise nonnegative. Then the result follows directly from Stoke's Theorem. □

Remark 3.8. In fact the differential defined above is *strictly* action decreasing [Hut14].

There are various natural maps defined on filtered ECH. First, if $L < L'$ then there is a map

$$\imath^{L,L'}_J : ECH^L_*(Y, \alpha; J) \longrightarrow ECH^{L'}_*(Y, \alpha; J)$$

induced by the inclusion of chain complexes. The usual ECH is recovered as the direct limit

$$ECH_*(Y, \alpha; J) = \lim_{L \to \infty} ECH^L_*(Y, \alpha; J).$$

There is an inclusion

$$\imath^L_J : ECH^L_*(Y, \alpha; J) \longrightarrow ECH_*(Y, \alpha; J)$$

Using a filtered version of Taubes' isomorphism (3.2)(see also [HT13, Section 3]), we have that the filtered embedded contact homology is independent of the choice of J so we drop it from the notations again. It is clear the full embedded contact homology chain complex $ECC_*(Y, \alpha; J)$ is usually infinite-dimensional but the filtered ones $ECC^s_*(Y,\alpha;J)$ is always finite dimensional. So each filtered chain complex $ECC^s_*(Y,\alpha;J)$ along with the differential and action filtration forms a finite-dimensional, orthogonalizable non-Archimedean normed vector space in the sense of Section 2.3(with the trivial valuation on the base field $\mathbb{Z}_2$).

One can use the filtration structure to define ECH spectral invariant as one usually does in Hamiltonian Floer theory. Let $\Lambda_{\text{nondeg}}(Y,\xi)$ be the set of non-degenerate contact forms supporting ξ, then for any $\alpha \in \Lambda_{\text{nondeg}}(Y,\xi)$ and $\sigma \in ECH_*(Y,\xi)\backslash\{0\}$, one can define

$$c_\sigma(\alpha) = \inf\{L \mid \sigma \in \operatorname{im}\imath^L\} \in \mathbb{R}_{\geq 0}.$$

ECH spectral invariants enjoy the following properties.

Theorem 3.9 ([Hut11, Kei25])**.** *The ECH spectral invariant $s(\cdot) = s_\sigma(\cdot)$ satisfies*

(Spectrality): *$s(\alpha) \in \mathcal{A}_+(Y,\alpha)$, where*

$$\mathcal{A}(Y,\alpha) := \{T_\gamma \mid \gamma \in \mathcal{P}(Y,\alpha)\},$$
$$\mathcal{A}_+(Y,\alpha) := \{0\} \cup \{a_1 + \cdots + a_m \mid m \geq 1,\, a_1,\ldots,a_m \in \mathcal{A}(Y,\alpha)\}.$$

Here $\mathcal{P}(Y,\alpha)$ is the set of simple periodic Reeb orbis of α and T_γ is the period of a periodic orbit γ.

(Conformality): *$s(c\alpha) = c \cdot s(\alpha)$ for any $c \in \mathbb{R}_{>0}$.*

(Monotonicity): *$s(e^h\alpha) \geq s(\alpha)$ for any $h \in C^\infty(Y,\mathbb{R}_{\geq 0})$.*

(C^0-continuity): *For any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that $|s(\alpha) - s(e^h\alpha)| \leq \varepsilon$ for any $h \in C^\infty(Y,[-\delta,\delta])$.*

Remark 3.10. By the C^0-continuity, the above ECH spectral invariants can be extended to all contact form(not necessarily non-degenerate) by C^0-approximation.

So to conclude, we have obtained a persistence module

$$V(Y,\alpha) = (\{ECH^L_*(Y,\alpha)\}_{L\in\mathbb{R}}, \{\imath^{L,L'}\}_{L\leq L'\in\mathbb{R}}).$$

The finite type and semicontinuity condition follows from the above properties of ECH spectral invariants. Then there is a barcode $B(Y,\alpha)$ associated to this persistence module, and the endpoints of the bars in the barcode lie in the set $\mathcal{A}_+(Y,\alpha)$.

4. Topological entropy and ECH-barcode entropy

In this section we first review the basic definitions and properties of topological entropy. After that we recall the construction of ECH-barcode entropy and give a brief analysis about the number of bars.

4.1. Topological entropy.

In this section we review the definition of topological entropy and recall some basic properties of it. One can refer to [KH95, FH19] for a comprehensive exposition on this important quantity in dynamical systems theory.

Let (X, d) be a compact metric space and let $f : X \to X$ be an automorphism. Define an increasing sequence $\{d_k^f\}$ of metrics for $k \in \mathbb{N}$ by

$$d_k^f(x, y) = \max_{1 \leq i \leq k-1} d(f^i(x), f^i(y))$$

where f^i denotes the composition of f with itself i times. Let $B_f(x, \varepsilon, k)$ be the open ball of radius(with respect to metric d_k^f) ε centered at x in X. A set $E \subset X$ is said to be (k, ε)**-spanning** if

$$X \subseteq \bigcup_{x \in E} B_f(x, \varepsilon, k).$$

Let $S_d(f, \varepsilon, k)$ be the minimal cardinality of an (k, ε)-spanning set. Then the **ε-topological entropy** $h_{\varepsilon,\mathrm{top}}(f)$ of f is defined by

$$h_{\varepsilon,\mathrm{top}}(f) = \limsup_{k \to \infty} \frac{1}{k} \log S_d(f, \varepsilon, k).$$

Then we let the **topological entropy** to be

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \searrow 0} h_{\varepsilon,\mathrm{top}}(f).$$

The above definition has an equivalent form by [Din70, Bow71] as follow. We set

$$\Gamma_k(f) := \left\{ \left(x, f(x), f^2(x), \ldots, f^{k-1}(X)\right) \in X^k | x \in X \right\}.$$

For a set $Y \subset X^k$, let $S_\varepsilon(Y)$ denotes the minimal number of ϵ-cubes in X^k needed to cover Y, where an ε-cube in X^k is a product of balls of radius ε from X. Then the ε-topological entropy is

$$h_{\varepsilon,\mathrm{top}}(f) = \limsup_{k \to \infty} \tfrac{1}{k} \log S_\varepsilon \left(\Gamma_k(\phi)\right)$$

and the topological entropy is the limit of ε-topological entropy for $\varepsilon \searrow 0$.

Here are some basic properties of topological entropy.

Proposition 4.1. *(1) The definition topological entropy does not depend on the metric structure of X.*

(2) If $\phi : X_1 \cup X_2 \to X_1 \cup X_2$ satisfies that $\phi(X_i) \subset X_i$, then

$$h_{top}(\phi) = \max\{h_{top}(X_1), h_{top}(X_2)\}.$$

(3) If $\phi : X \to X$ and $Y \subset X$ satisfy that $\phi(Y) \subseteq Y$, then

$$h_{top}(\phi) \geq h_{top}(\phi|_Y).$$

Proof. See [Gro87, Section 1.6]. □

Remark 4.2. For a definition of topological entropy without metric structure, see [AKM65].

The definition of topological entropy naturally extends to the flow. For a continuous flow $F = \{f^t\}_{t\in\mathbb{R}}$ on X, we define

$$h_{\text{top}}(F) = h_{\text{top}}(f^1).$$

Topological entropy of a flow has the following properties, which implies that the topological entropy of a flow $\{f^t\}_{t\in\mathbb{Z}}$ is controlled by its time-1 map.

Proposition 4.3. *Let $F = \{f^t\}_{t\in\mathbb{R}}$ be a continuous flow on X, then*

(1) For each $t \in \mathbb{R}$, $h_{top}(f^t) = |t| h_{top}(f^1)$.
(2) For each $m \in \mathbb{Z}$ and $t \in \mathbb{R}$, $h_{top}(f^{mt}) = |m| h_{top}(f^t)$.

Next we introduce the Yomdin theorem, which connects the topological entropy to the growth rate of volume, which is more computable intuitively. Let X be a smooth Riemannian manifold and $f : X \to X$ be a smooth map. Denote the graph of f by Γ_f, that is,

$$\Gamma_f = \{(x, f(x)) \in X \times X \mid x \in X\}.$$

For a smooth submanifold Y of X, the **exponential volume growth rate** $h_{\text{vol}}(f|_Y)$ is defined to be

$$h_{\text{vol}}(f|_Y) = \limsup_{\ell\to\infty} \frac{1}{\ell} \log \text{Vol}\left(\Gamma_{f^\ell|_Y}\right)$$

where Vol denotes the volume induced by the given Riemannian metric. Now we can state Yomdin's theorem.

Theorem 4.4 (Yomdin)**.** *Let X be a smooth Riemannian manifold and $f : X \to X$ be a smooth map. For a smooth submanifold Y of X, we have*

$$h_{vol}(f|_Y) \leq h_{top}(f|_Y).$$

Proof. See [Yom87].

□

4.2. **ECH-barcode entropy.** Now we focus on the barcode entropy. By the structure theorem, the persistence module $V(Y, \alpha)$ corresponds to a barcode $B(Y, \alpha)$. Let $V^T(Y, \alpha)$ and $B^T(Y, \alpha)$ be the T-truncation of the persistence module and barcode respectively as in Example 2.3(4) and we write $b^T_\varepsilon(Y, \alpha)$ be the number of the bars with length greater than ε in $B^T(Y, \alpha)$.

Definition 4.5. (1) Let B be a barcode. For any positive real number ε, $\boldsymbol{n_\varepsilon(B)}$ is defined as

$$\mathbb{Z}_{\geq 0} \cup \{\infty\} \ni n_\varepsilon(B) := \text{ the number of bars in } B \text{ whose length is greater than } \varepsilon.$$

(2) The **ε-ECH barcode entropy** of (Y, α), denoted by $\boldsymbol{h^{\mathrm{ECH}}_\epsilon(Y, \alpha)}$, is defined as

$$h^{\mathrm{ECH}}_\varepsilon(Y, \alpha) := \limsup_{T \to \infty} \frac{1}{T} \log \Big(n_\varepsilon\big(B^T(Y, \alpha)\big)\Big).$$

(3) The **symplectic homology barcode entropy** of (Y, α), denoted by $\boldsymbol{h^{\mathrm{ECH}}(Y, \alpha)}$, is defined as

$$h^{\mathrm{ECH}}(Y, \alpha) := \lim_{\varepsilon \searrow 0} h^{\mathrm{ECH}}_\varepsilon(Y, \alpha).$$

In view of our main result Theorem 1.3, we need to give some upper bound of the quantities $n_\varepsilon(B^T(Y, \alpha))$, that is, the number of bars in truncated barcode $B^T(Y, \alpha)$ with length larger than ε. Let $n^{T,\varepsilon}_1$ be the number of untruncated barcode $B(Y, \alpha)$ satisfying conditions below.

(1) The length of each of the bars is larger than ε.
(2) The left end point of each of the bars is smaller than $T - \varepsilon$.

Then

$$n_\varepsilon(B^T(Y, \alpha)) \leq n^{T,\varepsilon}_1.$$

For N sufficiently large, let $n^{T,\varepsilon}_2(N)$ be the number of bars in $B(Y, \alpha)$ so that

(1) The right end point of each of the bars is smaller than N.
(2) The length of each of the bars is larger than ε.

(3) The left end point of each of the bars is smaller than $T-\varepsilon$.

Recall that $(ECC^N(Y,\alpha;J),\partial,\mathcal{A})$ is a finite-dimensional and orthogonalizable non-Archimedean normed vector space. By Lemma 2.15, it admits a singular value decomposition. So there is an orthogonal basis

$$\mathcal{B}=\{x_1,\dots,x_n,y_1,\dots,y_M,z_1,\dots,z_M\}$$

of $(ECC^N(Y,\alpha;J),\partial)$ such that

$$\partial x_i=0 \text{ for all } i=1,\dots,n, \partial z_j=y_j \text{ for all } j=1,\dots,M,$$
$$\mathcal{A}(z_1)-\mathcal{A}(y_1)\le\mathcal{A}(z_2)-\mathcal{A}(y_2)\le\cdots\le\mathcal{A}(z_M)-\mathcal{A}(y_M).$$

Thus, $n_2^{T,\varepsilon}(\delta,N)$ is the number of $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,M\}$ satisfying.

(1) $\mathcal{A}(z_j)-\mathcal{A}(y_j)>\varepsilon$.
(2) $\mathcal{A}(x_i),\mathcal{A}(y_j)<T-\varepsilon$.

Let $n_3^{T,\varepsilon}(N)$ be the number of $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,M\}$ satisfying

$$\mathcal{A}(x_i),\mathcal{A}(y_j)<T-\varepsilon.$$

It directly follows that

$$n_2^{T,\varepsilon}(N)\le n_3^{T,\varepsilon}(N).$$

Since there is a natural orthogonal basis of $ECC^N(Y,\alpha;J)$ which consists of the ECH generators with action less than N. We denote this set of $\mathcal{P}(N)$. So by Lemma 2.16, there exists an action-preserving one-to-one correspondence between the basis $\mathcal{B}$ and $\mathcal{P}(N)$.

Now let n_4^T denote the number of ECH generators α of (Y,α) satisfying the following:

$$\mathcal{A}(\alpha)\le T.$$

Then

$$n_3^{T,\varepsilon}(N)\le N^T.$$

So to conclude we have

$$n_\varepsilon(B^T(Y,\alpha))\le n_1^{T,\varepsilon}\le\lim_{N\to\infty}n_2^{T,\varepsilon}(N)\le\lim_{N\to\infty}n_3^T(N)\le n_4^T. \tag{4.1}$$

In other words, if one can obtain an appropriate upper bound of n_4^T, i.e. the number of ECH generators with action less than T, then we can bound $n_\varepsilon(B^T(Y,\alpha))$, thus the barcode entropy. This is the main task of the next section.

5. Proof of Theorem 1.3

In the final section we give a proof of our main result, Theorem 1.3.

Recall that $\mathcal{P}(Y,\alpha)$ is the set of simple periodic Reeb orbits of (Y,α). For each $\gamma \in \mathcal{P}(Y,\alpha)$, write T_γ as its smallest period. For any $x>0$, define

$$\mathcal{P}(x) = \{\gamma \in \mathcal{P}(Y,\alpha) \mid T_\gamma \leq x\}.$$

Note that since α is non-degenerate, this set is always finite. Then we define $F(x)$ to be the number of elements in this set.

Now we recall that an ECH generator is a formal product $\theta = \prod_i \theta_i^{m_i}$, where θ_i are distinct simple orbits and $m_i \in \mathbb{A}$ are multiplicities so that $m_i = 1$ whenever θ_i is hyperbolic. The total action is $\mathcal{A}(\theta) = \sum m_i \mathcal{A}_{\theta_i}$.

We write $N(x)$ to be the number of ECH generators with action less than or equal to x, then $n_4^T = N(T)$, using the notations in Section 4.2.

Next we show that the exponential growth rates of $F(x)$ and $N(x)$ coincides, i.e.

$$\limsup_{T\to\infty} \frac{1}{T} \log n_4^T = \limsup_{T\to\infty} \frac{1}{T} \log F(T).$$

This follows from an almost combinatorical argument using the method of generating function.

Since $N(x) \geq F(x)$, the growth rate of $N(x)$ is greater than or equal to the growth rate of $F(x)$. It suffices to show the reversed inequality.

We define the "Euler product" as

$$\zeta_{ECH}(s) = \prod_{\gamma\in\mathcal{P}} \sum_{m=0}^{\infty} e^{-sm\mathcal{A}(\gamma)} = \prod_{\gamma\in\mathcal{P}} \frac{1}{1-e^{-s\mathcal{A}(\gamma)}}.$$

Since

$$\sum_{\theta} e^{-s\mathcal{A}(\theta)} \leq \zeta_{ECH}(s),$$

where in the sum θ ranges over all the ECH generators, the exponential growth rate of $N(x)$ is not larger than the abscissa of convergence of the series $\zeta_{ECH}(s)$.

The series $\log \zeta_{ECH}(s) = \sum_{\gamma\in\mathcal{P}} -\log(1-e^{-s\mathcal{A}(\gamma)})$ can be expanded using the Taylor series for $\log(1-x)$:

$$\log \zeta_{ECH}(s) = \sum_{\gamma\in\mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{k} e^{-ks\mathcal{A}(\gamma)} = \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{\gamma\in\mathcal{P}} e^{-ks\mathcal{A}(\gamma)} \right)$$

The inner sum $\sum_{\gamma\in\mathcal{P}} e^{-s\mathcal{A}(\gamma)}$ is the Laplace transform of the counting function $F(x)$. It converges for any $s > \Theta$.For $k \geq 2$, the terms $e^{-ks\mathcal{A}(\gamma)}$ represent multiple covers. The growth rate of these "higher order" terms is $\frac{\Theta}{k}$, which is strictly less than Θ.

So in general, for any $s > \Theta$, the series $\log \zeta_{ECH}(s)$ converges. Then the reversed inequality is obtained.

Finally we can finish the proof of the main result.

Since

$$h_\varepsilon^{\mathrm{ECH}}(Y,\alpha) = \limsup_{T\to\infty} \frac{1}{T} \log\Big(n_\varepsilon\big(B^T(Y,\alpha)\big)\Big),$$

by (4.1),

$$h_\varepsilon^{\mathrm{ECH}}(Y,\alpha) \leq \limsup_{T\to\infty} \frac{1}{T} \log n_4^T.$$

Recall by Theorem 3.3, one has

$$\limsup_{T\to\infty} \frac{1}{T} F(T) = h_{\mathrm{top}}(\phi_\alpha).$$

So

$$h_\varepsilon^{\mathrm{ECH}}(Y,\alpha) \leq \limsup_{T\to\infty} \frac{1}{T} F(T) = h_{\mathrm{top}}(\phi_\alpha).$$

Taking the limit $\varepsilon \searrow 0$ on the both sides of the inequality, we have

$$h^{\mathrm{ECH}}(Y,\alpha) \leq h_{\mathrm{top}}(\phi_\alpha).$$

Thus, Theorem 1.3 holds.

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