Floer Theory

In this note we study the theory of Floer, which is developed from the classical Morse theory, to partially solve the celebrated Arnold's conjecture in symplectic geometry.

Ref. «Morse Theory & Floer Homology» - Audin.
«Elliptic Methods in Symplectic Geometry» - McDuff

I. Introduction

1. What's Arnold's conjecture?

Consider a symplectic mfld (M, ω). a vector field X is called Hamiltonian if $\iota_X \omega$ is exact, i.e. $\iota_X \omega = dH$

in this case H is called the Hamiltonian function

A Hamiltonian isotopy is a family of diffeo φ_t with $\varphi_0 = Id$

s.t. $\frac{d}{dt}\varphi_t = X_t(\varphi_t)$. where X_t are Hamiltonian.

A symplecto φ is a Hamiltonian symplecto (also as exact diffeo) if $\exists$ Hamiltonian isotopy φ_t s.t. $\varphi_1 = \varphi$

A version of Arnold conjecture can be formulated as:

Conj. For cpt symp mfld (M,ω), $\varphi: M \to M$ Ham. symplecto.
then $\#\{\text{fixed pts of } \varphi\} \geq \#\{\text{crit pts of } f(\varphi),\ f(\varphi) \in C^\infty(M)\}$

There have been various important progress in this problem

① (Weinstein) If φ is C^1-closed to Id. then the conj. holds

(This is obtained using his Weinstein's Lagrangian Nbhd Thm)

② (Floer) If $\langle \omega, \pi_2(M) \rangle = \langle c_1(M), \pi_2(M) \rangle = 0$. then the conj. holds

③ (Non-degenerate case. Fukaya-Ono. Liu-Tian)

If the fixed pts of φ are non-degenerate. then

$\#\{\text{fixed pts of } \varphi\} \geq \#\{\text{crit. pt of a Morse function}\} \geq \sum_{i=0}^{n} \beta_i$

In this note we study Floer's idea and work.

2. Review on Morse theory

A Morse function is a smooth function which crit pts are all nondegenerate

Morse lemma: Let c be a nondegenerate crit pts of f, then in a nbhd of c there exists a coordinate $(x_1, \cdots x_n)$ s.t

$$f(x_1, \cdots x_n) = f(c) - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots x_n^2$$

here j is called the index of f at c.

Now for a Morse function $f: M \to \mathbb{R}$, a pseudo-gradient of f is a vector field X s.t.

① $Xf \leq 0$. "=" holds iff x is a crit. pt

② in a Morse chart of a crit. pt. $X = -\nabla f$ (∇: w.r.t $\mathbb{R}^n$-metric)

Let φ^s be the flow of X above. define the (un)stable mfld as

$$W^s(a) = \{x \in M \mid \lim_{s \to +\infty} \varphi^s(x) = a\}$$

$$W^u(a) = \{x \in M \mid \lim_{s \to -\infty} \varphi^s(x) = a\}$$

clearly $W^s(a) = W^u(a) = \phi$ if a is not a crit pt.

Prop. $\dim W^u(a) = \operatorname{codim} W^s(a) = \operatorname{Ind}(a)$

We additionally assume X satisfies the Smale condition, that is

$$W^u(a) \pitchfork W^s(b) \quad (\forall a, b \text{ crit pts})$$

(Fact: $\exists$ pseudo-gradient satisfying above.)

then $\dim(W^u(a) \cap W^s(b)) = \operatorname{Ind}(a) - \operatorname{Ind}(b)$.

Define $\mathcal{M}(a,b) = \{x \in M \mid \lim_{s \to -\infty} \varphi^s(x) = a,\ \lim_{s \to +\infty} \varphi^s(x) = b\}$

$\Rightarrow \dim \mathcal{M}(a,b) = \operatorname{Ind}(a) - \operatorname{Ind}(b)$

clearly $\mathbb{R}$ acts on $\mathcal{M}(a,b)$ freely by translation in time

$\Rightarrow$ quotient mfld $\mathcal{L}(a,b)$. $\dim \mathcal{L}(a,b) = \operatorname{Ind}(a) - \operatorname{Ind}(b) - 1$.

Now one can define the Morse complex.

$$\mathrm{Crit}_k(f) \triangleq \{a : \text{crit pt of } f.\ \mathrm{Ind}(a)=k\}$$

$$C_k(f) \triangleq \{\sum_{c\in \mathrm{Crit}_k(f)} a_c c \mid a_c \in \mathbb{Z}_2\}$$

$$\partial_X^k : C_k(f) \to C_{k-1}(f)$$

$$a \mapsto \sum_{b\in \mathrm{Crit}_{k-1}(f)} n_X(a,b) b \quad n_X(a,b) \in \mathbb{Z}_2 \ \text{ s.t. } n_X(a,b) \equiv |\mathcal{L}(a,b)| \bmod 2$$

Fact: ① ∂_X is well-defined: $|\mathcal{L}(a,b)| < +\infty$

② (C_k, ∂_k) is indeed a complex

③ The Morse homology $HM_k = \ker \partial_X^k / \mathrm{Im}\, \partial_X^{k+1}$ coincides with the $\mathbb{Z}_2$-homology of M

II. Floer Theory

Floer extended Morse's idea to an infinite-dim space ΩM

$$\Omega M = \{\text{smooth contractible loop on } M\}$$

here we assume $\langle \omega, \pi_2(M)\rangle = \langle c_1(M), \pi_2(M)\rangle = 0$

1. Action functional

Now fix a Hamiltonian symplecto φ, by def. $\exists\, \varphi_t$ s.t.

$$\varphi_0 = \mathrm{Id} \quad \varphi_1 = \varphi \qquad \frac{d}{dt}\varphi_t = X_t(\varphi_t)$$

X_t Hamiltonian $\Rightarrow$ $\iota_{X_t}\omega = dH_t$

Assume H_t varies smoothly w.r.t. time $t \in S^1 = \mathbb{R}/\mathbb{Z}$

Now $\forall z \in \Omega M$ define

$$a_H(z) = -\int_{D^2} \tilde{z}^* \omega - \int_{S^1} H_t(z(t))\,dt$$

where $\tilde{z} : D^2 \to M$ restricts to z on ∂D^2

Note that since $\langle \omega, \pi_2(M) \rangle = 0$, this is independent of choice of $\tilde{z}$.

Prop. For $\zeta = \Gamma^\infty(z^*(TM)) = T_z \Omega M$,

$$da_H(z)(\zeta) = \int_{S^1} \omega(\dot{z}(t), \zeta(t)) - dH_t(z(t))(\zeta(t))\,dt$$

pf. Take $\bar{z}(s,t)$ s.t. $\begin{cases} \bar{z}(0,t) = z(t) \\ \frac{\partial \bar{z}}{\partial s}(0,t) = \zeta(t) \end{cases}$

$\Rightarrow da_H(z)(\zeta) = \frac{\partial}{\partial s} a_H(\bar{z})\big|_{s=0}$

let $\tilde{z}'$ be so that $\begin{cases} \tilde{z}'(0,p) = \tilde{z}(p) \\ \tilde{z}'(s, e^{2\pi i t}) = \bar{z}(s,t) \end{cases}$

$\Rightarrow \tilde{z}' : D^2 \to M$ restricts to $\bar{z}$ on ∂D^2 $\qquad \zeta(p) \triangleq \frac{\partial \tilde{z}'}{\partial s}(0,p)$ (extends ζ to D^2)

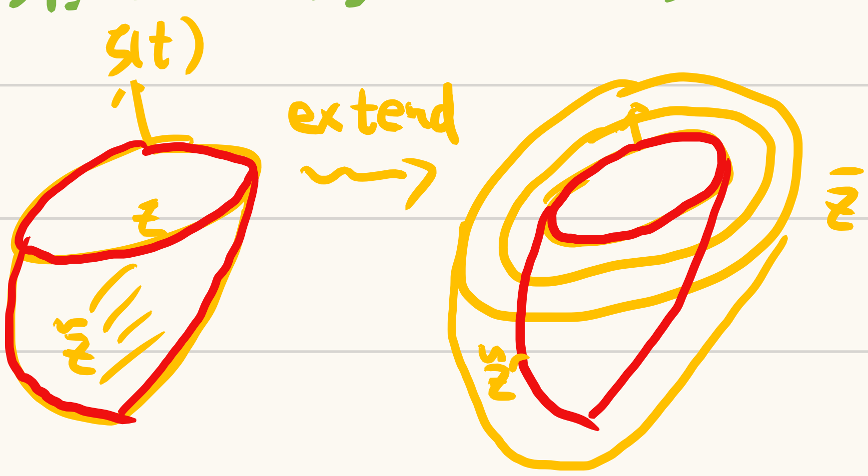
$\Rightarrow a_H(\bar{z}) = -\int_{D^2} \tilde{z}'^* \omega - \int_{S^1} H_t(\bar{z}(s,t))\,dt$

$-\int_{D^2} \frac{d}{ds} \tilde{z}'^* \omega = -\int_D \tilde{z}^* \mathcal{L}_{\zeta(p)} \omega = -\int_D \tilde{z}^* (d\iota_{\zeta(p)} \omega)$

$= -\int_{S^1} z^* \iota_{\zeta(t)} \omega = -\int_{S^1} \omega(\zeta(t), \dot{z}(t))\,dt$

$-\int_{S^1} \frac{\partial}{\partial s} H_t(\bar{z}(s,t))\big|_{s=0}\,dt = \int_{S^1} dH_t(z(t))(\zeta(t))\,dt$

#



So crit pt of $a_H \Leftrightarrow \omega(\dot{z}(t), \xi(t)) - dH_t(z(t))(\xi(t)) = 0 \quad \forall \xi$

$\overset{dH_t = \iota_{X_t}\omega}{\Leftrightarrow} \quad \dot{z}(t) = X_t(z(t))$

$\Leftrightarrow z(t)$ has the form $z(t) = \varphi_t(x)$

Let $\mu_J(J\cdot, \cdot) = \omega(\cdot, \cdot)$ be the induced Riem metric.

∇H_t be the gradient w.r.t. μ_J

then gradient of functional a_H is

$$\nabla_{J,H}(z)(t) = J(z(t))\dot{z}(t) - \nabla H_t(z)$$

If $z_\tau \in \Omega V$ is the trajectory $\quad \frac{\partial z_\tau}{\partial \tau} = -\nabla_{J,H}(z)$

let $u: \mathbb{R}\times S^1 \to M$ be $u(\tau, t) = z_\tau(t)$

one has $\quad \frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0 \quad (*)$

2. Floer's Complex

Main Thm: Any nondegenerate Hamiltonian symplecto φ on cpt monotone mfld (M, ω) satisfies the Morse ineq. over $\mathbb{Z}_2$

Here: (i) (M, ω) is monotone: $\exists k > 0$ s.t. $\langle \omega - kC_1(M), \pi_2(M) \rangle = 0$

(ii) satisfies Morse ineq. : $F(\varphi) = \{$fixed pts of $\varphi\}$

if $\exists\, \partial: F_* \xrightarrow{\text{homo}} F_*$ of free $\mathbb{Z}_2$-module F_* generated by $F(\varphi)$

s.t. $\partial^2 = 0$, $\ker\partial / \operatorname{Im}\partial \simeq H_*(M; \mathbb{Z}_2)$

Given $x, y \in \mathrm{crit}(a_H) \subset \Omega M$ and ω-compatible J, let

$$\mathcal{M}(x,y) = \{ u : \mathbb{R}\times S^1 \to M \mid \bar\partial_{J,H}(u) = 0,\ \lim_{\tau\to-\infty} u(\tau,t) = x(t),\ \lim_{\tau\to\infty} u(\tau,t) = y(t) \}$$

here $\bar\partial_{J,H}(u) = \frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u)$ as in (*)

Now let $\mathcal{J}$: C^∞ ω-compatible AC structures

$\mathcal{H}$: $C^\infty(M\times S^1, \mathbb{R})$ space of Hamiltonian

here is a result similar with Morse theory.

Thm (i) $\exists$ dense set $(\mathcal{J}\times\mathcal{H})_{reg}$ in $\mathcal{J}\times\mathcal{H}$ s.t. $\forall (J,H) \in (\mathcal{J}\times\mathcal{H})_{reg}$

$\mathrm{crit}(a_H) \triangleq Z$ is finite and $\mathcal{M}(x,y)$ above is a smooth mfld

Moreover, let $\Gamma = \mathbb{Z}\langle \langle c_1(M,J), \pi_2(M)\rangle \rangle$

$\exists$ an index function $\mathrm{Ind} : Z \to \mathbb{Z}/2\Gamma$

s.t. $\dim \mathcal{M}(x,y) \equiv \mathrm{Ind}(x) - \mathrm{Ind}(y) \mod 2\Gamma$

(ii) Let $\breve{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$, then $\forall x_1, x_2, x_3 \in Z$.

$\exists$ a local diffeo $\#$ from an open subset $O \subset \breve{\mathcal{M}}(x_1,x_2)\times\breve{\mathcal{M}}(x_2,x_3)\times\mathbb{R}$

into $\breve{\mathcal{M}}(x_1,x_3)$, s.t.

(T_1) $\forall K \overset{cpt}{\subset} \breve{\mathcal{M}}(x_1,x_2)\times\breve{\mathcal{M}}(x_2,x_3)$ $\exists \rho(K) > 0$ s.t. $K \times [\rho(K), \infty) \subset O$

(T_2) $i = 1,2$ $\exists$ lifting $\#_i : O \to \mathcal{M}(x_1,x_3)$ s.t. $\forall (u_1,u_2) \in \mathcal{M}(x_1,x_2)\times\mathcal{M}(x_2,x_3)$

$$\#_i(u_1,u_2,\rho) \xrightarrow{C^\infty_{loc}} u_i \quad \text{as } \rho\to\infty$$

(T_3) the 0- and 1-dim part of $\breve{\mathcal{M}} = \bigcup_{x,y} \breve{\mathcal{M}}(x,y)$ is cpt up to $\mathrm{Im}\#$.

Intutively: $\#$ "glues" trajectory from x_1 to x_2 and from x_2 to x_3

ρ: "gluing parameter" $\Rightarrow$ from x_1 to x_3

$$u_1 \# u_2 \# \rho(T, t) = \begin{cases} u_1(T+2\rho, t), & T \le -\rho \\ u_2(T-2\rho, t), & T \ge \rho \end{cases}$$ small elsewhere $\to$ "approximate trajectory"

when $\rho \to \infty$ $\exists$ D: approximate trajectories $\xrightarrow[\text{retract}]{\text{deformation}}$ trajectories

$\Rightarrow$ in (T_2) $\#_1(u_1, u_2, \rho)(T, t) = D(u_1 \# u_2 \# \rho)(T+2\rho, t)$

$\#_2(u_1, u_2, \rho)(T, t) = D(u_1 \# u_2 \# \rho)(T-2\rho, t)$

$\{D(u_1 \# u_2 \# N)\}$ converges weakly to the pair (u_1, u_2)

$\#$ local diffeo $\Rightarrow$ Im$\#$ $\subset$ a component of $\tilde{\mathcal{M}}(x_1, x_3)$ of dim $1 + \dim \tilde{\mathcal{M}}(x_1, x_2) + \dim \tilde{\mathcal{M}}(x_2, x_3)$

If u_1, u_2 isolated, they glue to a part of 1-dim component of $\tilde{\mathcal{M}}$ $\xRightarrow{(T_2)}$ this part is a nbhd of one of the ends

$\xRightarrow{(T_3)}$ this is the only noncptness in the 0- and 1-dim part

$\Rightarrow$ finite isolated trajectories and 1-dim component of $\tilde{\mathcal{M}}$

Also, any 1-dim component of $\tilde{\mathcal{M}}(x, y)$ has two ends in Im$\#$

Each end converges weakly to a pair of isolated trajectories.

Also note that by (i). $\exists$ isolated trajectory $\Leftrightarrow$ $\mathrm{Ind}\,x + \mathrm{Ind}\,y \equiv 1 \mod 2\Gamma$

$\exists$ 1-dim component in $M(x,y)$ $\Leftrightarrow \cdots \equiv 0 \mod 2\Gamma$

then define Floer complex (F_*, ∂) by

$$\partial x = \sum \langle x, y \rangle y$$

$\langle x, y \rangle$: # isolated trajectories from x to y mod 2.

Lemma. $\partial^2 = 0$

Pf. If not. $\exists\, \partial^2 x \neq 0 \Rightarrow \exists z \in Z$ and odd number of isolated trajectories u, v s.t.

$$x \xrightarrow{u} y \xrightarrow{v} z$$

By (T_1), for each pair (u,v). $\exists$ an arc $\alpha(u,v)$ which lies in a 1-dim component of $\breve{M}$.

One end converges weakly to the pair (u,v)

the other end converges weakly to another pair

$\Rightarrow$ even number of pairs Contradiction!

3. Dependence on (J, H)

A "continuation" is a smooth family of pairs $\{J_\lambda, H_\lambda\}_{\lambda \in \mathbb{R}}$

J_λ: ω-compatible AC structure

H_λ: time-dependent Hamiltonian

assume the pair are independent of λ if $\lambda \in [-R, R]^c$.

Recall the equation

$$\bar{\partial}_{J,H,\lambda}(u) = \frac{\partial u}{\partial \tau} + J_\tau(u)\frac{\partial u}{\partial t} - \nabla H_{t,\tau}(u) = 0$$

Let $x, y, \mathcal{M} \cdots$ correspond to the pair $(J, H) = (J_{-R}, H_{-R})$

$x', y', \mathcal{M}' \cdots$ - - - - - - $(J', H') = (J_R, H_R)$

Now for $(u_1, u_2) \in \mathcal{M}_\lambda(x, x') \times \mathcal{M}'(x', y')$ or $\mathcal{M}(x, y) \times \mathcal{M}_\lambda(x, y')$

if $\{J_\lambda, H_\lambda\}$ regular $\exists$ gluing maps from open subset of $\mathcal{M}_\lambda(x, x') \times \tilde{\mathcal{M}}'(x', y') \times \mathbb{R}$ and $\hat{\mathcal{M}}(x, y) \times \mathcal{M}_\lambda(y, y') \times \mathbb{R}$ to $\mathcal{M}_\lambda(x, y')$

Again, noncpt end of 1-dim component in $\mathcal{M}_\lambda$ converges weakly to a pair of isolated trajectories in $\tilde{\mathcal{M}} \times \mathcal{M}_\lambda$ or $\mathcal{M}_\lambda \times \tilde{\mathcal{M}}'$

And these account for the noncptness in 0- and 1-dim part of $\mathcal{M}_\lambda$

$\Rightarrow$ finitely many isolated trajectories and arcs in $\mathcal{M}_\lambda$

Then a regular continuation $\{J_\lambda, H_\lambda\}$ gives

$$h : F_* \to F_*' \quad \text{by} \quad h(x) = \sum \langle x, x' \rangle x'$$

$\langle x, x' \rangle$: #isolated trajectories in M_λ from x to x' mod 2

Prop. (1) $h\partial = \partial' h$ i.e. h is a chain map

(2) h induces isomorphism on homology

pf (1) $h\partial(x) = h\left(\sum \langle x, y \rangle y\right) = \sum \langle x, y \rangle \langle y, y' \rangle y'$ ⟶ in $\tilde{M} \times M_\lambda$

$\partial' h(x) = \partial'\left(\sum \langle x, x' \rangle x'\right) = \sum \langle x, x' \rangle \langle x', y' \rangle y'$ ⟶ in $M_\lambda \times \tilde{M}'$

(2) For $\{J_\lambda, H_\lambda\}$ from (J, H) to (J', H')

$\{J_\lambda', H_\lambda'\}$ from (J', H') to (J'', H'')

define $J_\lambda^\rho = \begin{cases} J_{\lambda+2\rho} & \lambda < 0 \\ J' & \lambda = 0 \\ J_{\lambda-2\rho}' & \lambda > 0 \end{cases}$ H_λ^ρ similarly for ρ large

slight perturbation to make $(J_\lambda^\rho, H_\lambda^\rho)$ regular

(doesn't change (F_*, ∂)) $\Rightarrow$ define chain map from (F_*, ∂) to (F_*'', ∂'')

$(J_\lambda^\rho, H_\lambda^\rho)$ trajectories $\Longleftrightarrow$ pairs of (J_λ, H_λ), (J_λ', H_λ') trajectories

$\Rightarrow$ the chain map $= h' \circ h$

$\Rightarrow$ continuation can be composed

Next we show that chain homotopy class of h only depends on the homotopy class of $\{J_\lambda, H_\lambda\}$ relative to its ends

For $\{J_\lambda(\nu), H_\lambda(\nu)\}$ a family of continuation with fixed ends

let $\overline{\mathcal{M}}(x, x') = \{(\nu, u) \mid \nu \in [0,1],\ \bar\partial_{J,H,\lambda}(u) = 0$ w.r.t. $\{J_\lambda(\nu), H_\lambda(\nu)\}\}$

define $S: F_* \to F_*'$

$$x \mapsto \sum \langle x, x' \rangle x'$$

but $\langle x, x' \rangle$ is # isolated trajectories in $\overline{\mathcal{M}}(x, x')$

It suffices to show S is a chain homotopy i.e. $h_1 - h_0 = S\delta - \delta S$

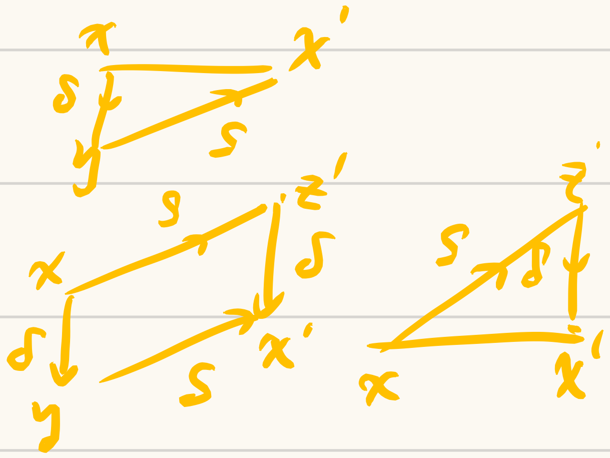
1-dim component of $\overline{\mathcal{M}}(x, x')$: $\alpha: [0,1] \to [0,1]$

① $\alpha(0)$ or $\alpha(1) = 0$ or $1 \Rightarrow$ end at $\{J_\lambda(\nu), H_\lambda(\nu)\}$-trajectory $\nu = 0$ or 1

② abuts in a pair of trajectories of the form

$$(S, \delta) \in \mathcal{M}_\lambda(\alpha(i), x, z') \times \tilde{\mathcal{M}}'(z', x')$$

$$\text{or } (\delta, S) \in \tilde{\mathcal{M}}(x, y) \times \mathcal{M}_\lambda(\alpha(i), y, y')$$



$\Rightarrow$ finite 1-dim $\tilde{\mathcal{M}}(x, x')$ in the form

Again by gluing together the pair (S, δ) or (δ, S)

one can show S is a chain homotopy.

$\Rightarrow \{J_\lambda, H_\lambda\}$ composed with $\{J_{-\lambda}, H_{-\lambda}\}$ $\underset{\text{homotopic}}{\sim}$ constant $\{J, H\}$

$\Rightarrow h$ induces isomorphism on homology #

Pf of Main Thm

It suffices to show $(F_*, \partial) \simeq H_*(M, \mathbb{Z}_2)$ for generic (J, H)

By the prop we'll show later. we may assume H is time-independent and small s.t. $Crit(a_H)$ = constant loops

$\Rightarrow \bar{\partial}_{J,H,\lambda}(u)=0$ has a class of time-independent solutions

trajectories are μ_J-gradient flow of H

$\Rightarrow$ the Floer homology generated by these trajectories $\simeq H_*(M, \mathbb{Z}_2)$

Again by the prop later. actually no other elements of $\mathcal{M}(x,y)$ contribute to (F_*, ∂). #.

4. Some Detailed Analysis

(i) On trajectory space $\mathcal{M}(x,y)$

$$\bar{\partial}_{J,H}(u) = \frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0 \qquad (*)$$

We work in the space $W^{loc}_{1,p}(\mathbb{R}\times S^1, M)$ for $p>2$.

Then define the length of u as

$$\ell^2(u) = \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial \tau}\right|^2 d\tau\wedge dt$$

$\mathcal{M}(x,y)$: $W^{loc}_{1,p}(\mathbb{R}\times S^1, M)$ solution of $(*)$ with $\lim\limits_{T\to-\infty} u(T,t) = x(t)$, $\lim\limits_{T\to\infty} u(T,t) = y(t)$

by elliptic theory. all the solutions are C^∞.

Lemma. $u \in W^{loc}_{1,p}(\mathbb{R}\times S^1, M)$ satisfies $\bar\partial_{J,H} u = 0$.

then u belongs to a trajectory space $\Leftrightarrow$ $\ell^2(u) < \infty$

pf. If $\bar\partial_{J,H} u = 0$. then $\frac{\partial u}{\partial \tau} = -\nabla_{J,H}(u(\tau))$

$$\Rightarrow \frac{\partial a_H(u(\tau))}{\partial \tau} = da_H(u(\tau))\left(\frac{\partial u}{\partial \tau}\right)$$

$$= -\langle \nabla_{J,H}(u(\tau)), \frac{\partial u}{\partial \tau}\rangle$$

$$= -\int_{S^1} \left|\frac{\partial u(\tau)}{\partial \tau}\right|^2 dt$$

$$\Rightarrow \ell^2(u) = -\int_{\mathbb{R}} \frac{\partial}{\partial \tau} a_H(u(\tau))\, d\tau.$$

So if $u \in \mathcal{M}(x,y)$ $\quad \ell^2(u) = a_H(y) - a_H(x) < \infty$.

Conversely. if $\ell^2(u) < \infty$ let $u_R = u|_{[-R,R]\times S^1}$

$$\ell^2(u_R) = \int_{-R}^{R} -da_H(u(\tau))\frac{\partial u}{\partial \tau}\, d\tau$$

$$= \int_{-R}^{R}\int_{S^1} dH_t(u(\tau))\left(\frac{\partial u}{\partial \tau}\right) - \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau}\right) dt\, d\tau$$

$$= \int_{S^1} H_t(u(R,t)) - H_t(u(R,t))\, dt + A(u_R) < +\infty$$

where $A(u_R) = \|du_R\|_2^2 = \int u_R^* \omega$ is the energy

$\Rightarrow A(u) < +\infty$

Let $v_\alpha = u(\cdot + P_\alpha)|_{[-2,2]\times S^1}$ where $P_\alpha \uparrow +\infty$

Since $A(u) < \infty$ $\quad \Rightarrow v_\alpha \xrightarrow{W_{1,2}} 0$

If it's not uniformly bounded in some $W_{1,p}$ norm

by "bubbling off" argument, this contradicts with $A(v_\alpha) \to 0$

①

$\Rightarrow \{v_\alpha\}$ uniformly bounded in $W^{1,p}$

② $\Rightarrow \exists\, v_{\alpha_k}$ s.t. $v_{\alpha_k}|_{[-1,1]\times S^1}$ converges in $W^{1,p}$ to v

Clearly $l^2(v)=0 \Rightarrow v$ independent of τ

Since $\bar\partial_{J,H} v = 0 \Rightarrow v(\tau,t) = x(t)$ $\quad x \in \mathrm{crit}(a_H)$

$\Rightarrow u(\tau) \to x$ as $\tau \to -\infty$ in $C^0(\Omega V)$

similarly $u(\tau) \to y$ $(\tau \to \infty) \Rightarrow u \in \mathcal{M}(x,y)$ #.

Two things in standard J-hol curve theory need to be explained.

① $k \geq 1$, $p > 2$. if $f \in W_{k,p}(S^2, M)$ and $\bar\partial_J f = 0$. then $f \in C^\infty(S^2, M)$

Moreover. every subset of $\bar\partial_J^{-1}(0)$ which is bounded in $W_{k,p}(S^2,M)$ has cpt closure in $C^m(S^2,M)$ $(\forall m)$

pf Recall $\quad W_{k+1,p}(S^2,M) \xrightarrow{cpt} W_{k,p}(S^2,M) \xrightarrow{cpt} C^{m,\alpha}(S^2,M)$

(if $k - \frac{2}{p} > m+\alpha$)

So it suffices to show if $S \subseteq W_{k,p}(S^2,M)$ s.t.

$\|f\|_{k,p}$ and $\|\bar\partial_J f\|_{k,p}$ are bounded by M. then

$f \in W_{k+1,p}(S^2,M)$ $\quad \|f\|_{k+1,p} \leq C = C(k,M,J)$

Since $W_{1,p}(S^2,M) \overset{cpt}{\hookrightarrow} C^\alpha(S^2,M)$, by p.o.u we can assume $f: D_\varepsilon \to \mathbb{C}^n$. where J is AC on $\mathbb{C}^n$ which is standard at 0

Smoothness: if $f \in W_{k,p}(D_\varepsilon, \mathbb{C}^n)$ $\bar\partial_J f = 0$

consider $L(u) = du + J f(z) \circ du \circ i$ elliptic operator for $u \in W_{1,p}$

then L has coefficients in $W_{k,p} \subset C^{m,\alpha}$ for $k - \frac{2}{p} > m + \alpha$

by elliptic theory. Solution of Lu are in $C^{m+1,\alpha}$

$\Rightarrow f \in C^{m+1,\alpha}$ .... $f \in C^\infty$.

$\|\cdot\|_{k,\varepsilon}$: $\|\ \|_{k,p}$ in D_ε

Cpt closure. Now if $S = \{f \in C^\infty(D_\varepsilon, \mathbb{C}^n) \mid \|f\|_{k,\varepsilon} \le M\}$

it suffices to show $\forall M$ $\exists \varepsilon > 0$, $C = C(\varepsilon, J, M)$ s.t.

(△) $\|f\|_{k+1,\frac{\varepsilon}{2}} \le C(\|\bar\partial_J f\|_{k,\varepsilon} + \|f\|_{k,\varepsilon})$ $(\forall f \in S)$

Since $|f(0)|$ is uniformly bounded. WLOG we let $f \in S_0$

$S_0 = \{f \in S \mid f(0) = 0\}$

then (△) is a result in standard $\bar\partial$-theory.

① If V_α is not $W_{1,p}$ uniformly bounded, then $\exists g: S^2 \to M$ continuous s.t. $A(g) \le A(V_\alpha)$

pf. By condition, $\exists z_\alpha \in [-2,2] \times S^1$ s.t. $|dV_\alpha(z_\alpha)| \to \infty$

Here we claim the stronger: $\exists z_\alpha \in [-2,2] \times S^1$ $\varepsilon_\alpha \in (0,1]$

s.t. $r_\alpha = \varepsilon_\alpha |dV_\alpha(z_\alpha)| \to \infty$ (i) $\sup_{z \in B(z_\alpha, \varepsilon_\alpha)} |dV_\alpha(z)| \le 2 |dV_\alpha(z_\alpha)|$ (ii) then define $g_\alpha(z) = V_\alpha(\psi_\alpha(z)) : B(0, r_\alpha) \to M$

where $\psi_\alpha : B(0, r_\alpha) \to B(z_\alpha, \varepsilon_\alpha)$ s.t $|d\psi_\alpha(z)| \le \frac{1}{|dV_\alpha(z_\alpha)|}$

$\Rightarrow \sup |dg_\alpha(z)| \le 2$

then $\forall R$. $g_{\alpha}|_{B(0,2R)}$ $W_{1,p}$ uniformly bounded

① $\Rightarrow \exists\, g_{\alpha_k} \xrightarrow{C^1} g_R : B(0, R) \to M$

$\Rightarrow \exists g : \mathbb{C} \to M$ J-hol s.t. $\|dg\|_2^2 \le A(V_\alpha)$

By removal of singularities. g extends to a continuous map $S^2 \to M$.

③

pf of claim: start from any $z_\alpha, \varepsilon_\alpha$ satisfying (i)

and if (ii) holds from any $\alpha < k$ let $(w_1, \delta_1) = (z_k, \varepsilon_k)$

if (ii) doesn't hold, take $w_2 \in B(w_1, \delta_1)$ s.t $|dV_k(w_2)| > 2|dV_k(w_1)|$

$\delta_2 = \frac{\delta_1}{2}$ (then (w_2, δ_2) still fits (i))

if this procedure never stops, then $w_n \to w \in B(z_k, 2\varepsilon_k)$

$\Rightarrow V_k$ is not differentiable at w

contradicts with ② ! #.

③ Removal of Singularity. g is a J-hol map with finite area from $D - \{0\}$ to M. then g continuously extends to D.

(pf omitted)

(ii) Fredholm theory

For $x, y \in \mathrm{crit}(a_H)$, define $P_{1,p}(x,y) \subset W^{loc}_{1,p}(\mathbb{R}\times S^1, M)$ be the functions which decay exponentially to x, y at the end of cylinder $\mathbb{R}\times S^1$.

If x, y nondegenerate, Floer defined a Banach bundle $\mathcal{L}$ over $P_{1,p}(x,y)$ and a Fredholm section $\bar{\partial}_{J,H}$ s.t.

$$\mathcal{M}(x,y) = \bar{\partial}_{J,H}^{-1}(0)$$

(iii) Index μ

$\mathrm{Ind}(x)$ ($x \in \mathrm{Crit}(a_H)$) not naturally defined

difference: $\mu(u) = \text{"}\mathrm{Ind}\text{"}(x) - \text{"}\mathrm{Ind}\text{"}(y)$ for $u \in \mathcal{M}(x,y)$

smooth flow of eigenvalues $a(\tau)$ (assume all have multiplicity 1)

If $\forall u$, one can define self-adjoint A_τ on Hilbert space L_τ $\tau \in [0,1]$, where A_0 (A_1) is the Hessian at x (y).

then let $\mu(u) = \#\{a : a(0) < 0 < a(1)\} - \#\{a : a(0) > 0 > a(1)\}$

L_z : $L^2(z^* TM)$ tangent space at $z \in \Omega M$

A_z: defined on a dense subset of $L^2(z^* TM)$ by $\nabla \cdot \nabla_{J,H}$

(if extended to $W_{1,2}(z^* TM)$, then adjoint)

If $z \in \mathrm{crit}(a_H)$. A_z is the Hessian, given by

$$A_z(\xi) = \nabla_\xi (J\dot{z}) - \nabla_\xi (\nabla H_t(z))$$

$$= J\nabla_z \xi + (\nabla_\xi J)\dot{z} - \nabla_\xi (\nabla H_t(z))$$

Floer shows if x, y nondegenerate, one can take A_τ be a slight perturbation of $A_{u(\tau)}$ with $a(\tau)$ independent of perturbation

And $\mu(u)$ is actually the dim of component of $\mathcal{M}(x,y)$ containing u.

Since ΩM consists of contractible loops. $\forall u, v$ path in ΩM from x to y, v homotopic to $u \# \underbrace{A}_{\text{image of a 2 sphere in } M}$ rel $x\ y$

write $v = u \# A$.

Prop. $\mu(u\#A) = \mu(u) + 2c_1(A)$

Lemma. If (M, ω) monotone, then $\forall K$, $\exists L_K$ s.t. if $\mu(u) \le K$ then $\ell(u) \le L_K$

pf. $\ell^2(u) = A(u) + \int_{S^1} H(x(t)) - H(y(t))\,dt$

$$\Rightarrow \ell^2(u\#A) = \ell^2(u) + \omega(A) = \ell^2(u) + kc_1(A) \le \ell^2(u) + \frac{k}{2}(K - \mu(u)) \le L_K.$$

(iv) The set $(\mathcal{J}\times\mathcal{H})_{reg}$

Prop. $\exists$ dense set $(\mathcal{J}\times\mathcal{H})_{reg}\subset\mathcal{J}\times\mathcal{H}$ s.t. $\forall (J,H)\subset(\mathcal{J}\times\mathcal{H})_{reg}$ a_H has a finite set Z of nondegenerate crit pts, and $\forall x,y\in Z$ $\mathcal{M}(x,y)$ is a smooth mfld. Further, the dimension of the component of $\mathcal{M}(x,y)$ containing u $=\mu(u)$

(This is done by analyzing Fredholm theory and transversality as in the J-hol curve theory)

For the sake of simplicity we want H to be time-independent but this is not generic. Further justifying arguments need to be done.

Lemma. Let H be any element in $\mathcal{H}$, x,y distinct nondegenerate crit pts of a_H, then the subset of $P_{1,p}(x,y)\times\mathcal{J}'$ formed by all simple (u,J) with $\bar\partial_{J,H}u=0$ is a Banach mfld.

Now we can consider the case when H is time-independent. For $x_0\in crit(H)$, let $x\in\Omega M$ be the constant loop at x_0

Let $S_J : T_{x_0}M \to T_{x_0}M$ be its Hessian w.r.t. μ_J

then $\mu_J(S_J(X), Y) = XYH = -\omega(JS_J(X), Y)$

$\Rightarrow JS_J$ is independent of J.

Lemma H time-independent. then x is a nondegenerate crit pt of a_H iff $\nexists\, k \in \mathbb{Z}$ s.t. $2\pi i k$ is in the spectrum of JS_J

In particular x_0 must be a nondegenerate crit pt of H.

pf. $\dot{x} = 0 \Rightarrow A_x(\xi(t)) = J(\nabla_t \xi(t)) - S_J(\xi(t))$. $\xi(t) \in W_{1,2}(S^1, T_{x_0}M)$

x nondegenerate $\Leftrightarrow$ $\underbrace{A_x(\xi) = 0}$ has no solution with $\xi(0) = \xi(1)$

$\Updownarrow$

$\xi(t) = \exp(-tJS_J)\xi(0)$

#

Prop Let H be a generic time-independent function on M

If (M, ω) monotone. λ sufficiently small. $\exists (J, \lambda H) \in (\mathcal{J} \times \mathcal{H})_{reg}$ s.t. the only components of $\mathcal{M}(x, y)$ w.r.t. $(J, \lambda H)$ which contribute to (F_*, ∂) consist of time-independent trajectories

pf. Choose $\lambda > 0$ small so that

(i) $\sup(\lambda H) - \inf(\lambda H) \leq k$ where $[\omega] = kc_1$

(ii) λH has no nonconstant orbits of period ≤ 1

Then $\mathrm{crit}(a_{\lambda H}) =$ constant loops at $\mathrm{crit}(H)$

H generic $\Rightarrow$ assume condition in the previous lemma holds

$\mathrm{Im}\,u$: 2-sphere on $M \Rightarrow E(u) = \omega([u]) - \lambda H(x) - \lambda H(y) > 0$

$\overset{(i)}{\Rightarrow} \omega([u]) > -k \overset{\omega = kc_1}{\Rightarrow} \omega([u]) \geq 0 \quad c_1([u]) \geq 0$

recall $\mu(u \# A) = \mu(u) + 2c_1(A)$

for $u^m = u(m\cdot, m\cdot) \qquad \mu(u^m) \geq \mu(u)$

Choose $J \in \mathcal{J}$ s.t. $\forall m \geq 1 \quad (J, \lambda H/m)$ is regular for all simple u

Then $\forall$ solution u of $\bar{\partial}_{J, \lambda H/m} u = 0$ which is not constant

has a 2-parameter family of reparametrization

$\Rightarrow \mu(u) \geq 2 \qquad \mu(u^m) \geq 2$

$\Rightarrow (J, \lambda H)$-trajectory which depends on time has local dim ≥ 2

$\hookrightarrow$ doesn't contribute to (F_*, ∂) !

(v) Compactness

Prop (M, ω) monotone and (J, H) regular. then the component of $\tilde{\mathcal{M}}(x, y)$ of dim 0 are cpt (hence finite)

of dim 1 are cpt except for sequences of trajectories which split up in two.

pf. Consider $\{u_\alpha\}$ in $\mathcal{M}(x,y)$ with $\mu(u_\alpha)=m$

then $E(u_\alpha)$ bounded $\Rightarrow u_\alpha$ uniformly bounded in $W_{1,2}(\mathbb{R}\times S^1)$

(i) If u_α uniformly bounded in $W_{1,p}^{loc}(\mathbb{R}\times S^1)$ $p>2$

then $\exists$ subsequence (also called u_α) $u_\alpha \xrightarrow{C^\infty_{loc}} u$

$\Rightarrow \forall \{u_\alpha \circ \sigma_\alpha\} = \{u_\alpha(\cdot + \sigma_\alpha)\}$ C^∞_{Local} converges to v

clearly $v \in C^\infty(\mathbb{R}\times S^1)$ $\bar\partial_{J,H} v = 0$ $E(v) < \infty$ $\Rightarrow v \in \mathcal{M}'(x', y')$

Since there are only finite possible limits $v_1 \sim v_k$ $v_i \in \mathcal{M}'(z_{i-1}, z_i)$

$$\mu(u_\alpha) = \sum \mu(v_i)$$

$\Rightarrow$ no splitting if $\mu(u_\alpha)=1$ only split into two if $\mu(u_\alpha)=2$

(ii) If u_α not uniformly bounded in $W_{1,p}^{loc}$ $(\forall p>2)$

then by bubbling off argument. $\exists$ subsequence u_α

converges to a k-trajectory v with finite J-hol sphere $A_1 \sim A_p$

$\Rightarrow \mu(v) + 2c_1(A_1) + \cdots + 2c_1(A_p) = \mu(u_\alpha) = m$ if $m \le 2$

if v is not constant $\Rightarrow \mu(v) \ge 1$

$\omega(A_i) > 0 \xRightarrow{\text{monotone}} c_1(A_i) > 0$ $\Rightarrow \times$

if v is constant $x=y$ then $p=1$ $c(A_1)=1$

then x is a single point on the bubble

but this doesn't happen for generic J' #