Geometry and Analysis on Manifold

Chap 1. Chern-Weil Theory of Characteristic Class

1.1 de Rham Cohomology

1.2 Super Vector Bundle

Def 1.1. V is a real/complex vector space. $T \in \mathrm{End}(V)$ is a linear map. (V, T) is a super vector space if $T^2 = 1_V$

$V_\pm$: eigenspace w.r.t. ± 1

$$V = \underbrace{V_+}_{\substack{\uparrow \\ \text{even} \\ \text{element}}} \oplus \underbrace{V_-}_{\substack{\uparrow \\ \text{odd} \\ \text{element}}}$$

Def 1.2 An algebra A is a superalgebra, if as a vector space A is equipped with a super structure T, and $A_\pm A_\pm \subseteq A_+$. $A_\pm A_\mp \subset A_-$

For super vector space (V,T). $\mathrm{End}(V)$ is naturally a super vector space with $\mathrm{End}_\pm(V) = \{A \in \mathrm{End}(V) \mid TA = \pm AT\}$

$A \in \mathrm{End}_+(V) \Leftrightarrow A(V_\pm) \subseteq V_\pm$

Def 1.3 (V,T). $A \in \mathrm{End}(V)$, the <u>super trace</u> of A is $\mathrm{str}[A] = \mathrm{tr}[TA]$

$A \in \mathrm{End}_-(V)$. $\mathrm{Str}[A] = \mathrm{tr}[TA] = -\mathrm{tr}[AT] = -\mathrm{tr}[TA]$
$= -\mathrm{Str}[A]$
$\Rightarrow \mathrm{Str}[A] = 0$

$A \in \mathrm{End}_+(V)$ $\quad \mathrm{str}[A] = \mathrm{tr}[A|_{V_+}] - \mathrm{tr}[A|_{V_-}]$

For $A, B \in \mathrm{End}(V)$ define <u>super bracket</u> as

$$[A,B]_s = AB - (-1)^{|A||B|} BA$$

Lemma 1.1 $\mathrm{str}[[A,B]_s] = 0$.

For $(V, T_V), (W, T_W)$. $T_V \otimes T_W$ gives super structure on $V \otimes W$.

$$(V\otimes W)_+ = (V_+ \otimes W_+) \oplus (V_- \otimes W_-)$$

$$(V\otimes W)_- = (V_+ \otimes W_-) \oplus (V_- \otimes W_+)$$

denote $(V\otimes W, T_V \otimes T_W)$ as $V \hat{\otimes} W$

element in $V\hat{\otimes} W$ as $a \hat{\otimes} b$.

If (V, T_V) (W, T_W) are superalgebra, then $T_V \hat{\otimes} T_W$ gives a superalgebra on $V \hat{\otimes} W$ with

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{(\deg a_2)(\deg b_1)} (a_1 a_2) \hat{\otimes} (b_1 b_2)$$

Lemma 1.2 (V, T_V) (W, T_W) are super vector spaces, then $\forall A \in End(V)$, $B \in End(B)$

$$str[A \hat{\otimes} B] = str(A)\, str[B]$$

Def 1.4 Vector bundle E on a manifold M is a super vector bundle if $\exists$ a $\mathbb{Z}_2$-grading

$$E = E_+ \oplus E_-$$

If E is an algebra bundle, and on each fiber

$$E_\pm E_\pm \subseteq E_+, \quad E_\pm E_\mp \subseteq E_-$$

then E is a superalgebra bundle

$\mathrm{End}(E) = \mathrm{End}_+(E) \oplus \mathrm{End}_-(E)$. one can extend the definition of supertrace to

$$\mathrm{str}\ \Gamma(\mathrm{End}(E)) \to C^\infty(M)$$

$\Omega^*(T^*M)$ is naturally a superalgebra bundle

then for any super vector bundle E

$\Lambda^*(T^*M) \hat{\otimes} E$ gives a super vector bundle.

$\Lambda^*(T^*M) \hat{\otimes} \mathrm{End}(E)$ gives a superalgebra bundle.

$\forall \alpha \in \Omega^*(M)$, $s \in \Omega^*(M, E)$, $T \in \Omega^*(M, \mathrm{End}(E))$

$$T(\alpha \wedge s) = (-1)^{(\deg \alpha)(\deg T)} \alpha \wedge (Ts)$$

$$\mathrm{str} : \Omega^*(M, \mathrm{End}(E)) \to \Omega^*(M)$$

$$\mathrm{str}(\alpha A) = \alpha\ \mathrm{str}(A) \qquad \left(\begin{array}{l} \forall \alpha \in \Omega^*(M) \\ A \in \Gamma(\mathrm{End}(E)) \end{array} \right)$$

Lemma 1.3 (E, T) is a super vector bundle then

$\forall A, B \in \Omega^*(M, \mathrm{End}(E))$ $\quad \mathrm{str}[[A, B]_s] = 0$

Def 1.5 A <u>connection</u> ∇^E is an operator

$$\nabla^E : \Gamma(E) \to \Omega^1(M, E).$$

s.t. $\nabla^E(fs) = (df)s + f\nabla^E s \qquad \left(\begin{array}{l} \forall f \in C^\infty(M) \\ s \in \Gamma(E) \end{array} \right)$

Covariant derivative : $\nabla^E_X : \Gamma(E) \to \Gamma(E)$

$$\nabla^E_X s = (\nabla^E s)(X) \quad (\forall s \in \Gamma(E))$$

Extend ∇^E to $\Omega^*(M,E) \to \Omega^{*+1}(M,E)$

s.t. $\forall \omega \in \Omega^*(M)$, $s \in \Gamma(E)$

$$\nabla^E(\omega s) = (d\omega)s + (-1)^{\deg \omega} \omega \wedge \nabla^E s$$

Def 1.6 $E = E_+ \oplus E_-$ is a super vector bundle

then the super connection

$$A : \Omega^*(M,E) \to \Omega^*(M,E)$$

is odd-graded linear operator

with $A(\alpha \wedge s) = (d\alpha)\wedge s + (-1)^{\deg \alpha} \alpha \wedge A s \quad \left(\begin{matrix} \alpha \in \Omega^*(M) \\ s \in \Omega^*(M,E) \end{matrix} \right)$

$E = E_+ \oplus E_-$. $\nabla^{E_\pm}$ are connections on $E_\pm$

then $A = \nabla^{E_+} \oplus \nabla^{E_-}$ is a super connection

Moreover, $\forall v \in \Gamma(\mathrm{End}_-(E)) = \Omega^0(M, \mathrm{End}_-(E))$

$$A_v = (\nabla^{E_+} \oplus \nabla^{E_-}) + v \text{ is a super connection.}$$

$$\Omega^*(M,E) = \bigoplus_{k=0}^{\dim M} \Omega^k(M,E)$$

$$A|_{\Omega^0(M,E)} = \sum_{k=0}^{\dim M} A_{[k]}, \text{ where}$$

$A_{[k]}: \Omega^0(M,E) \to \Omega^k(M,E)$ is $C^\infty(M)$-linear $(k \neq 1)$

$A_{[1]}: \Omega^0(M,E) \to \Omega^1(M,E)$ is a connection preserving $\mathbb{Z}_2$-grading.

Def 1.7 A is a super connection on $E = E_+ \oplus E_-$ then the curvature is

$$R^E = A^2 : \Omega^*(M,E) \to \Omega^*(M,E)$$

Prop 1.1 A is a super connection, then R^E is $\Omega^*(M)$-linear. i.e. $\forall \alpha \in \Omega^*(M)$, $s \in \Omega^*(M,E)$

$$R^E(\alpha \wedge s) = \alpha \wedge R^E s.$$

Pf.

$$\begin{aligned} R^E(\alpha \wedge s) &= A\left((d\alpha)\wedge s + (-1)^{\deg \alpha} \alpha \wedge As\right) \\ &= (-1)^{\deg d\alpha} d\alpha \wedge As + (-1)^{\deg \alpha} (d\alpha)\wedge As \\ &\quad + (-1)^{2\deg \alpha} \alpha \wedge A^2 s \\ &= \alpha \wedge R^E s \end{aligned}$$

#

Thus R^E can be regarded as an element in

$$\Omega^*_+(M, \mathrm{End}(E)) = \Gamma\left(\left(\Lambda^*(T^*M)\hat{\otimes} \mathrm{End}(E)\right)_+\right)$$

specifically, for vector bundle E, $R^E \in \Omega^2(M, \mathrm{End}(E))$

$$R^E(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}$$

Thm 1.1 (Bianchi) $[A, (A^2)^k]_s = 0 \quad (k \geq 1)$

1.3 Chern-Weil Theorem

Lemma 1.4 A is a super connection on $E = E_+ \oplus E_-$

then $\forall L \in \Omega^*(M, \mathrm{End}(E))$, $\mathrm{str}[[A, L]_s] = d\,\mathrm{str}[L]$

pf. By Lebiniz's rule it's easy to see ($\forall A_1$ is another super connection)

$$A - A_1 \in \Omega^*_-(M, \mathrm{End}(E))$$

thus by Lemma 1.3 $\mathrm{str}[[A - A_1, L]_s] = 0$

Then $\forall p \in M$ take $U_p \ni p$ s.t $E_\pm|_{U_p}$ is trivial.

take trivial super connection and the result holds #.

$f(x) = a_0 + a_1 x + \cdots a_k x^k$.

$R^E = A^2$ is the curvature

then since $\Omega^*_+(M, \mathrm{End}(E)) = \Omega^{even}(M, \mathrm{End}_+(E)) \oplus \Omega^{odd}(M, \mathrm{End}_-(E))$

$$\mathrm{str}[f(R^E)] \in \Omega^{even}(M)$$

Thm 1.2 (i) $d\,\mathrm{str}[f(R^E)] = 0$

(ii) $\hat{A}$ $\hat{R}^E$ then $\exists \omega \in \Omega^*(M)$, s.t

$$\mathrm{str}[f(R^E)] - \mathrm{str}[f(\hat{R}^E)] = d\omega.$$

Pf. (i) $d\,\mathrm{str}[f(R^E)] \overset{\text{Lemma}}{=} \mathrm{str}[[A, f(R^E)]_s]$

$$= \mathrm{str}[a_1[A, R^E]_s + \cdots + a_k[A, (R^E)^k]_s]$$

$$\overset{\text{Bianchi}}{=} 0$$

(ii) $A_t = (1-t)A + t\hat{A}$

$$\frac{dA_t}{dt} = \hat{A} - A \in \Omega^*_-(M, \mathrm{End}(E))$$

$$\frac{d}{dt}\mathrm{str}[f(R_t^E)] = \mathrm{str}\left[\frac{dR_t^E}{dt} f'(R_t^E)\right]$$

$$= \mathrm{str}\left[\frac{d(A_t)^2}{dt} f'(R_t^E)\right]$$

$$= \mathrm{str}\left[\left[A_t, \frac{dA_t}{dt}\right]_s f'(R_t^E)\right]$$

$$\overset{\text{Bianchi}}{=} \mathrm{str}\left[\left[A_t, \frac{dA_t}{dt} f'(R_t^E)\right]_s\right]$$

$$\overset{\text{Lemma}}{=} d\,\mathrm{str}\left[\frac{dA_t}{dt} f'(R_t^E)\right]$$

$$\Rightarrow \mathrm{str}[f(R^E)] - \mathrm{str}[f(\hat{R}^E)] = -d\int_0^1 \mathrm{str}\left[\frac{dA_t}{dt} f'(R_t^E)\right]dt$$

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Def 1.8. The cohomology class $[\mathrm{tr}[f(\frac{i}{2\pi}R^E)]]$ is the characteristic class of E w.r.t. f, denoted $f(E)$

Lemma 1.5. $\int_M f_1(E_1, \nabla^{E_1}) \cdots f_k(E_k, \nabla^{E_k})$

$$= \int_M \{f_1(E_1, \nabla^{E_1}) \cdots f_k(E_k, \nabla^{E_k})\}^{\max}$$

is free of choice of ∇^{E_i} $(1 \le i \le k)$, called the characteristic number $\langle f_1(E_1) \cdots f_k(E_k), [M]\rangle$

1.4 Some Examples

complex bundle:

Chern form (w.r.t ∇^E) is $c(E,\nabla^E) = \det\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)$
$= \exp\left(\operatorname{tr}\left[\log\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$

chern class $c(E)$.

$$c(E,\nabla^E) = 1 + c_1(E,\nabla^E) + \cdots + \cdots$$

$c_i(E,\nabla^E) \in \Omega^{2i}(M)$: i-th chern form

$c_i(E)$: i-th chern class.

real bundle:

Pontrjagin form (w.r.t. ∇^E) is $p(E,\nabla^E) = \det\left(\left(I-\left(\frac{R^E}{2\pi}\right)^2\right)^{\frac{1}{2}}\right)$

$$p(E,\nabla^E) = 1 + \underbrace{p_1(E,\nabla^E)}_{\in\Omega^{4i}(M)} + \cdots$$

$\in\Omega^{4i}(M)$: i-th pontrjagin form

$p_i(E)$: i-th Pontrjagin class

$$p_i(E) = (-1)^i c_{2i}(E\otimes\mathbb{C})$$

tangent bundle:

L-form (w.r.t. ∇^{TM})

$$L(TM,\nabla^{TM}) = \det\left(\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{2\pi}R^{TM}\right)}\right)^{\frac{1}{2}}\right)$$

L-class $L(TM)$

L-genus: $L(M) = \langle L(TM), [M] \rangle$
$$= \int_M L(TM, \nabla^{TM})$$

[eg. $\dim M = 4 \Rightarrow \{L(TM, \nabla^{TM})\}^{\max} = \frac{1}{3} p_1(TM, \nabla^{TM})$]

$\hat{A}$-form ($\hat{A}$-class)
$$\hat{A}(TM, \nabla^{TM}) = \det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi} R^{TM}\right)}\right)^{\frac{1}{2}}\right)$$

[eg. $\dim M = 4 \Rightarrow \{\hat{A}(TM, \nabla^{TM})\}^{\max} = -\frac{1}{24} p_1(TM, \nabla^{TM})$]
thus $L(M) = -8 \hat{A}(M)$

$\hat{A}$-genus: $\hat{A}(M) = \langle \hat{A}(TM), [M] \rangle = \int_M \hat{A}(TM, \nabla^{TM})$

Td-form. (Td-class)
$$Td(TM, \nabla^{TM}) = \det\left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{1 - \exp\left(-\frac{\sqrt{-1}}{2\pi} R^{TM}\right)}\right)$$

Td-genus: $Td(M) = \langle Td(TM), [M] \rangle$
$$= \int_M Td(TM, \nabla^{TM})$$

Now $E = E_+ \oplus E_-$ is a complex super vector bundle.

$$A = \nabla^{E_+} + \nabla^{E_-}$$

$$ch(E, A) = \mathrm{Str}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} A^2\right)\right] \in \Omega^{even}(M)$$

(chern characteristic form)

[Alt $\quad \mathrm{Str}[\exp(A^2)] = \sum_i \omega^i \qquad \omega^i \in \Omega^{2i}(M)$

$\quad ch(E, A) = \sum_i \left(\frac{\sqrt{-1}}{2\pi}\right)^i \omega^i$]

Prop $\quad ch(E \oplus F) = ch(E) + ch(F) \in H^{even}_{dR}(M; \mathbb{C})$

$Vect(M) = \{$ complex bundle on $M\}$.

$\quad E \sim F \Leftrightarrow \exists$ bundle G s.t. $E \oplus G \cong F \oplus G$.

K-Group: $\quad K(M) = Vect(M)/\sim$

Then the above property extends chern characteristic

$$ch : K(M) \to H^{even}_{dR}(M; \mathbb{C}).$$

chern-simons form: $-\int_0^1 \mathrm{Str}\left[\frac{dA_T}{dt} f'(R_T^E)\right]dt$

For M is a orientable closed 3-manifold

TM is topologically trivial: $\exists$ global basis $e_1 \sim e_3$

$$\text{s.t. } \forall X \in \Gamma(TM) \quad X = \sum_{i=1}^{3} f_i e_i$$

$$d^{TM} \triangleq d^{TM}(f_1 e_1 + f_2 e_2 + f_3 e_3) = df_1 \cdot e_1 + df_2 \cdot e_2 + df_3 \cdot e_3$$

then $\forall \nabla^{TM} = d^{TM} + A$. $\quad A \in \Omega^1(M, \mathrm{End}(TM))$

$\forall t \in [0,1] \quad \nabla_t^{TM} \triangleq d^{TM} + tA$

let $f(x) = -x^2$

$$-\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^{TM}}{dt} f'(R_t^{TM})\right]dt = -\int_0^1 \mathrm{tr}\left[A(-2)(d^{TM}+tA)^2\right]dt$$

$$= 2\int_0^1 \mathrm{tr}\left[tA \wedge d^{TM}A + t^2 A \wedge A \wedge A\right]dt$$

$$= \mathrm{tr}\left[A \wedge d^{TM}A + \frac{2}{3} A \wedge A \wedge A\right]$$

1.5 Bott Vanishing Thm of Foliation

$F \subseteq TM$ is integrable $\Rightarrow \forall p \in F$ $\exists$ a maximal submanifold $\mathcal{F}_p$ s.t. $T_p\mathcal{F}_p = F|_p$

$P_{i_1}(TM/F), \cdots P_{i_k}(TM/F)$ Pontrjagin class

Thm 1.3 If $i_1 + \cdots i_k > \frac{\dim M - \operatorname{rank}(F)}{2}$, then

$$P_{i_1}(TM/F) \cdots P_{i_k}(TM/F) = 0 \quad \text{in } H_{dR}^{4(i_1+\cdots i_k)}(M;\mathbb{R})$$

pf. Choose a Riemannian metric g^{TM}.

then $TM = F \oplus F^{\perp}$. $TM/F \simeq F^{\perp}$

∇^{TM} is the Levi-Civita connection

($P, P^{\perp}$ projection $\quad g^F, g^{F^{\perp}}$: restricted metric)

$$\nabla^F = P\nabla^{TM}P \quad \nabla^{F^{\perp}} = P^{\perp}\nabla^{TM}P^{\perp}$$

so it suffices to show $\exists \omega \in \Omega^*(M)$

$$\text{s.t. } P_{i_1}(F^{\perp}, \nabla^{F^{\perp}}) \cdots P_{i_k}(F^{\perp}, \nabla^{F^{\perp}}) = d\omega \quad (*)$$

Def 1.9 $\forall X \in \Gamma(TM)$, $u \in \Gamma(F^{\perp})$.

(i) $X \in \Gamma(F)$: $\hat{\nabla}_X^{F^{\perp}} u \triangleq P^{\perp}[X, u]$

(ii) $X \in \Gamma(F^{\perp})$: $\hat{\nabla}_X^{F^{\perp}} u = \nabla_X^{F^{\perp}} u$

$\hat{\nabla}^{F^{\perp}}$: Bott connection

[Lemma 1.6 $\forall X, Y \in \Gamma(F)$ $\tilde{R}^{F\perp}(X,Y)=0$]

So $\tilde{R}^{F\perp} \in \Gamma(F^{\perp,*}) \wedge \Omega^*(M, \mathrm{End}(F^{\perp}))$

Then $\forall k \le j \le R$. $P_{i_j}(F^{\perp}, \overset{\sim}{\nabla}^{F\perp}) \in \Gamma(\Lambda^{2i_j}(F^{\perp,*})) \wedge \Omega^*(M)$

thus $P_{i_1}(F^{\perp}, \overset{\sim}{\nabla}^{F\perp}) \cdots P_{i_k}(F^{\perp}, \overset{\sim}{\nabla}^{F\perp})$

$$\in \Gamma(\Lambda^{2(i_1+\cdots+i_k)}(F^{\perp,*})) \wedge \Omega^*(M)$$
$$=0$$

Then $(*)$ follows from Chern-Weil Thm #

$g^{TM,\varepsilon} = g^F \oplus \frac{1}{\varepsilon} g^{F^\perp}$ w.r.t. $g^{TM,\varepsilon}$ one has $\nabla^{TM,\varepsilon}$
$\nabla^{F,\varepsilon}$, $\nabla^{F^\perp,\varepsilon}$

$\varepsilon \to 0$ adiabatic limit

Thm 1.4 $\forall X \in \Gamma(F)$ $\lim_{\varepsilon \to 0} \nabla_X^{F^\perp,\varepsilon} = \overset{\sim}{\nabla}_X^{F^\perp}$

(pf omitted here)

1.6 Odd-dim Chern-Weil Theory

$g: M \to GL(N, \mathbb{C})$ is a linear map

$\mathbb{C}^N|_M$ is the trivial rank-N complex bundle

d is the trivial connection on $\mathbb{C}^N|_M$

$$g^{-1}dg \in \Omega^1(M, \mathrm{End}(\mathbb{C}^N|_M))$$

If n is even, $\mathrm{tr}[(g^{-1}dg)^n] = \frac{1}{2}\mathrm{tr}[(g^{-1}dg)^{n-1}, g^{-1}dg] = 0$

$$gg^{-1} = 1 \Rightarrow dg^{-1} = -g^{-1}(dg)g^{-1}$$

If n is odd $\quad d\,\mathrm{tr}[(g^{-1}dg)^n] = n\,\mathrm{tr}[d(g^{-1}dg)\,(g^{-1}dg)^{n-1}]$

$$= -n\,\mathrm{tr}[(g^{-1}dg)^{n+1}] = 0$$

Lemma 1.7 $g_t: M \to GL(N, \mathbb{C})$ $(t \in [0,1])$, then for n odd

$$\frac{\partial}{\partial t}\mathrm{tr}[(g_t^{-1}dg_t)^n] = n\,d\,\mathrm{tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}(g_t^{-1}dg_t)^{n-1}\right]$$

Pf $\frac{\partial}{\partial t}(g_t^{-1}dg_t) = -g_t^{-1}\frac{\partial g_t}{\partial t}g_t^{-1}dg_t + \underbrace{g_t^{-1}d\frac{\partial g_t}{\partial t}}_{\substack{\| \\ d(g_t^{-1}\frac{\partial g_t}{\partial t}) + (g_t^{-1}dg_t)(g_t^{-1}\frac{\partial g_t}{\partial t})}}$

$$d(g_t^{-1}dg_t)^2 = d(g_t^{-1}dg_t)\,g_t^{-1}dg_t - g_t^{-1}dg_t\,d(g_t^{-1}dg_t) = 0$$

$\Rightarrow \forall k$ even $\quad d(g_t^{-1}dg_t)^k = 0$

Thus $\frac{\partial}{\partial t} \operatorname{tr}[(g_t^{-1} dg_t)^n]$

$$= n \operatorname{tr}[\tfrac{\partial}{\partial t}(g_t^{-1} dg_t)(g_t^{-1} dg_t)^{n-1}]$$

$$= n \operatorname{tr}\Big[[g_t^{-1} dg_t , g_t^{-1}\tfrac{\partial g_t}{\partial t}](g_t^{-1} dg_t)^{n-1}\Big] + n \operatorname{tr}[d(g_t^{-1}\tfrac{\partial g_t}{\partial t})(g_t^{-1} dg_t)^{n-1}]$$

$$= n \operatorname{tr}\Big[[g_t^{-1} dg_t , g_t^{-1}\tfrac{\partial g_t}{\partial t}(g_t^{-1} dg_t)^{n-1}]\Big] + n \operatorname{tr}[d(g_t^{-1}\tfrac{\partial g_t}{\partial t}(g_t^{-1} dg_t)^{n-1})]$$

$$= n d \operatorname{tr}[g_t^{-1}\tfrac{\partial g_t}{\partial t}(g_t^{-1} dg_t)^{n-1}]$$

#.

Cor 1.1. $f, g: M \to GL(N, \mathbb{C})$. then $\forall n$ odd. $\exists \omega_n \in \Omega^{n-1}(M)$

s.t. $\operatorname{tr}[((fg)^{-1} d(fg))^n] = \operatorname{tr}[(f^{-1}df)^n] + \operatorname{tr}[(g^{-1}dg)^n] + d\omega_n$

pf. $\mathbb{C}^{2N}|_M = \mathbb{C}^N|_M \oplus \mathbb{C}^N|_M$

$u \in [0, \frac{\pi}{2}]$ $\qquad h(u): M \to GL(2N, \mathbb{C})$

$$h(u) = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$$

$h(u)$ gives a deformation from $(fg, 1)$ to (f, g)

then use Lemma 1.7 #.

Cor 1.2 $g \in \Gamma(\operatorname{Aut}(\mathbb{C}^N|_M))$ d' is another trivial connection

then $\forall n$ odd. $\exists \omega_n \in \Omega^{n-1}(M)$ s.t.

$$\operatorname{tr}[(g^{-1}dg)^n] = \operatorname{tr}[(g^{-1}d'g)^n] + d\omega_n$$

Pf. $\quad d' = A^{-1} d A$

$g^{-1}d'g = g^{-1} d' g - d'$

$= g^{-1} A^{-1} d A g - A^{-1} d A$

$$= A^{-1}(A g^{-1} A^{-1} d \cdot A g A^{-1} - d) A$$
$$= A^{-1}\left((AgA^{-1})^{-1} d (AgA^{-1}) \right) A$$

then $\exists \omega_n \in \Omega^{n-1}(M)$ s.t.

$$tr[(g^{-1}d g)^n] = tr[(A^{-1}((AgA^{-1})^{-1} d(AgA^{-1}))A)^n]$$
$$\overset{Cor 1.1}{=} tr[(AdA^{-1})^n] + tr[(g^{-1}dg)^n] + tr[(A^{-1}dA)^n] - d\omega_n$$
$$= tr[(g^{-1}dg)^n] - d\omega_n$$

#

For n odd. closed form $\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{n+1}{2}} tr[(g^{-1}dg)^n]$

is the n-th chern form (w.r.t. g. d)

denoted $c_n(g, d)$

chern class $c_n([g])$.

odd chern characteristic form:

$$ch(g, d) = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} c_{2n+1}(g, d)$$

Chapter 2 Bott's Formula & Duistermaat-Heckman's Formula

2.1 Berline-Vergne Localize Formula

M: even-dim closed, equipped with a S^1-action
a S^1-invariant metric g^{TM}

S^1 acts on $C^\infty(M)$ naturally: $(g\cdot f)(x) = f(xg)$ $\left(\begin{array}{l} x\in M \\ g\in S^1 \\ f\in C^\infty(M)\end{array}\right)$

$t\in \mathrm{Lie}(S^1)$ define $K\in \Gamma(TM)$ as

$$(Kf)(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x\exp(\varepsilon t)) \quad \left(\begin{array}{l} x\in M \\ f\in C^\infty(M)\end{array}\right)$$

K is a Killing field $\Rightarrow$ $\begin{cases} \langle \nabla_X^{TM} K, Y\rangle + \langle X, \nabla_Y^{TM} K\rangle = 0 \\ K\langle X, Y\rangle = \langle L_K X, Y\rangle + \langle X, L_K Y\rangle \\ L_K = di_K + i_K d \ \text{(Cartan)} \end{cases}$

$$\Omega_K^*(M) = \{\omega\in\Omega^*(M) : L_K\omega = 0\}$$

$$d_K = d + i_K : \Omega^*(M)\to\Omega^*(M)$$

$$d_K^2 = di_K + i_K d = L_K \quad \Rightarrow d_K^2|_{\Omega_K^*(M)} = 0$$

$\Rightarrow$ S^1-equivariant cohomology: $H_K^*(M) = \dfrac{\ker(d_K|_{\Omega_K^*(M)})}{\mathrm{Im}(d_K|_{\Omega_K^*(M)})}$

Prop 2.1 If K has no zero on M, then for any d_K-closed form ω $\int_M \omega = 0$

pf. $\theta \in \Omega^1(M)$ defined as $i_X\theta = \langle X, K\rangle$ $\forall X \in \Gamma(TM)$

$L_K\theta = 0 \Rightarrow (d + i_K)\theta$ is d_K-closed

[Lemma 2.1 $\forall T \geq 0$ $\omega \in \Omega^*_K(M)$ is d_K-closed.

$$\int_M \omega = \int_M \omega \exp(-T d_K\theta)$$

since $\exp(-Td_K\theta) - 1 \xlongequal{d_K(d_K\theta)=0} d_K\left(\sum_{i=1}^{\infty} \frac{(-1)^i}{i!} T^i \theta \wedge (d_K\theta)^{i-1}\right)$

$$\Rightarrow \int_M \omega \exp(-Td_K\theta) - \int_M \omega$$
$$= (-1)^{\deg\omega} \int_M d_K\left(\omega \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} T^i \theta \wedge (d_K\theta)^{i-1}\right) = 0 \quad]$$

$$d_K\theta = d\theta + i_K\theta = d\theta + |K|^2$$

$$\Rightarrow \int_M \omega \exp(-Td_K\theta) = \int_M \omega \exp(-T|K|^2) \sum_{i=1}^{\dim M} \frac{(-1)^i}{i!} T^i (d\theta)^i$$

since $|K| \geq \delta$ let $T \to \infty$. use Lemma 2.1. #

Now $\forall p \in \mathrm{zero}(K)$. take normal coordinates $(x^1, \cdots x^{2l})$ at p $(l = \frac{1}{2}\dim M)$

$$g^{TM} = (dx^1)^2 + \cdots (dx^{2l})^2$$

$\lambda(p) \triangleq \lambda_1 \cdots \lambda_l$

$$K = \lambda_1\left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}\right) + \cdots + \lambda_l\left(x^{2l} \frac{\partial}{\partial x^{2l-1}} - x^{2l-1} \frac{\partial}{\partial x^{2l}}\right)$$

Thm 2.1 (Berline-Vergne).

$\forall d_K$-closed form $\omega \in \Omega^*(M)$.

$$\int_M \omega = (2\pi)^{2l} \sum_{p \in zero(K)} \frac{\omega^{[0]}(p)}{\lambda(p)}$$

pf. $\int_M \omega = \sum_{p \in zero(K)} \int_{U_p} \omega \exp(-T d_K \theta)$

in U_p. $\theta = \lambda_1(x^2 dx^1 - x^1 dx^2) + \cdots + \lambda_l(x^{2l} dx^{2l-1} - x^{2l-1} dx^{2l})$

$d\theta = -2(\lambda_1 dx^1 \wedge dx^2 + \cdots \lambda_l dx^{2l-1} \wedge dx^{2l})$

$|K|^2 = \lambda_1^2((x^1)^2 + (x^2)^2) + \cdots + \lambda_l^2((x^{2l-1})^2 + (x^{2l})^2)$

$$\int_{U_p} \omega \exp(-T d_K \theta) = \sum_{i=0}^{l} \frac{(-1)^i}{i!} \int_{U_p} \omega^{[2l-2i]} \exp(-T|K|^2) T^i (d\theta)^i$$

$x = (x^1, \cdots x^{2l}) \to \sqrt{T} x = (\sqrt{T} x^1, \cdots \sqrt{T} x^{2l})$

If $0 \le i \le l-1$. $T \to \infty \Rightarrow \int_{U_p} \omega^{[2l-2i]} \exp(-T|K|^2) T^i (d\theta)^i$

$$= \int_{\sqrt{T} U_p} \left(\frac{1}{T}\right)^{l-i} \omega^{[2l-2i]}\left(\frac{x}{\sqrt{T}}\right) \exp(-|K|^2) (d\theta)^i$$

$$\to 0$$

$i = l$: $T \to \infty \Rightarrow \frac{(-1)^i}{i!} \int_{U_p} \omega^{[2l-2i]} \exp(-T|K|^2) T^i (d\theta)^i$

$$= \int_{\sqrt{T} U_p} \omega^{[0]}\left(\frac{x}{\sqrt{T}}\right) \exp\left(-\left(\lambda_1^2((x^1)^2 + (x^2)^2) + \cdots + \lambda_l^2((x^{2l-1})^2 + (x^{2l})^2)\right)\right) \cdot 2^l \lambda_1 \cdots \lambda_l dx^1 \cdots dx^{2l}$$

$$\to (2\pi)^l \frac{\omega^{[0]}}{\lambda_1 \cdots \lambda_l}.$$

#

2.2 Bott's Formula

R^{TM} is the curvature of Levi-Civita connection ∇^{TM}

$i_1, \cdots i_k$ are even.

$\forall p \in \mathrm{zero}(K)$ $\quad \lambda^{i_j}(p) = \lambda_1^{i_j} + \cdots + \lambda_\ell^{i_j}$

Thm 2.2 If $i_1 + \cdots + i_k = \ell$.

$$\int_M \mathrm{tr}[(R^{TM})^{i_1}] \cdots \mathrm{tr}[(R^{TM})^{i_k}] = (2\pi\sqrt{-1})^{\ell} \sum_{p \in \mathrm{zero}(K)} \frac{2^k \lambda^{i_1}(p) \cdots \lambda^{i_k}(p)}{\lambda(p)}$$

if $i_1 + \cdots + i_k < \ell$ then $\sum_{p \in \mathrm{zero}(K)} \cdots = 0$.

2.3 Duistermaat-Heckman Formula

M is equipped with a sympletic form $\omega \in \Omega^2(M)$

and the S^1-action preserves ω.

S^1-action is a Hamilton-action: $\exists \mu \in C^\infty(M)$ s.t. $d\mu = i_K \omega$

Thm 2.3. $$\int_M \exp(\sqrt{-1}\mu) \frac{\omega^\ell}{(2\pi)^\ell \ell!} = (\sqrt{-1})^\ell \sum_{p \in \mathrm{zero}(K)} \frac{\exp(\sqrt{-1}\mu(p))}{\lambda(p)}$$

pf. $d\mu = i_K \omega \Rightarrow (d + i_K)(\omega - \mu) = 0$

thus $\exp(\sqrt{-1}\mu - \sqrt{-1}\omega)$ is d_K-closed.

$$\overset{2.1}{\Rightarrow} \int_M \exp(\sqrt{-1}\mu - \sqrt{-1}\omega) = (2\pi)^\ell \sum_{p \in \mathrm{zero}(K)} \frac{\exp(\sqrt{-1}\mu(p))}{\lambda(p)}$$

#

Chap3 Gauss–Bonet–Chern Thm

3.1 Berezin Integration

Firstly $E=\mathbb{R}^m$, regard it as a bundle on a single point

$x=(x^1,\dots x^m)$ $\quad u(x)\triangleq e^{-\frac{|x|^2}{2}}dx^1\wedge\cdots\wedge dx^m$

$\Rightarrow \left(\frac{1}{2\pi}\right)^{\frac{m}{2}}\int_E u=1$

Now define **Berezin integration** on E as

$$\int^B : \Lambda^*(E)\to\mathbb{R}$$

$$\omega\mapsto\langle\omega, dx^1\wedge\cdots\wedge dx^m\rangle$$

Now lift $\Lambda^*(E)$ to a bundle on E.

Then extend $\int^B$ to $\Lambda^*(E,\Lambda^*(E))$

$$\int^B : \alpha\wedge\beta\in\Lambda^*(E,\Lambda^*(E))\mapsto\alpha\int^B\beta\in\Lambda^*(E)$$

$$(\alpha\in\Lambda^*(E)\quad \beta\in\Gamma(\Lambda^*(E)))$$

Prop3.1 $\quad U(x)=(-1)^{\frac{m(m+1)}{2}}\int^B\exp\left(-\frac{|x|^2}{2}-dx\right)\quad x\in\Lambda^*(E)$

Pf. $(-1)^{\frac{m(m+1)}{2}}\int^B e^{-dx}=(-1)^{\frac{m(m+1)}{2}}\int^B\prod_{k=1}^m(1-dx^k\wedge e_k)$

$$=dx^1\wedge\cdots\wedge dx^m$$

#

Now E is the m-rank Euclidean bundle on manifold M

extend $\int^B$ to $\int^B: \Omega^*(M, \Lambda^*(E)) \to \Omega^*(M)$

∇^E is an Euclidean connection on E (∇^E preserves g^E)

One can extend ∇^E to ∇ on $\Omega^*(M, \Lambda^*(E))$

Prop 3.2 $\forall \alpha \in \Omega^*(M, \Lambda^*(E))$. $d\int^B \alpha = \int^B \nabla\alpha$.

pf. $e_1, \cdots e_m$ is an orthonormal basis of E

WLOG. $\alpha = \omega \wedge e_1 \wedge \cdots \wedge e_m \quad \omega \in \Omega^*(M)$

$$\nabla\alpha = (d\omega) \wedge e_1 \wedge \cdots \wedge e_m + (-1)^{\deg\omega} \omega \wedge \nabla(e_1 \wedge \cdots \wedge e_m)$$

$$= (d\omega) \wedge e_1 \wedge \cdots \wedge e_m$$

#

3.2 Thom form of Mathai–Quillen

M closed $\quad p: E \to M$ m-rank orientable bundle

∇^E can be lifted to p^*E and induce a derivation
∇ on $\Omega^*(E, \Lambda^*(p^*E))$.

By 3.2 $\quad d\int^B \alpha = \int^B (\nabla + i_s)\alpha \qquad \alpha \in \Omega^*(E, \Lambda^*(p^*E))$
$s \in \Gamma(E, p^*E)$

$$A \in \underbrace{so(E)}_{\text{skew self-adjoint}} \mapsto \sum_{i<j} \langle Ae_i, e_j \rangle e_i \wedge e_j \in \Lambda^2(E)$$

Lemma 1.1 $\quad \mathcal{A} = \frac{|x|^2}{2} + \nabla x - p^*R^E \in \Omega^*(E, \Lambda^*(p^*E))$
$$(\nabla + i_x)\mathcal{A} = 0$$

Then define $\qquad U = (-1)^{\frac{m(m+1)}{2}} \int^B e^{-\mathcal{A}}$

Prop 3.3 $\quad U$ is a closed m-form on E. and
$$\left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_{E/M} U = 1$$

Thus U is the Thom form

3.3 Transgression Formula

$$A = \frac{|x|^2}{2} + \nabla x - p^* R^E \Rightarrow A_t = \frac{t^2|x|^2}{2} + t\nabla x - p^* R^E$$

$$U = (-1)^{\frac{m(m+1)}{2}} \int^B e^{-A} \Rightarrow U_t = (-1)^{\frac{m(m+1)}{2}} \int^B e^{-A_t}$$

Prop 3.4 (Transgression Formula)

$$\frac{dU_t}{dt} = -(-1)^{\frac{m(m+1)}{2}} d\int^B (x e^{-A_t})$$

pf

$$\frac{dA_t}{dt} = t|x|^2 + \nabla x = (\nabla + t i_x) x$$

$$(\nabla + t i_x) A_t = 0$$

$$\Rightarrow \frac{d}{dt} e^{-A_t} = -\frac{dA_t}{dt} e^{-A_t}$$

$$= -(\nabla + t i_x)(x e^{-A_t})$$

$$\Rightarrow \frac{dU_t}{dt} = -(-1)^{\frac{m(m+1)}{2}} \int^B (\nabla + t i_x)(x e^{-A_t})$$

$$= -(-1)^{\frac{m(m+1)}{2}} d\int^B (x e^{-A_t})$$

#

3.4 Euler Form Euler Class

Real bundle E with rank $m=2n$.

$v\in\Gamma(E)$ then v^*U is a $2n$-dim closed form on M

$$\text{with } v^*U=(-1)^n\int^B e^{-\left(\frac{|v|^2}{2}+\nabla^E v-R^E\right)} \qquad (3.3)$$

If let $v=0$ $\Rightarrow$ we get Euler form (w.r.t. E, g^E, ∇^E)

$$e(E,\nabla^E)=\left(\frac{-1}{2\pi}\right)^n Pf(R^E)$$
$$=\left(\frac{-1}{2\pi}\right)^n\int^B \exp(R^E)$$

$e_1\cdots e_{2n}$ is an orthonormal basis of E

$$\Omega_{ij}\triangleq g^E(R^E e_i, e_j)=\langle R^E e_i, e_j\rangle\in\Omega^2(M)$$
$$\Rightarrow R^E=\frac{1}{2}\sum_{i,j=1}^{2n}\Omega_{ij}e_i\wedge e_j\in\Omega^2(M,\Lambda^2(E))$$

$$Pf(R^E)=\int^B\exp\left(\frac{1}{2}\sum_{i,j=1}^{2n}\Omega_{ij}e_i\wedge e_j\right)$$
$$=\frac{1}{2^n n!}\int^B\left(\sum_{i,j=1}^{2n}\Omega_{ij}e_i\wedge e_j\right)^n$$
$$=\frac{1}{2^n n!}\sum_{i_1\cdots i_{2n}}\varepsilon_{i_1\cdots i_{2n}}\Omega_{i_1i_2}\cdots\Omega_{i_{2n-1}i_{2n}}$$

Prop 3.5 $(\tilde{g}^E, \tilde{\nabla}^E, \tilde{R}^E)$. then $\exists\,\omega\in\Omega^{2n-1}(M)$

$$\text{s.t. } Pf(R^E)-Pf(\tilde{R}^E)=d\omega$$

(Pf Later)

Thus one can attain a class. Euler class $e(E)$

3.5 Proof of Gauss-Bonnet-Chern Thm

(M, g^{TM}): $2n$-dim oriented closed Riemannian manifold

∇^{TM}: Levi-Civita connection $\quad R^{TM}$

Thm 3.1 (Gauss-Bonnet-Chern)

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M Pf(R^{TM})$$

Pf $V \in \Gamma^\infty(TM)$ and is non-trivial.

$\forall p \in Zero(V)$, take U_p and coordinates

$y = (y^1, \cdots y^{2n})$ $\quad y(p) = (0, \cdots 0)$

$V(y) = yA_p\partial_y + O(|y|^2)$

$\partial_y = \left(\frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^{2n}}\right)^T$ $\quad A_p$: $2n\times 2n$ nonsingular matrix (independent of y)

$$\left(-\frac{1}{2\pi}\right)^n \int_M Pf(R^{TM})$$

$$= \left(-\frac{1}{2\pi}\right)^n \int_M \int^B e^{\left(-\left(\frac{t^2|V|^2}{2} + t\nabla^{TM}V - R^{TM}\right)\right)}$$

Adjust g^{TM} to $g^{TM} = (dy^1)^2 + \cdots (dy^{2n})^2$

$$\Rightarrow RHS = \left(-\frac{1}{2\pi}\right)^n \sum_{p\in Zero(V)} \int_{U_p} \int^B e^{-\left(\frac{t^2|V|^2}{2} + tdV\right)}$$

$$+ \left(-\frac{1}{2\pi}\right)^n \int_{M\setminus \bigcup_p U_p} \int^B \underline{e^{-\left(\frac{t^2|V|^2}{2} + t\nabla^{TM}V - R^{TM}\right)}}_{\;|V|\geq\delta \quad t\to\infty}$$

$$\overset{t\to\infty}{=} \left(-\frac{1}{2\pi}\right)^n \int_{U_p} \int^B e^{-\left(\frac{t^2|yA_p|^2}{2} + td(yA_p)\partial_y\right)}$$

$$= t^{2n}\det(A_p) \left(-\frac{1}{2\pi}\right)^n \int_{U_p} e^{-\frac{t^2|yA_p|^2}{2}} dy^1\wedge\cdots\wedge dy^{2n}$$

$\to \operatorname{sign}(\det(A_p))$ $\quad$ Then use Hopf-index Thm.

3.6 Review Gauss-Bonnet-Chern Thm

V is a m-dim real Euclid space with metric g^V.

$\forall v \in V \quad v^*(X) = g^V(v, X) \quad v^* \in V^*$

Define Clifford-action on $\Lambda^*(V^*)$

$$c(v) = v^* \wedge - i_v \qquad \hat{c}(v) = v^* \wedge + i_v$$

$\forall v, v' \in V \quad [i_{v'}, v^*\wedge] = i_{v'} v^*\wedge + v^*\wedge i_{v'} = g^V(v, v')$

Lemma 3.2
$$\begin{cases} c(v)c(v') + c(v')c(v) = -2g^V(v, v') \\ \hat{c}(v)\hat{c}(v') + \hat{c}(v')\hat{c}(v) = 2g^V(v, v') \\ c(v)\hat{c}(v') + \hat{c}(v')c(v) = 0 \end{cases}$$

Lemma 3.3 $\{e_1, \cdots, e_m\}$ is an orthonormal basis of V

then $c(e_i)$ $\hat{c}(e_i)$ $(i = 1, \cdots m)$ generates $\mathrm{End}(\Lambda^*(V^*))$

Then the super structure of $\mathbb{Z}_2$-grading of $\Lambda^*(V^*)$

is $\hat{\tau} = \hat{c}(e_1)c(e_1) \cdots \hat{c}(e_m)c(e_m)$

(independent of $\{e_1, \cdots e_m\}$)

Now $\dim V = 2n$ $\quad \tau = (\sqrt{-1})^n c(e_1) \cdots c(e_{2n})$

$\tau^2 = 1$

τ is the super structure of $\Lambda^*_{\mathbb{C}}(V^*)$

$\Lambda^*_{\mathbb{C}}(V^*) = \Lambda_+(V^*) \oplus \Lambda_-(V^*)$: signature grading and one has $\begin{cases} c(e)\widehat{\tau} = -\widehat{\tau}\, c(e) & \hat{c}(e)\widehat{\tau} = -\widehat{\tau}\,\hat{c}(e) \\ c(e)\tau = -\tau c(e) & \hat{c}(e)\tau = \tau\hat{c}(e) \end{cases}$

Lemma 3.4 In $\Lambda^*(V^*)$. $\quad tr[c(e_I)\hat{c}(e_J)] = \begin{cases} 2^{\dim V} & I=J=\phi \\ 0 & \text{other} \end{cases}$

Cor 3.1 (i) $\dim V = m$. In $\Lambda^*(V^*)$

$$Str[c(e_I)\hat{c}(e_J)] = \begin{cases} (-1)^{\frac{m(m+1)}{2}} 2^m & I=J=N_m \\ 0 & \text{other} \end{cases}$$

(ii) $\dim V = 2n$ In $\Lambda^*_{\mathbb{C}}(V^*)$

$$str[c(e_I)\hat{c}(e_J)] = \begin{cases} (-\sqrt{-1})^n 2^{2n} & I=N_{2n} \ J=\phi \\ 0. & \text{other} \end{cases}$$

Now $E\to M$: rank $2n$ real bundle

similarly one can give the grading on $\Lambda^*_{\mathbb{C}}(E^*)$

$\nabla^E \xrightarrow{\text{lift}} \nabla^{\Lambda^*_{\mathbb{C}}(E^*)}$ $\quad \overline{\omega} = (\omega_{ij})$. connection matrix

$\nabla^E e_i = \omega_{ij} \otimes e_j \qquad \omega_{ij} = -\omega_{ji}$

then $\nabla^{\Lambda^*_{\mathbb{C}}(E^*)} = d - \frac{1}{4}\sum_{i,j=1}^{2n} \omega_{ij}\,(\hat{c}(e_i)+c(e_i))(\hat{c}(e_j)-c(e_j))$

$$= d + \frac{1}{4}\sum_{i,j=1}^{2n} \omega_{ij}\,(c(e_i)c(e_j) - \hat{c}(e_i)\hat{c}(e_j))$$

$\forall \omega \in \Gamma(\Lambda^*_{\mathbb{C}}(E^*))$ $\quad X \in \Gamma(TM)$ $\quad e \in \Gamma(E)$

$$\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}(e^* \wedge \omega) = (\nabla_X^{E^*} e^*) \wedge \omega + e^* \wedge \nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)} \omega$$

$$\Rightarrow [\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, e^* \wedge] \omega = (\nabla_X^{E^*} e^*) \wedge \omega = (\nabla_X^E e)^* \wedge \omega \qquad ①$$

$\forall e' \in \Gamma(E)$ $\quad (\nabla_X^{E^*} e^*)(e') = X(e^*(e')) - e^*(\nabla_X^E e')$

$$= X g^E(e, e') - g^E(e, \nabla_X^E e')$$

$$= g^E(\nabla_X^E e, e') = (\nabla_X^E e)^*(e')$$

Also one can check

$$[\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, i_e](\eta \wedge \omega) = ([\nabla_X^{E^*}, i_e]\eta) \wedge \omega + (-1)^{k-1} \eta \wedge ([\nabla_X^{E^*}, i_e]\omega)$$

$\forall \eta \in \Gamma(E^*)$ $\quad [\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, i_e]\eta = \nabla_X^{E^*}(i_e \eta) - i_e(\nabla_X^{E^*} \eta)$

$$= X(\eta(e)) - i_e(\nabla_X^{E^*} \eta)$$

$$= (\nabla_X^{E^*} \eta)(e) + \eta(\nabla_X^E e) - (\nabla_X^{E^*} \eta)(e)$$

$$= \eta(\nabla_X^E e) = i_{\nabla_X^E e} \eta$$

$$\Rightarrow [\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, i_e] = i_{\nabla_X^E e} \qquad ②$$

$$① + ② \Rightarrow \quad [\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, c(e)] = c(\nabla_X^E e)$$

$$[\nabla_X^{\Lambda^*_{\mathbb{C}}(E^*)}, \hat{c}(e)] = \hat{c}(\nabla_X^E e)$$

$$\Rightarrow [\nabla^{\Lambda^*_{\mathbb{C}}(E^*)}, \tau] = 0.$$

Thus $\nabla^{\Lambda^*_{\mathbb{C}}(E^*)}$ is a super connection on

$$\Lambda^*_{\mathbb{C}}(E^*) = \Lambda_+(E^*) \oplus \Lambda_-(E^*)$$

$$\mathrm{ch}\left(\Lambda^*_{\mathbb{C}}(E^*), \nabla^{\Lambda^*_{\mathbb{C}}(E^*)}\right) = \mathrm{str}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^{\Lambda^*_{\mathbb{C}}(E^*)}\right)\right]$$

$$R^{\Lambda^*_{\mathbb{C}}(E^*)} = -\frac{1}{4}\sum_{i,j=1}^{2n} \Omega_{ij}\left(\hat{c}(e_i)+c(e_i)\right)\left(\hat{c}(e_j)-c(e_j)\right)$$

$$= \frac{1}{4}\sum_{i,j=1}^{2n} \Omega_{ij}\left(c(e_i)c(e_j) - \hat{c}(e_i)\hat{c}(e_j)\right)$$

$$\Rightarrow \mathrm{ch}\left(\Lambda^*_{\mathbb{C}}(E^*), \nabla^{\Lambda^*_{\mathbb{C}}(E^*)}\right) \overset{3.1(ii)}{=} \frac{1}{2^n n!}\left(\frac{\sqrt{-1}}{2\pi}\right)^n \mathrm{str}\left[\left(\sum_{i,j=1}^{2n}\Omega_{ij}c(e_i)c(e_j)\right)^n\right]$$

$$= \frac{1}{2^n n!}\left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_{i_1\cdots i_{2n}} \varepsilon_{i_1\cdots i_{2n}}\Omega_{i_1 i_2}\cdots\Omega_{i_{2n-1}i_{2n}} \mathrm{Str}[c(e_1)\cdots c(e_{2n})]$$

$$= \frac{1}{(2\pi)^n n!}\sum_{i_1\cdots i_{2n}} \varepsilon_{i_1\cdots i_{2n}}\Omega_{i_1 i_2}\cdots\Omega_{i_{2n-1}i_{2n}} = \frac{1}{\pi^n}\mathrm{Pf}(R^E)$$

$$\Rightarrow \left(-\frac{1}{2\pi}\right)^n \mathrm{Pf}(R^E) = \left(-\frac{1}{2}\right)^n \mathrm{ch}\left(\Lambda^*_{\mathbb{C}}(E^*), \nabla^{\Lambda^*_{\mathbb{C}}}(E^*)\right)$$

Now for $v \in \Gamma(TM)$, $T>0 \Rightarrow$ super connections

$$A_T = \nabla^{\Lambda^*_{\mathbb{C}}(T^*M)} + Tc(v) : \Omega(M, \Lambda^*_{\mathbb{C}}(T^*M)) \to \Omega(M, \Lambda^*_{\mathbb{C}}(T^*M))$$

Then again $\left(-\frac{1}{2\pi}\right)^n \int_M \mathrm{Pf}(R^{TM})$

$$= \left(-\frac{\sqrt{-1}}{4\pi}\right)^n \int_M \mathrm{Str}\left[\exp\left(\left(\nabla^{\Lambda^*_{\mathbb{C}}(T^*M)} + Tc(v)\right)^2\right)\right]$$

$$= \left(-\frac{\sqrt{-1}}{4\pi}\right)^n \int_M e^{-T^2|v|^2}\mathrm{str}\left[\exp\left(\nabla^{\Lambda^*_{\mathbb{C}}(T^*M)} + Tc(\nabla^{TM}v)\right)\right]$$

Similarly one can prove Gauss-Bonnet-Chern.

Generally for ∇^E, there might be no g^E preserving ∇^E
so the two methods of representing Euler form
introduced before fail

First $Y \in \Gamma(\pi^* E)$: $Y(y) = (\pi(y), y) \in \pi^*(E)$ $(y \in E)$

$\forall g^E$ define Clifford action

$$c(Y) = Y^* \wedge - i_Y \quad \pi^* \Lambda^{even/odd}(E^*) \to \pi^* \Lambda^{odd/even}(E^*)$$

$$Y^* = (\pi^* g^E)(Y, \cdot)$$

$$A_T = \pi^* \nabla^{\Lambda^*(E^*)} + T c(Y) \cdot \quad \Omega^*(E, \pi^* \Lambda^*(E^*)) \to \Omega^*(E, \pi^* \Lambda^*(E^*))$$

$$A_T^2 = \pi^* R^{\Lambda^*(E^*)} + T[\pi^* \nabla^{\Lambda^*(E^*)}, c(Y)] - T^2 |Y|^2$$

$$\int_{E/M} \mathrm{str}[\exp(A_T^2)] = \int_{E/M} e^{-T^2|Y|^2} \mathrm{str}[\exp(\pi^* R^{\Lambda^*(E^*)} + T[\pi^* \nabla^{\Lambda^*(E^*)}, c(Y)])]$$

$\forall \tilde{\nabla}^E, \tilde{g}^E$ let $\nabla_u^E = (1-u)\nabla^E + u\tilde{\nabla}^E \Rightarrow \nabla_u^{\Lambda^*(E^*)}$ $c_u(Y)$

$g_u^E = (1-u) g^E + u \tilde{g}^E$ $A_{u,T}$

$$\int_{E/M} \mathrm{str}[\exp(A_{1,T}^2)] - \int_{E/M} \mathrm{str}[\exp(A_{0,T}^2)]$$

$$= \int_0^1 \left\{ \frac{\partial}{\partial u} \int_{E/M} \mathrm{str}[\exp(A_{u,T}^2)] \right\} du$$

$$= d \int_0^1 \left\{ \int_{E/M} \mathrm{str}\left[\frac{\partial A_{u,T}}{\partial u} \exp(A_{u,T}^2)\right] \right\} du$$

Now take ∇^E preserving g^E

$\forall x\in M$ take $\{e_1, \dots e_{2n}\}$ s.t. $(\nabla^E e_i)(x)=0$.

$$\int_{E_x}\{\mathrm{str}[\exp(A_T^2)]\}^{(4n)}$$
$$=\int_{E_x}\{e^{-T^2|Y|^2}\mathrm{str}[\exp(\pi^* R^{\Lambda^*(E^*)}+Tc(\pi^*\nabla^E Y)]\}^{(4n)}$$
$$=\int_{E_x} e^{-T^2\sum_{i=1}^{2n}(y^i)^2}\{\mathrm{str}[\exp(\tfrac{1}{4}\sum_{i,j}^{2n}\Omega_{ij}(c(e_i)c(e_j)-\hat{c}(e_i)\hat{c}(e_j))$$
$$+T\sum_{i=1}^{2n}dy^i c(e_i)]\}^{(4n)}$$
$$=\int_{E_x}\frac{(-1)^n}{2^n}e^{-T^2\sum_{i=1}^{2n}(y^i)^2}\{\mathrm{str}[\frac{1}{2^n n!}(\sum_{i,j=1}^{2n}\Omega_{ij}\hat{c}(e_i)\hat{c}(e_j))^n\prod_{i=1}^{2n}(1+Tdy^i c(e_i))]\}^{(4n)}$$
$$=\int_{E_x}\frac{(-1)^n T^{2n}}{2^n}e^{-T^2\sum_{i=1}^{2n}(y^i)^2}\{\mathrm{str}[\pi^*\mathrm{Pf}(R^E)(x)\,\hat{c}(e_1)\cdots\hat{c}(e_{2n})\prod_{i=1}^{2n}dy^i c(e_i)]\}^{(4n)}$$
$$=(\tfrac{-1}{2})^n\int_{E_x}T^{2n}e^{-T^2\sum_{i=1}^{2n}(y^i)^2}\pi^*\mathrm{Pf}(R^E)(x)\wedge dy^1\cdots\wedge dy^{2n}$$
$$\mathrm{str}[\hat{c}(e_1)c(e_1)\cdots\hat{c}(e_{2n})c(e_{2n})]$$
$$=(-2\pi)^n\mathrm{Pf}(R^E)(x)$$
$$\Rightarrow(-\tfrac{1}{2\pi})^{2n}\int_{E/M}\{\mathrm{str}[\exp(A_T^2)]\}^{(4n)}=(-\tfrac{1}{2\pi})^n\mathrm{Pf}(R^E)$$

Thm 1.2 (i) $\pi: E\to M$ $\dim M=2n$. $\mathrm{rank}E=2n$.

$\forall\nabla^E$ g^E $\forall T>0$ $\quad e(E,\nabla^E)=(\frac{1}{2\pi})^{2n}\int_{E/M}\{\mathrm{str}[\exp(A_T^2)]\}^{(4n)}$

(ii) $E=TM\Rightarrow\chi(M)=(\frac{1}{2\pi})^{2n}\int_{TM}\mathrm{str}[\exp(A_T^2)]$

Then for ∇^E preserving g^E $\quad e(E,\nabla^E)=(-\frac{1}{2\pi})^n\mathrm{Pf}(R^E)$

another $\tilde{\nabla}^E$ preserving g^E: $\quad e(E,\nabla^E)-e(E,\tilde{\nabla}^E)=d\omega$

Chapter 4 Poincaré-Hopf Formula

4.1 Weitzenböck Formula

Orthonormal basis $\{e_1 \cdots e_n\}$ of TM

then $d = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)}$ $\quad \Omega^*(M) \to \Omega^*(M)$

$\Rightarrow \forall \alpha\ \beta \in \Omega^*(M)$ $\quad \int_M (d\alpha \wedge *\beta + \alpha \wedge * \sum_{i=1}^{n} i_{e_i} \nabla_{e_i}^{\Lambda^*(T^*M)} \beta)$

$$= \int_M \sum_{i=1}^{n} (e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)} \alpha \wedge *\beta + e^i \wedge \alpha \wedge * \nabla_{e_i}^{\Lambda^*(T^*M)} \beta)$$

$$= \int_M \sum_{i=1}^{n} e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)} (\alpha \wedge *\beta)$$

$$= \int_M d(\alpha \wedge *\beta) = 0$$

$$\Rightarrow d^* = -\sum_{i=1}^{n} i_{e_i} \nabla_{e_i}^{\Lambda^*(T^*M)}$$

$$d + d^* = \sum_{i=1}^{n} c(e_i) \nabla_{e_i}^{\Lambda^*(T^*M)} \quad \Omega^*(M) \to \Omega^*(M)$$

Define $\Delta_0^{\Lambda^*(T^*M)} = \sum_{i=1}^{n} \left(\nabla_{e_i}^{\Lambda^*(T^*M)} \nabla_{e_i}^{\Lambda^*(T^*M)} - \nabla_{\nabla_{e_i}^{TM} e_i}^{\Lambda^*(T^*M)} \right)$

is free of choice of $\{e_1 \cdots e_n\}$

It is self-adjoint two-order elliptic operator.

called Laplace-Beltrami operator

Thm 4.1 (Weitzenböck Formula)

$$\Box = (d+d^*)^2 = -\Delta_0^{\Lambda^*(T^*M)} + \sum_{i,j,k,l} R_{ijkl}\, e_i^* \wedge e_k^* \wedge i_{e_j} i_{e_l} + \sum_{i,j} Ric_{ij}\, e_i^* \wedge i_{e_j}$$

pf. $\Box = \sum_{i,j} c(e_i) \nabla_{e_i}^{\Lambda^*(T^*M)} c(e_j) \nabla_{e_j}^{\Lambda^*(T^*M)}$

see P30

$$= \sum_{i,j} c(e_i)c(e_j) \nabla_{e_i}^{\Lambda^*(T^*M)} \nabla_{e_j}^{\Lambda^*(T^*M)} + \sum_{i,j} c(e_i) c(\nabla_{e_i}^{TM} e_j) \nabla_{e_j}^{\Lambda^*(T^*M)}$$

$$= -\sum_i \left(\nabla_{e_i}^{\Lambda^*(T^*M)}\right)^2 + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\left[\nabla_{e_i}^{\Lambda^*(T^*M)}, \nabla_{e_j}^{\Lambda^*(T^*M)}\right]$$

$$+ \underbrace{\sum_{i,j,k} c(e_i)c(e_k) \nabla^{\Lambda^*(T^*M)}_{\langle \nabla_{e_i}^{TM} e_j, e_k\rangle e_j}}_{} = \sum_{i,k} c(e_i)c(e_k) \nabla^{\Lambda^*(T^*M)}_{\nabla_{e_i}^{TM} e_k}$$

$$= -\Delta_0^{\Lambda^*(T^*M)} + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j) R^{\Lambda^*(T^*M)}(e_i, e_j)$$

$$\sum_{i,j} c(e_i)c(e_j) R^{\Lambda^*(T^*M)}(e_i, e_j)$$

$$= \sum_{i,j,k,l} R_{ijkl} (e_i^* \wedge - i_{e_i})(e_j^* \wedge - i_{e_j})\, e_k^* \wedge i_{e_l}$$

$$= \sum_{i,j,k,l} R_{ijkl}\, e_i^* \wedge e_j^* \wedge e_k^* \wedge i_{e_l} + \sum_{i,j,k,l} R_{ijkl}\, i_{e_i} i_{e_j} e_k^* \wedge i_{e_l}$$

$$- \sum_{i,j,k,l} R_{ijkl} (e_i^* \wedge i_{e_j} + i_{e_i} e_j^* \wedge) e_k^* \wedge i_{e_l}$$

$$= \sum_l \Big(\sum_{i,j,k} R_{ijkl}\, e_i^* \wedge e_j^* \wedge e_k^* \wedge\Big) i_{e_l} - \sum_k \Big(\sum_{i,j,l} R_{ijkl}\, i_{e_i} i_{e_j} i_{e_l}\Big) e_k^* \wedge$$

$$- 2\sum_{i,j,k,l} R_{ijkl}\, e_i^* \wedge i_{e_j} e_k^* \wedge e_l$$

Bianchi

$$= -2\sum_{i,j,k,l} e_i^* \wedge i_{e_j} e_k^* \wedge e_l$$

$$= 2\sum_{i,j,k,l} R_{ijkl}\, e_i^* e_k^* \wedge i_{e_j} i_{e_l} - \sum_{i,k,l} R_{ikkl}\, e_i^* \wedge e_l$$

$$= 2\sum_{i,j,k,l} R_{ijkl}\, e_i^* e_k^* \wedge i_{e_j} i_{e_l} + \sum_{i,j} Ric_{ij}\, e_i^* \wedge i_{e_j}. \ \#$$

Cor 4.2 On $\Omega^1(M)$ $\square = -\Delta_0^{\Lambda^*(T^*M)} + \sum_{i,j} Ric_{ij}\, e_i^* \wedge i e_j$

Thm 4.3 (Lichnerowicz Formula)

$$\square = -\Delta_0^{\Lambda^*(T^*M)} + \frac{1}{8}\sum_{i,j,k,l} R_{ijkl}\, c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l) + \frac{k_M}{4}$$

pf. $R^{\Lambda^*(T^*M)}(e_i, e_j) = \sum_{k,l} R_{ijkl}\, e_k^* \wedge i e_l$

$$= \frac{1}{4}\sum_{k,l} R_{ijkl} \left(\hat{c}(e_k)\hat{c}(e_l) - c(e_k) c(e_l) \right)$$

$$\Rightarrow \square = -\Delta_0^{\Lambda^*(T^*M)} + \frac{1}{8}\sum_{i,j,k,l} R_{ijkl}\, c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l)$$
$$- \frac{1}{8}\sum_{i,j,k,l} R_{ijkl}\, c(e_i) c(e_j) c(e_k) c(e_l)$$

Using Bianchi's identity one can simplify

$$R_{ijkl}\, c(e_i) c(e_j) c(e_k) c(e_l)$$
$$= -R_{jkil}\, c(e_j) c(e_k) c(e_i) c(e_l) - R_{kijl}\, c(e_k) c(e_i) c(e_j) c(e_l)$$
$$-2R_{kiil}\, c(e_k) c(e_l) + 2R_{kikl}\, c(e_i) c(e_l) + 2R_{ijil}\, c(e_j) c(e_l)$$

$$3R_{ijkl}\, c(e_i) c(e_j) c(e_k) c(e_l)$$
$$= 2R_{ikil}\, c(e_k) c(e_l) + 2R_{ijil}\, c(e_j) c(e_l) + 2R_{kikl}\, c(e_i) c(e_l)$$
$$= 6R_{ikil}\, c(e_k) c(e_l)$$
$$= 3R_{ikil}\left[c(e_k) c(e_l) + c(e_l) c(e_k) \right]$$
$$= -6R_{ijij} = -6k_M$$

#

4.2 Poincaré-Hopf Index Thm

$v \in \Gamma^{\infty}(TM)$ $\quad \forall p \in \mathrm{zero}(v)$ $\quad \exists U_p \ni p$ $\quad v(y) = \sum_{i=1}^{n} v^i(y) \frac{\partial}{\partial y_i}$.

$\det\left(\frac{\partial v^i}{\partial y_j}(p)\right)$ is free of choice of basis $\quad v^i(p) = 0$.

If $\det\left(\frac{\partial v^i}{\partial y_j}(p)\right) \neq 0$. then we say p is non-singular

$$\mathrm{ind}(v; p) \triangleq \mathrm{sign}\left(\det\left(\frac{\partial v^i}{\partial y_j}(p)\right)\right)$$

WLOG. assume in U_p $\quad v(y) = y A_p = y\left(\frac{\partial v^i}{\partial y_j}(p)\right)$

Thm 4.3 (Poincaré-Hopf) $\quad \chi(M) = \sum_{p \in \mathrm{zero}(v)} \mathrm{ind}(v; p)$

To prove this define $D_T = d + d^* + T\hat{c}(v) : \Omega^*(M) \to \Omega^*(M)$

$$D_T : \Omega^{\text{even/odd}}(M) \to \Omega^{\text{odd/even}}(M)$$

$$D_{T,\text{even/odd}} = D_T\big|_{\Omega^{\text{even/odd}}}$$

$$\mathrm{ind}(D_{T,\text{even}}) = \mathrm{ind}(D_{\text{even}}) = \chi(M).$$

Prop 4.1 $\quad D_T^2 = D^2 + T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i}^{TM} v) + T^2|v|^2$

pf. $D_T^2 = D^2 + T\sum_{i=1}^{n}\left(c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\hat{c}(v) + \hat{c}(v)c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\right) + T^2|v|^2$

$$= D^2 + T\sum_{i=1}^{n}\Big(c(e_i)\hat{c}(v)\nabla_{e_i}^{\Lambda^*(T^*M)} + c(e_i)\hat{c}(\nabla_{e_i}^{TM} v) + \hat{c}(v)c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\Big) + T^2|v|^2$$

$$= D^2 + T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i}^{TM} v) + T^2|v|^2$$

4.3 Estimate outside of $\bigcup_{p\in zero(v)} U_p$

$\langle \alpha,\beta\rangle = \int_M \alpha\wedge *\beta$ on $\Omega^*(M)$

$\|\ \|_0$ is the induced Sobolev norm on $\Omega^*(M)$

$H^0(M)$ Sobolev space

Prop 4.2 $\exists C>0\ T_0>0$ s.t. $\forall s\in\Omega^*(M)$ $T\geq T_0$ with $\mathrm{Supp}(s)\subset M\setminus \bigcup_{p\in zero(v)} U_p$

$$\|D_T s\|_0 \geq C\sqrt{T}\|s\|_0$$

pf. $\exists C_1$ s.t. $|v|^2\geq C_1$ on $M\setminus\bigcup_{p\in zero(v)} U_p$

then $\|D_T s\|_0^2 = \langle D_T^2 s, s\rangle$

$$= \langle D^2 s + T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i}^{TM} v)s + T^2|v|^2 s, s\rangle$$

$$\geq (C_1T^2 - C_2T)\|s\|_0^2$$

#

4.4 Harmonic Oscillator

WLOG $g^{TM} = (dy^1)^2 + \cdots (dy^n)^2$ on U_p $V = yA_p$

Regrad U_p as a nbhd in Euclid space E_n

$\{e_i = \frac{\partial}{\partial y_i}\}$ is the basis of E_n.

$$D_T^2 = -\sum_{i=1}^n \left(\frac{\partial}{\partial y_i}\right)^2 + T\sum_{i=1}^n c(e_i)\hat{c}(e_i A) + T^2 \langle yAA^*, y\rangle$$

$$= -\sum_{i=1}^n \left(\frac{\partial}{\partial y_i}\right)^2 - T\,\mathrm{Tr}[\sqrt{AA^*}] + T^2\langle yAA^*, y\rangle$$

$$+ T\left(\mathrm{Tr}[\sqrt{AA^*}] + \sum_{i=1}^n c(e_i)\hat{c}(e_i A)\right)$$

$$\triangleq \underbrace{K_T}_{\text{harmonic oscillator}} + T\left(\mathrm{Tr}[\sqrt{AA^*}] + \sum_{i=1}^n c(e_i)\hat{c}(e_i A)\right)$$

Lemma 4.1 (i) $T>0 \Rightarrow K_T$ is nonnegative.

$$\mathrm{Ker}(K_T) = \left\langle \exp\left(-\frac{T\langle y\sqrt{AA^*}, y\rangle}{2}\right)\right\rangle$$

(ii) $L = \mathrm{Tr}[\sqrt{AA^*}] + \sum_{i=1}^n c(e_i)\hat{c}(e_i A)$ on $\Lambda^*(E_n^*)$

$L \geq 0$ and $\dim(\mathrm{Ker}(L)) = 1$

If $\det A > 0$. $\mathrm{Ker}(L) \subset \Lambda^{\text{even}}(E_n^*)$

$\det A < 0$ $\mathrm{Ker}(L) \subset \Lambda^{\text{odd}}(E_n^*)$

Prop 4.3. $\forall T>0$. $D_T^2 \geq 0$ on $\Gamma(\Lambda^*(E_n^*))$

and $\mathrm{Ker}(D_T) = \left\langle \exp\left(\frac{-T\langle y\sqrt{AA^*}, y\rangle}{2}\right) \rho\right\rangle$

And $\exists C>0$ s.t. $\forall \lambda_{D_T^2} \geq CT$ if $\lambda_{D_T^2} \neq 0$

4.5 Proof of Poincaré-Hopf

WLOG. assume $U_p = B_p(4a)$

$\gamma : \mathbb{R} \to [0,1]$ s.t. $\gamma(z) = \begin{cases} 1 & |z| \le a \\ 0 & |z| \ge 2a \end{cases}$

$\forall p \in \mathrm{zero}(V)$, $T > 0$ let $\alpha_{p,T} = \int_{U_p} \gamma(|y|)^2 \exp(-T\langle y, \sqrt{A_p A_p^*} \cdot y\rangle) \, dV_{U_p}$

$$\rho_{p,T} = \frac{\gamma(|y|)}{\sqrt{\alpha_{p,T}}} \exp\left(\frac{-T\langle y, \sqrt{A_p A_p^*} \cdot y\rangle}{2}\right) \rho_p$$

$$E_T = \bigoplus_p \langle \rho_{p,T} \rangle = E_{T,\mathrm{even}} \oplus E_{T,\mathrm{odd}}$$

$\Updownarrow$ $\det(A_p) > 0$

$H^0(M) = E_T \oplus E_T^\perp$ $\quad P_T, P_T^\perp$: projection

$$\begin{cases} D_{T,1} = P_T D_T P_T & D_{T,2} = P_T D_T P_T^\perp \\ D_{T,3} = P_T^\perp D_T P_T & D_{T,4} = P_T^\perp D_T P_T^\perp \end{cases}$$

Prop 4.4 $\exists T_0 > 0$ s.t.

(i) $\forall T \ge T_0$, $0 \le u \le 1$

$$D_T(u) = D_{T,1} + D_{T,4} + u(D_{T,2} + D_{T,3}) : H^1(M) \to H^0(M)$$

is Fredholm

(ii) $D_{T,4} : E_T^\perp \cap H^1(M) \to E_T^\perp$ is invertible

$\Rightarrow$ By the homotopy invariance of Fredholm operator

$$\chi(M) = \mathrm{ind}(D_T : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M))$$

$$= \mathrm{ind}(D_T(0) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M))$$

$$= \mathrm{ind}(D_{T,1} : E_{T,\mathrm{even}} \to E_{T,\mathrm{odd}})$$

$$= \sum_{p \in \mathrm{zero}(V)} \mathrm{sign}(\det(A_p))$$

Chap 5 Morse's Inequality

5.1 Witten Deformation

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^{\dim M}(M) \to 0$$

$\forall$ Morse function f. $T>0$ define $d_{Tf} = e^{-Tf} d e^{Tf}$

$$d_{Tf}^2 = 0$$

$$H^*_{Tf,dR}(M,\mathbb{R}) = \frac{\mathrm{Ker}(d_{Tf})}{\mathrm{Im}(d_{Tf})} = \bigoplus_{i=0}^{n} H^i_{Tf,dR}(M,\mathbb{R})$$

Prop 5.1 $\dim(H^i_{Tf,dR}(M;\mathbb{R})) = \dim(H^i_{dR}(M,\mathbb{R}))$

Pf $\forall d\alpha = 0$, $\alpha \in \Omega^i(M)$

$$d_{Tf}(e^{-Tf}\alpha) = e^{-Tf} d\alpha = 0$$

$\forall \beta \in \Omega^{i-1}(M)$ $\quad e^{-Tf} d\beta = d_{Tf}(e^{-Tf}\beta)$

$$\alpha \in \Omega^i(M) \mapsto e^{-Tf}\alpha \in \Omega^i(M)$$

induced a homomorphism from $H^i_{dR}(M,\mathbb{R})$ to $H^i_{Tf,dR}(M,\mathbb{R})$

$\alpha \mapsto e^{Tf}\alpha$ $\quad \cdots\cdots$ $H^i_{Tf,dR}(M,\mathbb{R})$ to $H^i_{dR}(M,\mathbb{R})$ #

5.2 Hodge Thm for $(\Omega^*(M), d_{Tf})$

$$\langle d_{Tf}\alpha, \beta\rangle = \langle e^{-Tf} d e^{Tf}\alpha, \beta\rangle = \langle \alpha, e^{Tf} d^* e^{-Tf}\beta\rangle$$

$$\exists\, d^*_{Tf} = e^{Tf} d^* e^{-Tf}$$

$D_{Tf} \triangleq d_{Tf} + d^*_{Tf}$ $\quad \square_{Tf} = D^2_{Tf} = d_{Tf}d^*_{Tf} + d^*_{Tf}d_{Tf}$

Again one can obtain $\dim(\ker(\square_{Tf}|_{\Omega^i(M)})) = \dim(H^i_{dR}(M,\mathbb{R}))$

5.3 $\Box_{Tf}$ near $\mathrm{Crit}(f)$

$x\in \mathrm{Crit}(f)$ $\quad x\in U_x$ $\quad \forall y=(y^1,\cdots,y^n)\in U_x$

$$g^{TM}=(dy^1)^2+\cdots(dy^n)^2$$

$$d_{Tf}=d+T\,df\wedge \qquad d^*_{Tf}=d^*+T\,i_{(df)^*}$$

$$D_{Tf}=D+T\hat{c}(df) \qquad (df\rightsquigarrow (df)^*\in\Gamma^\infty(TM))$$

$$df(x)=-y^1dy^1-\cdots-y^{n_f(x)}dy^{n_f(x)}+y^{n_f(x)+1}dy^{n_f(x)+1}+\cdots+y^ndy^n$$

(Morse Lemma)

$$e_i=\frac{\partial}{\partial y_i}$$

$$\Box_{Tf}\overset{\text{Prop4.1}}{=}-\sum_{i=1}^n\left(\frac{\partial}{\partial y_i}\right)^2-nT+T^2|y|^2$$
$$+T\sum_{i=1}^{n_f(x)}\left(1-c(e_i)\hat{c}(e_i)\right)+T\sum_{i=n_f(x)+1}^{n}\left(1+c(e_i)\hat{c}(e_i)\right)$$
$$=-\sum_{i=1}^n\left(\frac{\partial}{\partial y_i}\right)^2-nT+T^2|y|^2$$
$$+2T\underbrace{\left(\sum_{i=1}^{n_f(x)}i_{e_i}e_i^*\wedge+\sum_{i=n_f(x)+1}^{n}e_i^*\wedge i_{e_i}\right)}_{L}$$

$L\geq 0$ and has a 1-dim kernel $\langle(dy^1\wedge\cdots\wedge dy^{n_f(x)})\rangle$

Prop5.2 $\forall T>0$. $\Box_{Tf}\geq 0$ on $\Gamma(\Lambda^*(E_n^*))$

with 1-dim kernel $\langle \exp(-\frac{T|y|^2}{2})\,dy^1\wedge\cdots\wedge dy^{n_f(x)}\rangle$

And $\exists C>0$ s.t. all the nonzero eigenvalues $>CT$

5.4 Pf of Morse Inequality

Prop 5.3 $\forall c>0\ \exists T_0>0$ s.t. $\forall T\geq T_0$. $\#\{$eigenvalues of $\square_{Tf|\Omega^i(M)})\cap[0,c]\} = m_i \quad (0\leq i\leq n)$

$\forall 0\leq i\leq n$ $F^{[0,c]}_{Tf,i}\subset\Omega^i(M)$ is the eigenspace of eigenvalues in $[0,c]$.

$$\dim F^{[0,c]}_{Tf,i} = m_i$$

$$d_{Tf}\square_{Tf} = \square_{Tf}d_{Tf} = d_{Tf}d^*_{Tf}d_{Tf}$$

$$d^*_{Tf}\square_{Tf} = \square_{Tf}d^*_{Tf} = d^*_{Tf}d_{Tf}d^*_{Tf}$$

$\Rightarrow d_{Tf}$ (or d^*_{Tf}) maps $F^{[0,c]}_{Tf,i}$ to $F^{[0,c]}_{Tf,i+1}$ (or $F^{[0,c]}_{Tf,i-1}$)

$$(F^{[0,c]}_{Tf}, d_{Tf}) \quad 0\to F^{[0,c]}_{Tf,0}\xrightarrow{d_{Tf}} F^{[0,c]}_{Tf,1}\to\cdots\xrightarrow{d_{Tf}} F^{[0,c]}_{Tf,n}\to 0$$

again
$$\beta^{[0,c]}_{Tf,i} = \dim\left(\frac{\mathrm{Ker}(d_{Tf}|_{F^{[0,c]}_{Tf,i}})}{\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i-1}})}\right)$$
$$= \dim(\ker(\square_{Tf|\Omega^i(M)})) = \beta_i$$

$$\Rightarrow \dim(F^{[0,c]}_{Tf,i}) = \beta_i + \dim(\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i-1}})) + \dim(\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i}}))$$

$$\Rightarrow \sum_{j=0}^{i}(-1)^j m_{i-j}$$
$$= \sum_{j=0}^{i}(-1)^j\left(\beta_{i-j} + \dim(\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i-j-1}})) + \dim(\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i-j}}))\right)$$
$$= \sum_{j=0}^{i}(-1)^j\beta_{i-j} + \dim(\mathrm{Im}(d_{Tf}|_{F^{[0,c]}_{Tf,i}}))$$
$$\geq \sum_{j=0}^{i}(-1)^j\beta_{i-j}.$$

Chap 7 Atiyah's Thm on Kervaire Semi-Characteristic

7.1 Kervaire Semi-characteristic

Kervaire Semi-Characteristic: $k(M) = \sum_{i=0}^{2q} \dim(H_{dR}^{2i}(M;\mathbb{R})) \mod 2$

M: $4q+1$-dim closed

Take an orthonormal basis $e_1, \cdots e_{4q+1}$

$$D_{sig} \triangleq \hat{c}(e_1) \cdots \hat{c}(e_{4q+1})(d+d^*) \quad \Omega^{even}(M) \longrightarrow \Omega^{even}(M)$$

$$\langle D_{sig} s, s' \rangle = -\langle s, D_{sig} s' \rangle.$$

$$\dim(\ker(D_{sig})) = \sum_{i=0}^{2q} \dim(H_{dR}^{2i}(M;\mathbb{R}))$$

$\forall$ skew self-adjoint elliptic operator D define

$$\mathrm{ind}_2(D) = \dim(\ker(D)) \mod 2$$

Fact: ind_2 is homotopy-invariant

i.e. $\{D(u)\}$ is a family of such operators. One has

$$\mathrm{ind}_2(D(1)) = \mathrm{ind}_2(D(0))$$

$$k(M) = \mathrm{ind}_2(D_{sig})$$

72 Original Proof By Atiyah

If $V_1, V_2 \in \Gamma^{p,q}(TM)$ are independent. WLOG let V_1, V_2 be orthonormal.

$$D' \triangleq \frac{1}{2}\left(D_{sig} + \hat{c}(V_1)\hat{c}(V_2) D_{sig} \hat{c}(V_2)\hat{c}(V_1)\right)$$

$$= D_{sig} + \frac{1}{2}\hat{c}(e_1)\cdots\hat{c}(e_{4q+1}) \sum_{i=1}^{4q+1} c(e_i)\hat{c}(V_1)\hat{c}(\nabla^{TM}_{e_i} V_1)$$

$$+ \frac{1}{2}\hat{c}(e_1)\cdots\hat{c}(e_{4q+1}) \sum_{i=1}^{4q+1} c(e_i)\hat{c}(V_1)\hat{c}(V_2)\hat{c}(\nabla^{TM}_{e_i} V_2)\hat{c}(V_1)$$

is also skew-adjoint elliptic.

$\Rightarrow D(u) = (1-u) D_{sig} + uD' \Rightarrow \mathrm{ind}_2(D_{sig}) = \mathrm{ind}_2(D')$

One can check that $\begin{cases} \hat{c}(V_1)\hat{c}(V_2) D' = D'\hat{c}(V_1)\hat{c}(V_2) \\ (\hat{c}(V_1)\hat{c}(V_2))^2 = -1 \end{cases}$

thus $\hat{c}(V_1)\hat{c}(V_2)$ gives a complex structure on $\ker(D')$

$\Rightarrow 2 \mid \dim(\ker(D')) \Rightarrow k(M) = 0$

7.3 Proof from Witten Deformation

$V = V_1 \quad X = V_2$

Define $D_V = \frac{1}{2}\left(\hat{c}(V)(d+d^*) - (d+d^*)\hat{c}(V)\right) : \Omega^{even}(M) \to \Omega^{even}(M)$

$$= \hat{c}(V)(d+d^*) - \frac{1}{2}\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V)$$

Thm 7.1 $\mathrm{ind}_2(D_V) = k(M)$

pf $D'' = D_{sig} - \frac{1}{2}\hat{c}(e_1)\cdots\hat{c}(e_{4q+1})\hat{c}(V)\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V) : \Omega^{even}(M) \to \Omega^{even}(M)$

Since V is unit $\Rightarrow \langle V, \nabla^{TM}_{e_i} V\rangle = 0$

$\Rightarrow \hat{c}(V)\hat{c}(\nabla^{TM}_{e_i} V) + \hat{c}(\nabla^{TM}_{e_i} V)\hat{c}(V) = 0$

$\Rightarrow D''$ is skew-adjoint $\Rightarrow \mathrm{ind}_2(D'') = \mathrm{ind}_2(D_{sig})$

$\mathrm{Ker}(D'') = \mathrm{Ker}\left(\hat{c}(e_1)\cdots\hat{c}(e_{4q+1})\left(d+d^* - \frac{1}{2}\hat{c}(V)\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V)\right)\right)$

$= \mathrm{Ker}\left(\hat{c}(V)\left(d+d^* - \frac{1}{2}\hat{c}(V)\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V)\right)\right) = \mathrm{Ker}(D_V)$ #

$\forall T \in \mathbb{R} \quad D_{V,T} = D_V + T\hat{c}(V)\hat{c}(X) : \Omega^{even}(M) \to \Omega^{even}(M)$

is skew-adjoint

$\Rightarrow k(M) = \lim_{T\to\infty} \mathrm{ind}_2(D_{V,T})$

Prop 7.1 $-D_{V,T}^2 = -D_V^2 + T\sum_{i=1}^{4q+1}\left(c(e_i)\hat{c}(\nabla^{TM}_{e_i} X) - \langle \nabla^{TM}_{e_i} X, V\rangle c(e_i)\hat{c}(V)\right) + T^2|X|^2$

$\Rightarrow \exists T_0 > 0$ s.t. $\forall T \geq T_0 \quad D_V^2 - D_{V,T}^2 > 0$

Since $-D_V^2 \geq 0 \Rightarrow -D_{V,T}^2 \geq 0$

$\Rightarrow \mathrm{Ker}(D_{V,T}) = \{0\} \Rightarrow k(M) = 0$

7.4 A Counting Formula of $k(M)$

By Hopf Index Thm. $\exists$ non-vanishing field V on a $4q+1$-dim orientable manifold M

$[V]$ is the 1-dim vector bundle.

$TM/[V]$ is a $4q$-rank bundle. X is a section.

zero(X) is formed by non-intersecting circles on M

F is such a circle. then $\forall y \in F$ X induces a homeomorphism on $T_yM/[V_y]$

Then this gives a 1-dim subspace of $\Lambda^*((T_yM/[V_y])^*)$

these spaces form a real bundle. denoted as $O_F(X)$

$$\mathrm{ind}_2(X,F) \triangleq \begin{cases} 1. & O_F(X) \text{ is orientable on } F \\ 0. & \text{otherwise} \end{cases}$$

Thm 7.2 $\quad k(M) = \sum_{F \in \mathrm{zero}(X)} \mathrm{ind}_2(X, F)$

7.5 $k(M)$ is NOT multiplicative

Assume $H^1(M,\mathbb{Z}_2)\neq 0$. take $\alpha\in H^1(M;\mathbb{Z}_2)$ $\alpha\neq 0$

$\pi_\alpha: \hat{M}_\alpha \to M$ is the corresponding double cover

Thm 7.3 $k(\hat{M}_\alpha) = \langle \alpha \cdot w_{4q}(TM), [M]\rangle$

Pf. $\hat{V} = \pi_\alpha^* V$. $\hat{X} = \pi_\alpha^* X$ $\Rightarrow \mathrm{zero}(\hat{X}) = \pi_\alpha^{-1}(\mathrm{zero}(X))$

L_α is the real bundle on M s.t. $w_1(L_\alpha) = \alpha$.

$\forall F \in \mathrm{zero}(X)$ is a circle

(i) $L_\alpha|_F$ is orientable. then $\pi_\alpha^{-1}(F) = \hat{F}_1 \cup \hat{F}_2$

$\Rightarrow \mathrm{ind}_2(\hat{X}, \pi_\alpha^{-1}(F)) = \mathrm{ind}_2(\hat{X}, \hat{F}_1) + \mathrm{ind}_2(\hat{X}, \hat{F}_2) = 0$

(ii) $L_\alpha|_F$ is not orientable

$\pi_\alpha: \pi_\alpha^{-1}(F) \to F$ is a double cover between circles

then $\pi_\alpha^*(O_F(X))$ is orientable on $\pi_\alpha^{-1}(F)$

$\mathrm{ind}_2(\hat{X}, \pi_\alpha^{-1}(F)) = 1$

$$\Rightarrow k(\hat{M}_\alpha) = \sum_{F\in \mathrm{zero}(X)} \langle w_1(L_\alpha|_F), [F]\rangle$$

$$= \langle \alpha\, w_{4q}(TM).\ [M]\rangle.$$

#

Chap 9 Pf of Gauss-Bonnet-Chern (Heat Kernel Method)

9.0 About Heat Kernel

$\forall F \in \Gamma(End(E))$ is self-adjoint

$H = -\Delta_0^E + F : \Gamma(E) \to \Gamma(E)$ is a Laplace operator

Def. $\{P_t(x,y) : E_y \to E_x \mid (t,x,y) \in (0,\infty) \times M \times M\}$ s.t.

(i) $\forall y \in M$, $v \in E_y$ $\quad \left(\frac{\partial}{\partial t} + H\right)(P_t(x,y)v) = 0$

(ii) $\forall \phi \in \Gamma(E)$ $\quad \lim\limits_{t \to 0^+} \int_M P_t(x,y)\phi(y)\, dV_M(y) = \phi(x)$

is a heat kernel of H.

Thm ① M is closed oriented Riemannian manifold

then (i) $\{P_t(x,y)\}$ $\exists$, and $P_t(x,y)$ is C^∞ w.r.t t, x, y

(ii) If $\exists$ heat kernel $P_t^*(x,y)$ for H^*,

then the heat kernel of H is unique

② $\forall \lambda \in \mathbb{R}$. if $\phi \in \overline{\Gamma(E)}$ $\quad H\phi = \lambda\phi$ then $\phi \in \Gamma(E)$

$\{e^{-tH} : t > 0\}$ heat operator

$$e^{-tH} : \overline{\Gamma(E)} \to \Gamma(E)$$

$$(e^{-tH}\phi)(x) = \int_M P_t(x,y)\phi(y)\, dV_M(y)$$

Thm ③ e^{-tH} is a compact operator on $\overline{\Gamma(E)}$

and e^{-tH} is self-adjoint with $e^{-t_1 H}e^{-t_2 H} = e^{-(t_1+t_2)H}$

$\exists\ \lambda_1 \le \lambda_2 \le \cdots$, 1-dim orthogonal $V_1, V_2 \cdots \subset \Gamma(E)$, s.t.

$$e^{-tH}|_{V_i} = e^{-t\lambda_i}, \qquad H|_{V_i} = \lambda_i$$

select unit vector $\phi_i \in V_i$

$$\Rightarrow \phi = \sum_i \langle \phi, \phi_i \rangle \phi_i, \quad e^{-tH}\phi = \sum_i \langle \phi, \phi_i \rangle e^{-t\lambda_i}\phi_i$$

$$|P_t(x,y)|^2 = \sum_{\alpha,\beta=1}^{rk(E)} |(P_t(x,y)e_\alpha(y), e_\beta(x))|^2$$

$$= \sum_{\alpha=1}^{rk(E)} |P_t(x,y)e_\alpha(y)|^2$$

Thm ④ $\mathrm{Tr}[e^{-tH}] = \sum_{i=1}^{\infty} e^{-t\lambda_i} < \infty$

$$= \int_M \mathrm{tr}[P_t(x,x)]\, dV_M(x)$$

⑤ $P_t(x,y) = \sum_{i=1}^{\infty} e^{-t\lambda_i} (\cdot, \phi_i(y))\, \phi_i(x)$

9.1 Mckean–Singer Conjecture

(M, g^{TM}): $2n$-dim closed orientable

$$D_{even} = d + d^* : \Omega^{even}(M) \to \Omega^{odd}(M)$$

$$\chi(M) = \dim(\ker \Box_{even}) - \dim(\ker \Box_{odd})$$

$P_t(x,y)$ is the heat kernel of $\Box$

$$\mathrm{Tr}[e^{-t\Box}] = \int_M \mathrm{tr}[P_t(y,y)]\, dv_M(y)$$

$$\mathrm{Tr}[e^{-t\Box_{even/odd}}] = \int_M \mathrm{tr}\left[P_t(y,y)\big|_{\Omega^{even/odd}(T^*M)}\right] dv_M(y)$$

$$\mathrm{Str}[e^{-t\Box}] \triangleq \mathrm{Tr}[e^{-t\Box_{even}}] - \mathrm{Tr}[e^{-t\Box_{odd}}]$$
$$= \int_M \mathrm{str}[P_t(y,y)]\, dv_M(y)$$

Thm 9.1 (Mckean–Singer) $\forall t > 0$. $\mathrm{Str}[e^{-t\Box}] = \mathrm{ind}(D_{even})$

$$\Rightarrow \chi(M) = \lim_{t \to 0^+} \int_M \mathrm{str}[P_t(y,y)]\, dv_M(y)$$

To calculate the limit we use a lemma

about the heat kernel which is not proved here.

Lemma. For N sufficiently large $t \to 0^+$

$$P_t(x,x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{i=0}^{N} t^{i-\frac{n}{2}} u^{(i)}(x,x) + o(t^{N-\frac{n}{2}})$$

where $u^{(i)}(x,y)$ is a linear map from $E_y \to E_x$ s.t.

$$\begin{cases} \left(\nabla^E_{\hat{a}} + \frac{\hat{a}G}{4G}\right) u^{(0)}(x,y)v = 0 \\ \left(\nabla^E_{\hat{a}} + i + \frac{\hat{a}G}{4G}\right) u^{(i)}(x,y)v = -H_x u^{(i-1)}(x,y)v \end{cases} \quad (\forall v \in E_y)$$

thus $P_t(y,y) = \frac{1}{(4\pi)^n} \sum_{i=0}^{n} t^{i-n} u^{(i)}(y,y) + o(1)$

$$\Rightarrow \begin{cases} \int_M \operatorname{Str}[u^{(i)}(y,y)]\, dv_M(y) = 0 \quad (i<n) \\ \chi(M) = \frac{1}{(4\pi)^n} \int_M \operatorname{str}[u^{(n)}(y,y)]\, dv_M(y) \end{cases}$$

McKean-Singer Conjecture:

$$\begin{cases} \frac{1}{(4\pi)^n} \operatorname{Str}[u^{(n)}(y,y)]\, dV_M(y) = -\left(\frac{1}{2\pi}\right)^n Pf(R^{TM}) \\ \int_M \operatorname{Str}[u^{(i)}(y,y)]\, dV_M(y) = 0 \quad (i<n) \end{cases} \qquad (*)$$

(*): local index formula of de Rham-Hodge operator

9.2 Proof of (*)

$\forall y \in M$. $(O_y, x=(x^1,\cdots x^{2n}))$. normal coordinate

and trivialize $\Lambda^*(T^*M)|_{O_y}$ using parallel transport

$$\Rightarrow \Lambda^*(T^*M) \cong O_y \times \Lambda^*(T_y^*M) = C^\infty(O_y) \times \Lambda^*(T_y^*M)$$

For $\omega = (\varphi_1(x)\frac{\partial}{\partial x^{i_1}}(x) \cdots \varphi_m(x)\frac{\partial}{\partial x^{i_m}}(x)\, \varphi_{m+1}(x))\, c(e_{j_1})\cdots c(e_{j_p})\hat{c}(e_{k_1})\cdots\hat{c}(e_{k_q})$

define $\chi(\omega) = p+q+m-\nu(\varphi_1,\cdots \varphi_m \varphi_{m+1})$

$$\chi(\omega_1+\omega_2) \stackrel{\Delta}{=} \max\{\chi(\omega_1),\chi(\omega_2)\}$$

Lemma 9.1 If $\chi(\omega) < 4n \Rightarrow \mathrm{Str}[\omega(0)]=0$

then $\begin{cases} (\nabla_{\rho\frac{\partial}{\partial\rho}} + i + \frac{d\sqrt{G}}{4G})\, u^{(i)} = -\Box u^{(i-1)}, \ i\geq 1 \\ u^{(0)}(0,y) = \mathrm{Id}_{\Lambda^*(T_y^*M)} \end{cases}$

by Lichnerowicz's formula

$$\Box = \frac{1}{8}\sum_{k,l,p,q} R_{klpq}(y)\, c(e_k)c(e_l)\hat{c}(e_p)\hat{c}(e_q) + (\chi<4)$$

$$\Rightarrow i\, u^{(i)}(0,y) = \{-\frac{1}{8}\sum_{k,l,p,q} R_{klpq}(y)\, c(e_k)c(e_l)\hat{c}(e_p)\hat{c}(e_q) + (\chi<4)\}\, u^{(i-1)}(0,y)$$

$$u^{(n)}(0,y) = \frac{(-1)^n}{2^{3n}n!}\sum R_{i_1i_2j_1j_2}(y)\cdots R_{i_{2n-1}i_{2n}j_{2n-1}j_{2n}}(y)\, c(e_{i_1})c(e_{i_2})\cdots c(e_{i_{2n}})\hat{c}(e_{j_1})\cdots\hat{c}(e_{j_{2n}}) + (\chi<4n)$$

$\Rightarrow \chi(u^{(i)}(0,y)) \leq 4i$.

$\Rightarrow \mathrm{Str}[u^{(i)}(y,y)]=0 \quad (i<n)$

$$\Omega_{ij} = -\frac{1}{2}\sum_{i,j,k,l} R_{ijkl}\, e^k\wedge e^l$$

$$\mathrm{Str}[u^{(n)}(0,y)]\,dv_M(y) = \mathrm{str}[u^{(n)}(y,y)]\,dv_M(y)$$

$$= (\frac{1}{2})^n\frac{1}{n!}\sum \varepsilon_{i_1\cdots i_{2n}}\varepsilon_{j_1\cdots j_{2n}} R_{i_1i_2j_1j_2}\cdots R_{i_{2n-1}i_{2n}j_{2n-1}j_{2n}}\,dv_M(y)$$

$$= \frac{(-1)^n}{n!}\sum \varepsilon_{i_1\cdots i_{2n}}\,\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{2n-1}i_{2n}}(y)$$

$$= (-2)^n\, \mathrm{Pf}(R^{TM})(y).$$

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Chap 10 Hirzebruch Signature Thm (Heat Kernel Method)

10.1 Interpretation of Signature

M: $4m$-dim closed oriented Riemannian manifold

$\mathrm{Sign}(M)$ is the signature of B where

$$B: H^{2m}_{dR}(M;\mathbb{R}) \times H^{2m}_{dR}(M;\mathbb{R}) \to \mathbb{R}$$
$$([\omega],[\omega']) \mapsto \int_M \omega\wedge\omega'$$

$$\Longleftrightarrow \quad B_0: \mathcal{H}^{2m}(M;\mathbb{R}) \times \mathcal{H}^{2m}(M;\mathbb{R}) \to \mathbb{R}$$
$$(\omega,\omega') \mapsto \int_M \omega\wedge\omega'$$

$$\mathrm{Sign}(M) = \mathrm{Sign}(B) = \mathrm{Sign}(B_0)$$

Def 10.1 $\tau: \Omega^*(M) \to \Omega^{4m-*}(M)$

$$\omega \mapsto (-1)^{\frac{k(k-1)}{2}+m} * \omega \qquad \omega\in\Omega^k(M) \quad 0\le k\le 4m$$

then $\tau^2 = 1$. $\quad \tau|_{\Omega^{2m}(M)} = *|_{\Omega^{2m}(M)}$

Also $\quad d^*\omega = (-1)^{4mk+4m+1} *d*\omega = -*d*\omega. \quad \forall \omega\in\Omega^k(M)$

$$*d = (-1)^{k+1} d^* *. \qquad *d^* = (-1)^k d*$$

$$\Rightarrow \tau D = -D\tau. \qquad \tau\Box = \Box\tau$$

$$\Rightarrow \mathcal{H}^{2m}_{\pm}(M) = \{\omega\in\mathcal{H}^{2m}(M) \mid *\omega = \pm\omega\}$$

$$\mathcal{H}^{2m}(M) = \mathcal{H}^{2m}_{+}(M) \oplus \mathcal{H}^{2m}_{-}(M)$$

$$\Rightarrow \forall \omega\in\mathcal{H}^{2m}_{\pm}(M) \qquad B_0(\omega,\omega) = \int_M \omega\wedge*(*\omega)$$
$$= \langle\omega, *\omega\rangle = \pm\langle\omega,\omega\rangle$$

Thus $\mathrm{sign}(M) = \dim \mathcal{H}_+^{2m}(M) - \dim \mathcal{H}_-^{2m}(M)$

Do the $\mathbb{Z}_2$-grading using τ:

$$\Omega^*(M) = \Omega_+(M) \oplus \Omega_-(M)$$

$$\Omega_\pm(M) = \{\omega \in \Omega^*(M) \mid \tau\omega = \pm\omega\}$$

$$D_\pm(M): \Omega_\pm(M) \to \Omega_\mp(M)$$

D_+: **Hirzebruch Signature Operator**

Thm 10.1 $\quad \mathrm{sign}(M) = \mathrm{ind}(D_+) \triangleq \dim(\ker D_+) - \dim(\ker D_-)$

pf $\square_\pm = \square|_{\Omega_\pm(M)}$ $\qquad \mathcal{H}(M) = \mathcal{H}(M)_+ \oplus \mathcal{H}(M)_-$

$$\mathcal{H}(M)_\pm = \ker(\square_\pm)$$

$$\mathrm{ind}(D_+) = \dim(\ker D_+) - \dim(\ker D_-)$$

$$= \dim(\ker \square_+) - \dim(\ker \square_-) = \dim(\mathcal{H}(M)_+) - \dim(\mathcal{H}(M)_-)$$

$$(\mathcal{H}^k(M) \oplus \mathcal{H}^{4m-k}(M))_\pm = \{\omega \in \mathcal{H}^k(M) \oplus \mathcal{H}^{4m-k}(M) \mid \tau\omega = \pm\omega\}$$

$$\Rightarrow \mathrm{ind}(D_+) = \dim\Big(\mathcal{H}_+^{2m}(M) \oplus \Big(\bigoplus_{k=0}^{2m-1} \mathcal{H}^k(M) \oplus \mathcal{H}^{4m-k}(M)\Big)_+\Big)$$

$$- \dim\Big(\mathcal{H}_-^{2m}(M) \oplus \Big(\bigoplus_{k=0}^{2m-1} \mathcal{H}^k(M) \oplus \mathcal{H}^{4m-k}(M)\Big)_-\Big)$$

$$= \dim(\mathcal{H}_+^{2m}(M)) - \dim(\mathcal{H}_-^{2m}(M))$$

[$\omega \mapsto \frac{1}{2}(\omega \pm \tau\omega)$ gives an isomorphism between $\mathcal{H}^k(M)$ and $(\mathcal{H}^k(M) \oplus \mathcal{H}^{4m-k})_\pm$]. #

Thm 10.2 (Hirzebruch Signature Thm)

$$\mathrm{sign}(M) = L(M) = \int_M L(TM, \nabla^{TM})$$

10.2 Local Index Formula of D_+

$P_t(x,y)$: heat kernel of $\square$

$$\mathrm{Str}[e^{-t\square}] = \int_M \mathrm{str}[P_t(y,y)]\,dV_M(y)$$

Thm 10.3 $\mathrm{Str}[e^{-t\square}] = \mathrm{Sign}(M) = \mathrm{Ind}(D_+)$

The proof is similar with McKean–Singer theorem.

$$\Rightarrow \mathrm{Sign}(M) = \lim_{t\to 0^+} \int_M \mathrm{Str}[P_t(y,y)]\,dV_M(y)$$

Again $\quad P_t(y,y) = \frac{1}{(4\pi)^{2m}} \sum_{i=0}^{2m} t^{i-2m} u^{(i)}(y,y) + o(1) \quad (t\to 0^+)$

$$\Rightarrow \begin{cases} \int_M \mathrm{str}[u^{(i)}(y,y)]\,dV_M(y) = 0 & i<2m \\ \mathrm{Sign}(M) = \frac{1}{(4\pi)^{2m}} \int_M \mathrm{str}[u^{(2m)}(y,y)]\,dV_M(y) \end{cases}$$

Thm 10.4 (Local Index Formula of D_+)

$$\begin{cases} \mathrm{Str}[u^{(i)}(y,y)] = 0 & i<2m \quad ① \\ \frac{1}{(4\pi)^{2m}} \mathrm{str}[u^{(2m)}(y,y)]\,dV_M(y) = \{L(TM, \nabla^{TM})\}^{\max} & ② \end{cases}$$

10.3 Proof of Thm 10.4

Define $\chi(\omega) = n+p-2\,|\varphi_1 \cdots \varphi_n \varphi_{n+1}|$ $\left[\omega = (\varphi_1(x)\frac{\partial}{\partial x^i} \cdots \varphi_n \frac{\partial}{\partial x^{i_n}} \varphi_{n+1}(x)) \, c(e_{j_1}) \cdots c(e_{j_p}) \, \hat{c}(e_{k_1}) \cdots \hat{c}(e_{k_q})\right]$

$$\begin{cases} \left(\nabla_{\rho\frac{\partial}{\partial\rho}} + i + \frac{\hat{a}G}{4G}\right) u^{(i)} = -\square u^{(i-1)} \\ u^{(0)}(0;y) = Id_{\Lambda^*(T_y^*M)} \end{cases}$$

In normal nbhd $e_i = \frac{\partial}{\partial x^i} + (\chi \le -1)$

$$\nabla_{e_i}^{\Lambda^*(T^*M)} = e_i + \frac{1}{4}\sum_{j,k=1}^{4m} \Gamma_{ij}^k \, c(e_j)\, c(e_k)$$

$$= \frac{\partial}{\partial x_i} - \frac{1}{8}\sum_{j,k,l} x^l R_{lijk}(y)\, c(e_j) c(e_k) + (\chi \le 0)$$

$$\Rightarrow \Delta_0^{\Lambda^*(T^*M)} = \sum_i \left(\nabla_{e_i}^{\Lambda^*(T^*M)}\right)^2 - \sum_i \nabla^{\Lambda^*(T^*M)}_{\nabla_{e_i}^{TM} e_i}$$

$$= \sum_i \left(\frac{\partial}{\partial x_i} - \frac{1}{8}\sum_{j,k,l} x^l R_{lijk}(y)\, c(e_j) c(e_k)\right)^2 + (\chi < 2)$$

$$\begin{cases} \square_0 \triangleq -\sum_i \left(\frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{j,k,l} x^l R_{lijk}(y)\, c(e_j)\, c(e_k)\right)^2 \\ F = \frac{1}{8}\sum_{k,l,p,q} R_{klpq}(y)\, c(e_k) c(e_l) \hat{c}(e_p) \hat{c}(e_q) \end{cases}$$

$$\xrightarrow{\text{Lichnerowicz}} \square = \square_0 + F + (\chi < 2) \qquad \chi(F) = 2, \quad \chi(\square_0) = 2$$

And $\nabla^{\Lambda^*(T^*M)}_{\rho\frac{\partial}{\partial\rho}} = \nabla^{\Lambda^*(T^*M)}_{\hat{a}} = \hat{a}$ on $\mathcal{O}_y$

$h \triangleq \hat{a}(\log G^{\frac{1}{4}})$ $\Rightarrow$
$$\begin{cases} (\hat{a} + i + h)\, u^{(i)} = -(\square_0 + F + (\chi < 2))\, u^{(i-1)} \\ u^{(0)}(0;y) = Id_{\Lambda^*(T_y^*M)} \end{cases}$$

Lemma 10.2 $\forall i \ge 0$ $\quad \chi(u^{(i)}(x,y)) \le 2i$

Thus we proved ①

Consider $\begin{cases} (\hat{\partial}+i)V^{(i)} = -(\Box_0+\hat{F})V^{(i-1)}, & i\geq 1 \\ V^{(0)}(0,y) = \mathrm{Id}_{\Lambda^*(T^*M)|_y} \end{cases}$

similarly $\chi\left(V^{(i)}(x,y)\right)\leq 2i$

Lemma 10.3 $\forall i\geq 0$ $\chi\left(u^{(i)}(x;y)-V^{(i)}(x;y)\right)<2i$.

specifically. $\mathrm{Str}[u^{(2m)}(0;y)] = \mathrm{Str}[V^{(2m)}(0;y)]$

One can calculate

$$(4\pi)^{-2m}\,\mathrm{Str}[V^{(2m)}(0;y)]\,dV_M(y)$$

$$= \frac{(-1)^m}{(4\pi)^{2m}}\left\{\det{}^{\frac{1}{2}}\left(\frac{\frac{\Omega(y)}{2}}{\sinh\left(\frac{\Omega(y)}{2}\right)}\right)\mathrm{tr}\left[\exp(-\hat{F})\right]\right\}^{\max}$$

where $\hat{F} = -\frac{1}{4}\sum_{k,l}\Omega_{kl}(y)\,\hat{c}(e_k)\,\hat{c}(e_l)$

When $\Omega(y) = \begin{pmatrix} 0 & \Omega_{12} & & \\ -\Omega_{12} & 0 & & \\ & & \ddots & \\ & & 0 & \Omega_{4m-1,4m} \\ & & -\Omega_{4m-1,4m} & 0 \end{pmatrix}$

$$\det{}^{\frac{1}{2}}\left(\cosh\frac{\Omega(y)}{2}\right) = \det{}^{\frac{1}{2}}\left(\sum_{n\geq 0}\frac{1}{(2n)!}\left(\frac{\Omega(y)}{2}\right)^{2n}\right)$$

$$= \left[\prod_{i=1}^{2m}\left(\sum_{n\geq 0}\frac{(-1)^n}{(2n)!}\left(\frac{\Omega_{2i-1,2i}}{2}\right)^{2n}\right)^2\right]^{\frac{1}{2}}$$

$$= \prod_{i=1}^{2m}\cos\left(\frac{\Omega_{2i-1,2i}}{2}\right)$$

$$\mathrm{tr}[\exp(-\tfrac{t}{F})]$$

$$= \mathrm{tr}[\exp(\sum_{i=1}^{2m} \frac{\Omega_{2i-1,2i}}{2} \hat{c}(e_{2i-1})\hat{c}(e_{2i})]$$

$$= \mathrm{tr} \prod_{i=1}^{2m} \exp(\frac{\Omega_{2i-1,2i}}{2} \hat{c}(e_{2i-1})\hat{c}(e_{2i})]$$

$$= \mathrm{tr}[\prod_{i=1}^{2m} (\cos\frac{\Omega_{2i-1,2i}}{2} + \hat{c}(e_{2i-1})\hat{c}(e_{2i}) \sin(\frac{\Omega_{2i-1,2i}}{2})]$$

$$= 2^{4m} \prod_{i=1}^{2m} \cos(\frac{\Omega_{2i-1,2i}}{2})$$

$$\Rightarrow \frac{1}{(4\pi)^{2m}} \mathrm{str}[U^{(2m)}(y,y)]\, dv_M(y)$$

$$= (\frac{\sqrt{-1}}{\pi})^{2m} [\det{}^{\frac{1}{2}}(\frac{\frac{\Omega(y)}{2}}{\sinh\frac{\Omega(y)}{2}}) \det{}^{\frac{1}{2}}(\cosh\frac{\Omega(y)}{2})]^{\max}$$

$$= (\frac{\sqrt{-1}}{\pi})^{2m} [\det{}^{\frac{1}{2}} \frac{\frac{\Omega(y)}{2}}{\tanh\frac{\Omega(y)}{2}}]^{\max}$$

$$= [\det{}^{\frac{1}{2}}(\frac{\frac{\sqrt{-1}}{2\pi}\Omega(y)}{\tanh(\frac{\sqrt{-1}}{2\pi}\Omega(y))})]^{\max}$$

$$= \{L(TM, \nabla^{TM})\}^{\max}.$$

thus we proved ②

#.