VARIOUS MEASUREMENTS ON THE SPACE OF CONTACT FORMS

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Abstract. There are various types of numerical measurements on the space of contact forms on a Liouville fillable contact manifold. These measurements include the interleaving distance between a logarithmic version of the contact persistence modules of Cant[Can23], the contact Banach-Mazur distance d_{CBM} between contact forms[RZ21], the fine symplectic Banach-Mazur distance d_{SBM} between the graph domains, and so on. In this paper we give some comparison results between them.

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1. Introduction and main results

The numerical measurements on geometric objects have been intensely studied by symplectic and contact geometers. In terms of symplectic geometry, firstly one can define measurements on the space of Hamiltonian diffeomorphism. Hofer's distance d_H [Hof90] founded the base of Hofer geometry. Many have adapted various tools in Hamiltonian dynamics to study Hofer geometry (for instance [LM95, Ush13, PS16]), and conversely, many phenomena in Hofer geometry have also indicated plenty of applications to Hamiltonian dynamics [Pol01, Sch00, Sch06]. Using spectral invariants extracted from Floer homology, one can also define spectral norm and study some properties of it, for example C^0-continuity[Sey12, BHS21, Kaw22] and large-scale geometry[FZ25]. Besides, one can also define distances on the space of symplectic domains. Fine and coarse symplectic Banach-Mazur distances, which can be seen as non-linear analogues of the standard Banach-Mazur distance in convex geometry[Rud00], on the star-shaped domains in a Liouville manifold, have interesting large-scale geometry properties[Ush22, SZ21, CGH23].

As for contact geometry, Shelukhin[She17] defined a Hofer-like distance d_α between contactomorphisms on a contact manifold (M, α), and Rosen-Zhang[RZ21] constructed a contact Banach-Mazur distance between "contact domains" of some contact manifolds.

In this article, we place our attention on the space of contact forms supporting a certain contact structure. We describe our set-up first.

Let (M, ξ) be a closed contact manifold and ξ is co-orientable. We fix a contact 1-form α_0 and assume $(M, \xi = \ker \alpha_0)$ is Liouville fillable. To be precise, we suppose that there is a Liouville domain (W, ω) with complete Liouville flow X for $t < 0$, satisfying $\partial W = M$ and $(\iota_X \omega)|_M = \alpha_0$. Define the following family of contact forms

$$O_\xi(\alpha_0) = \{e^f \alpha_0 \mid f \in C^\infty(M, \mathbb{R})\}.$$

There are many numerical measurements on this family.

1.1. **Symplectic Banach-Mazur distance.** For a Liouville domain (W, ω) with Liouville flow X, let $\hat{W}$ be its completion. For any open star-shaped domain U, V in $\hat{W}$ and real number $C > 0$, write $CV := r_C(V) = \phi_X^{\ln C}(V)$, where ϕ_X is the flow along Liouville flow X and CU is defined similarly. Then we can define the **coarse** symplectic Banach-Mazur distance as

$$d_{\mathrm{BM}}(U,V) := \inf\{\ln C \mid C > 1, \exists \phi, \psi, \text{s.t.} \frac{1}{C}U \overset{\phi}{\hookrightarrow} V \overset{\psi}{\hookrightarrow} CU\},$$

where $\frac{1}{C}U \overset{\phi}{\hookrightarrow} V$ means there is Hamiltonian isotopy ϕ on $\hat{W}$ with $\phi_0 = \mathrm{id}$ and $\phi_1(\frac{1}{C}U) \subseteq V$, similarly we impose the conditions on ψ.

We focus more on the **fine** symplectic Banach-Mazur distance, denoted by d_{SBM}, in this article. It is defined in a same pattern as above, but we require that:

(1) ϕ and ψ are *exact Liouville embeddings*. Here an embedding ϕ is said to be an exact Liouville embeddings if $\phi^*\lambda - \lambda_0 = df$ for some smooth function f, and $\lambda = \iota_X\omega$ is the Liouville primitive.

(2) Both $\psi\circ\phi : \frac{1}{C}U \to CU$ and $\phi(C)\circ\psi(C^{-1}) : \frac{1}{C}V \to CV$ are strongly unknotted. Here $\phi(C) = r_C \circ \phi \circ r_{1/C}$ and $\psi(C^{-1}) = r_{1/C} \circ \psi \circ r_C$. And an exact Liouville embedding $\eta : A \to B$ is called strongly unknotted if there exists an isotopy of exact Liouville embeddings $\eta_t : A \to B, t \in [0,1]$ so that $\eta_0 = i_{A,B}, \eta_1 = \eta$.

Clearly one has $d_{\mathrm{BM}} \leq d_{\mathrm{SBM}}$.

Remark 1.1. Here our definition of fine symplectic Banach-Mazur distance uses the unknottedness condition introduced by [GU19, Ush22]. Note that [RZ21] required that the homotopy in the unknottedness condition should be inside the corresponding target domain.

Remark 1.2. One can impose different kinds of restriction in the above definition to obtain many kinds of symplectic Banach-Mazur distance. For example a $\tilde{\pi}_1$-trivial condition is required in the definition of [SZ21], and a homological Banach-Mazur distance is defined in [ZZ26] by adding H_1-trivial restriction.

For the contact manifold (M, α_0) with the Liouville filling $(\hat{W}, \omega)$ above, by [AM19], there is a decomposition

$$\hat{W} = SM \sqcup \mathrm{Core}(M),$$

where SM is the symplectization of M, and the coordinate (x, u) defines an identification between SM and $M \times \mathbb{R}_+$.

For any $\alpha \in O_\xi(\alpha_0)$, there is a smooth function f on M with $\alpha = e^f\alpha_0$, then there is an open Liouville domain in $\hat{W}$ as follow:

$$W_\alpha = \{(x, u) \in \hat{W} \mid e^{f(x)} - u > 0\}.$$

Then both $d_{\mathrm{BM}}(W_{\alpha_1}, W_{\alpha_2})$ and $d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2})$ give numerical measurements on the space of supporting contact forms. We will also called these measurements as (coarse or fine) symplectic Banach-Mazur distances.

There is also a contact analogue of symplectic Banach-Mazur distance.

Definition 1.3. ([RZ21, Definition 1.12]) Let (M, ξ) be a closed contact manifold with ξ co-orientable, and $O_\xi(\alpha_0)$ is as above for a fixed contact form α_0.

(1) If $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$ with $\alpha_i = e^{f_i}\alpha_0 (i = 1, 2)$, then we say $\alpha_1 \preceq \alpha_2$ if and only if $f_1 \leq f_2$.
(2) For any $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$, the *contact Banach-Mazur distance* d_{CBM} is defined as

$$d_{\mathrm{CBM}}(\alpha_1, \alpha_2) = \inf\{\ln C \mid C \geq 1, \exists \mathrm{Cont}_0(M, \xi), \mathrm{s.t.} \frac{1}{C}\alpha_1 \preceq \phi^*\alpha_2 \preceq C\alpha_1\},$$

where $\mathrm{Cont}_0(M, \xi)$ is the identity component of the contactomorphism group of (M, ξ).

Remark 1.4. For any Liouville manifold W, Rosen-Zhang[RZ21] constructed a candidate for contact Banach-Mazur distance between fiberwise star-shaped domains in $W \times S^1$. Their distance is sometimes not well-defined and the non-triviality of this distance has connection with non-squeezing domains.

There is also a simple C^0-type distance measurement on $O_\xi(\alpha_0)$. Let $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$, so there exists $f \in C^\infty(M; \mathbb{R}_+)$ so that $\alpha_1 = f \cdot \alpha_2$. Then we can define

$$\delta(\alpha_1, \alpha_2) = \|\log f\|_{C^0}.$$

1.2. **The bottleneck distance.** In the persistent homology theory, the isometry theorem gives an isometry to the space of persistence modules (a persistence module can be roughly regarded as a filtered family of vector spaces), equipped with the interleaving distance d_{inter}, and the space of barcodes (a barcode is a multiset of intervals), equipped with the bottleneck distance d_{bottle}.

Since there is an action functional and thus energy filtration in Floer theory, a Floer theory usually induces a persistence module, so also a barcode by the isometry theorem. See [UZ16, PRSZ20] for barcodes associated to classical Hamiltonian Floer theory, [CGG24, Die25] for Lagrangian Floer theory, [Fer24] for wrapped Floer theory and [FLS26] for symplectic homology.

We consider two types of barcodes in this article. Recall that $W_\alpha = \{(x,u) \in \hat{W} \mid e^{f(x)} - u > 0\}$, so $\partial W_\alpha = \{(x,u) \in \hat{W} \mid e^{f(x)} = u\}$, and

$$\Gamma_\alpha : (M,\alpha) \to (\partial W_\alpha, \lambda|_{\partial W_\alpha})$$

is a contactomorphism, where λ is the Liouville form on $\hat{W}$. So W_α is a non-degenerate Liouville domain if and only if α is a non-degenerate contact form on M. Then one can define a filtered version of positive symplectic homology $\mathrm{SH}_*^a(W_\alpha)$ for any $a > 0$, and let $\mathbb{SH}_*(W_\alpha)$ and $\mathcal{B}_*(W_\alpha)$ be the corresponding persistence module and barcode. Write

$$\log \mathrm{SH}_*^s(W_\alpha) = \mathrm{SH}_*^{e^s}(W_\alpha), \forall s \in \mathbb{R}.$$

Then there is also a resulting persistence module $\log \mathbb{SH}_*(W_\alpha)$ and barcode $\log \mathcal{B}_*(W_\alpha)$. If both $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$ are non-degenerate, we can consider the bottleneck distance between the logarithmic symplectic homology barcode

$$d_{\mathrm{bottle,SH}}(\alpha_1,\alpha_2) := d_{\mathrm{bottle}}\big(\log \mathcal{B}_*(W_{\alpha_1}), \log \mathcal{B}_*(W_{\alpha_2})\big).$$

The other barcode arises from the persistence module defined by Dylan Cant[Can23]. Recall that a *discriminant point* of $\phi \in \mathrm{Cont}_0(M,\xi)$ is a point $x \in M$ with

$$\phi(x) = x, (\phi^*\alpha)_x = \alpha_x.$$

For a contact form α, let $R_s = R_s^\alpha$ be the corresponding Reeb flow. A *translated point* of **length** s is a discriminant point of $\phi^{-1} \circ R_s$. The *spectrum* of ϕ, written as $\mathrm{Spec}_\alpha(\phi)$, is the set of lengths of translated points. See [San12, AM13, She17] for more work on translated points.

We let $\mathrm{Cont}_0^*(M,\xi)$ be the subset of $\mathrm{Cont}_0(M,\xi)$ consisting of contactomorphisms without discriminant point. For any contact isotopy $\phi_t \in \mathrm{Cont}_0(M,\xi)$ with $\phi_1 \in \mathrm{Cont}_0^*(M,\xi)$, there is an associated Floer cohomology $\mathrm{HF}(\phi)$ given by the Floer cohomology on $\hat{W}$ with respect to any Hamiltonian system whose ideal restriction agrees with ϕ. Then for any $s \notin \mathrm{Spec}_\alpha(\phi_1)$, define

$$V_{\alpha,s}(\phi) = \mathrm{HF}(\phi_t^{-1} \circ R_{st}^\alpha),$$

and the continuation map $V_{\alpha,s}(\phi) \to V_{\alpha,s'}(\phi), s \le s'$ is given by counting certain twisted cylinders. The detailed construction will be discussed in Section 3.

The family $\{V_{\alpha,s}(\phi)\}_s$ along with the continuation maps form a persistence module $\mathbb{V}_\alpha(\phi)$ and there is a corresponding barcode $\mathcal{B}_\alpha(\phi)$.

For any $s \in \mathbb{R}$, define

$$\log V_{\alpha,s}(\phi) = V_{\alpha,e^s}(\phi),$$

then there is also a continuation map $\log V_{\alpha,s}(\phi) \to \log V_{\alpha,s'}(\phi)$ for any $s \le s'$. We write $\log V_\alpha(\phi)$ and $\log \mathcal{B}_\alpha(\phi)$ be the associated persistence module and barcode. For different contact forms $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$, it is also natural to consider the bottleneck distance $d_{\text{bottle}}(\log \mathcal{B}_{\alpha_1}(\phi), \log \mathcal{B}_{\alpha_2}(\phi))$. In particular, we write

$$d_{\text{bottle,cont}}(\alpha_1, \alpha_2) = d_{\text{bottle}}\big(\log \mathcal{B}_{\alpha_1}(\text{id}), \log \mathcal{B}_{\alpha_2}(\text{id})\big)$$

1.3. **Main results.** The main result of this paper is the following.

Theorem 1.5. *Let (M, ξ) be a closed contact manifold and ξ is a co-oriented contact structure, then for a fixed contact form α_0 and any $\alpha_1, \alpha_2 \in O_\xi(\alpha_0)$, we have the following.*

(1) For every Cant system ϕ_t, $t \in [0,1]$, with $\phi_0 = \text{id}$ and $\phi_1 \in \text{Cont}_0^*(M, \xi)$, and for any $\varepsilon > 0$, there exists $\chi_\varepsilon \in Cont_0(M, \xi)$ so that

$$d_{\text{bottle}}\big(\log \mathcal{B}_{\alpha_1}(\phi), \log \mathcal{B}_{\alpha_2}(\chi_\varepsilon \phi \chi_\varepsilon^{-1})\big) \le d_{\text{CBM}}(\alpha_1, \alpha_2) + \varepsilon. \tag{1.1}$$

(2) If both α_1 and α_2 are non-degenerate, there is

$$d_{\text{bottle,SH}}(\alpha_1, \alpha_2) \le d_{\text{SBM}}(W_{\alpha_1}, W_{\alpha_2}) \le d_{\text{CBM}}(\alpha_1, \alpha_2) \le \delta(\alpha_1, \alpha_2). \tag{1.2}$$

Theorem 1.5 can be interpreted in two complementary ways. First, it gives upper bounds on Floer-theoretic measurements in terms of Banach-Mazur type measurements of contact forms. Second, after reversing the inequalities, the barcode distances provide computable lower bounds for d_{CBM} and, in the symplectic homology case, for d_{SBM}.

Let us spell out a few immediate consequences.

Corollary 1.6 (Lipschitz continuity with respect to conformal changes)**.** *Let $\alpha' \in \mathcal{O}_\xi(\alpha_0)$ be of the form $\alpha' = e^g \alpha$. Then, for every Cant system ϕ_t,*

$$d_{\text{bottle}}\big(\log \mathcal{B}^+_\alpha(\phi), \log \mathcal{B}^+_{\alpha'}(\phi)\big) \le \|g\|_{C^0}.$$

In particular, the map

$$\alpha \longmapsto \log \mathcal{B}^+_\alpha(\text{id})$$

is 1-Lipschitz with respect to the elementary C^0-measurement $\delta(\alpha, \alpha') = \|\log(\alpha'/\alpha)\|_{C^0}$.

Proof. If $\alpha' = e^g\alpha$, then $e^{-\|g\|_{C^0}}\alpha \preceq \alpha' \preceq e^{\|g\|_{C^0}}\alpha$. Applying Theorem 1.7 with $C = e^{\|g\|_{C^0}}$ gives the result. □

Example 1.7 (Constant rescaling). Let $\alpha_a := e^a\alpha$ for a constant $a \in \mathbb{R}$. Since $R^{\alpha_a} = e^{-a}R^\alpha$, one has

$$V_{\alpha_a,s}(\phi) = \mathrm{HF}(\phi_t^{-1} \circ R^{\alpha_a}_{st}) = \mathrm{HF}(\phi_t^{-1} \circ R^{\alpha}_{e^{-a}st}) = V_{\alpha,e^{-a}s}(\phi).$$

Thus the logarithmic persistence module satisfies

$$\log V^t_{\alpha_a}(\phi) = \log V^{t-a}_\alpha(\phi),$$

and consequently

$$\log \mathcal{B}^+_{\alpha_a}(\phi) = \log \mathcal{B}^+_\alpha(\phi) + a.$$

In particular,

$$d_{\mathrm{bottle}}\big(\log \mathcal{B}^+_\alpha(\phi), \log \mathcal{B}^+_{\alpha_a}(\phi)\big) \leq |a|,$$

which agrees with the estimate $\delta(\alpha, \alpha_a) = |a|$.

Example 1.8 (Small non-constant conformal perturbations). Suppose $\alpha' = e^g\alpha$ with $|g(x)| \leq \varepsilon$ for all $x \in M$. Then

$$e^{-\varepsilon}\alpha \preceq \alpha' \preceq e^\varepsilon\alpha.$$

Hence

$$d_{\mathrm{bottle}}\big(\log \mathcal{B}^+_\alpha(\phi), \log \mathcal{B}^+_{\alpha'}(\phi)\big) \leq \varepsilon.$$

Equivalently, every bar endpoint in the logarithmic barcode can move by at most ε under such a perturbation, up to the usual bottleneck matching. In the original action variable, this means that relevant action levels are controlled up to a multiplicative factor e^ε.

Corollary 1.9 (Lower bounds for Banach–Mazur measurements)**.** *For $\alpha_1, \alpha_2 \in \mathcal{O}_\xi(\alpha_0)$, Theorem 1.5 implies*

$$d_{\mathrm{CBM}}(\alpha_1, \alpha_2) \geq d_{\mathrm{bottle}}\big(\log \mathcal{B}^+_{\alpha_1}(\mathrm{id}), \log \mathcal{B}^+_{\alpha_2}(\mathrm{id})\big).$$

If α_1 and α_2 are non-degenerate, then it also implies

$$d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2}) \geq d_{\mathrm{bottle}}(\log B_*(W_{\alpha_1}), \log B_*(W_{\alpha_2})).$$

Thus barcode computations can be used as obstructions to contact or symplectic squeezing.

1.4. **Outline of the proof.** Now we give a sketch of the proof. The first part is to show the inequality (1.1). The main ingredient is to connect the interleaving distance of the Cant persistence module to the squeezing constant controlling two contact forms. The precise statement is the following.

Theorem 1.10. *For $\alpha, \beta \in \mathcal{O}_\xi(\alpha_0)$, if for some real number $C > 1$, there is*

$$C^{-1}\alpha \preceq \beta \preceq C\alpha,$$

then for any contact isotopy ϕ, $\log V_\alpha(\phi)$ and $\log V_\beta(\phi)$ are $\log C$-interleaved.

To construct interleaving persistence morphism, we need the order and rescaling funtoriality of Cant's persistence module. This would lead to Theorem 1.10 as in Section 4.2.

Theorem 1.11. *The following properties hold.*

(A1) If $\alpha_1 \preceq \alpha_2$, then there exists a canonical persistence morphism

$$h^\phi_{\alpha_2,\alpha_1} : V^+_{\alpha_2}(\phi) \longrightarrow V^+_{\alpha_1}(\phi).$$

That is, for every $0 < s \leq s'$, the square

$$\begin{array}{ccc} V_{\alpha_2,s}(\phi) & \xrightarrow{h^{\phi,s}_{\alpha_2,\alpha_1}} & V_{\alpha_1,s}(\phi) \\ {\scriptstyle c^{\alpha_2,\phi}_{s,s'}}\downarrow & & \downarrow{\scriptstyle c^{\alpha_1,\phi}_{s,s'}} \\ V_{\alpha_2,s'}(\phi) & \xrightarrow{h^{\phi,s'}_{\alpha_2,\alpha_1}} & V_{\alpha_1,s'}(\phi) \end{array}$$

commutes.

(A2) If $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$, then

$$h^\phi_{\alpha_3,\alpha_1} = h^\phi_{\alpha_2,\alpha_1} \circ h^\phi_{\alpha_3,\alpha_2}$$

as persistence morphisms

$$V^+_{\alpha_3}(\phi) \longrightarrow V^+_{\alpha_1}(\phi).$$

In particular,

$$h^\phi_{\alpha,\alpha} = \mathrm{id}_{V^+_\alpha(\phi)}.$$

(A3) For every constant $C > 0$, there exists a canonical multiplicative-shift persistence isomorphism

$$\rho^{\eta,\phi}_C : V^+_\eta(\phi) \xrightarrow{\cong} (m_C)^* V^+_{C\eta}(\phi), \qquad m_C(s) = Cs.$$

Moreover, for every $0 < s \leq s'$,

$$c^{C\eta,\phi}_{Cs,Cs'} \circ \rho^{\eta,\phi}_{C,s} = \rho^{\eta,\phi}_{C,s'} \circ c^{\eta,\phi}_{s,s'}.$$

(A4) If $\alpha_1 \preceq \alpha_2$*, then, for every* $C > 0$ *and every* $s > 0$*, the following diagram commutes:*

$$\begin{array}{ccc} V_{\alpha_2,s}(\phi) & \xrightarrow{h^{\phi,s}_{\alpha_2,\alpha_1}} & V_{\alpha_1,s}(\phi) \\ \rho^{\alpha_2,\phi}_{C,s} \big\downarrow \cong & & \big\downarrow \rho^{\alpha_1,\phi}_{C,s} \\ V_{C\alpha_2,Cs}(\phi) & \xrightarrow{h^{\phi,Cs}_{C\alpha_2,C\alpha_1}} & V_{C\alpha_1,Cs}(\phi). \end{array}$$

Remark 1.12. The item (A4) is actually not necessary for our proof of Theorem 1.10, thus Theorem 1.5. However, we include the statement and proof of it here.

The above theorem will be proved in Section 4.1. Along with the following conjugation invariance of Cant's barcode, it would yield (1.1).

Theorem 1.13. *Let* $\chi \in Cont_0(M,\xi)$ *and* $\alpha \in \mathcal{O}_\xi(\alpha_0)$*. If* $\beta = \chi^*\alpha$*, then there is a persistence isomorphism*

$$\log V_\beta(\phi) \cong \log V_\alpha(\chi\phi\chi^{-1}).$$

Proof of (1.1)*, assuming Theorem 1.10 and Theorem 1.13.* Now let $\varepsilon > 0$. By the definition of $d_{\mathrm{CBM}}(\alpha,\alpha')$, choose $C_\varepsilon \geq 1$ and $\chi_\varepsilon \in \mathrm{Cont}_0(M,\xi)$ such that $\log C_\varepsilon \leq d_{\mathrm{CBM}}(\alpha,\alpha') + \varepsilon$ and

$$C_\varepsilon^{-1}\alpha \preceq \chi_\varepsilon^*\alpha' \preceq C_\varepsilon\alpha.$$

Set $\beta_\varepsilon := \chi_\varepsilon^*\alpha'$. Then

$$C_\varepsilon^{-1}\alpha \preceq \beta_\varepsilon \preceq C_\varepsilon\alpha.$$

By Theorem 1.10,

$$d_{\mathrm{bottle}}\big(\log\mathcal{B}^+_\alpha(\phi), \log\mathcal{B}^+_{\beta_\varepsilon}(\phi)\big) \leq \log C_\varepsilon.$$

By Theorem 1.13,

$$\log\mathcal{B}^+_{\beta_\varepsilon}(\phi) = \log\mathcal{B}^+_{\alpha'}\left(\chi_\varepsilon\phi\chi_\varepsilon^{-1}\right).$$

Therefore

$$d_{\mathrm{bottle}}\big(\log\mathcal{B}^+_\alpha(\phi), \log\mathcal{B}^+_{\alpha'}\left(\chi_\varepsilon\phi\chi_\varepsilon^{-1}\right)\big) \leq \log C_\varepsilon.$$

Using the choice of C_ε, we obtain

$$d_{\mathrm{bottle}}\big(\log\mathcal{B}^+_\alpha(\phi), \log\mathcal{B}^+_{\alpha'}\left(\chi_\varepsilon\phi\chi_\varepsilon^{-1}\right)\big) \leq d_{\mathrm{CBM}}(\alpha,\alpha') + \varepsilon.$$

This proves the desired statement. □

Then we need to show $d_{\text{bottle,SH}}(\alpha_1, \alpha_2) \leq d_{\text{SBM}}(W_{\alpha_1}, W_{\alpha_2})$. This inequality is a direct result from the following more general stability theorem.

Theorem 1.14. *Let $U, V \subset \widehat{W}$ be nondegenerate star-shaped Liouville domains in $\hat{W}$. Then*

$$d_{\text{bottle}}\big(\log \mathbb{SH}_*(U), \log \mathbb{SH}_*(V)\big) \leq d_{\text{SBM}}(U, V).$$

This result is based on the following properties of symplectic homology barcode, the proof is given in Section 4.3.

Theorem 1.15. *Let $U, V, W \subset \widehat{W}$ be nondegenerate star-shaped Liouville domains. The following statements hold.*

(B1) If $\phi : U \hookrightarrow V$ is an exact Liouville embedding, then for every regular action value $a > 0$ there is a Viterbo restriction map

$$h^a_\phi : \mathrm{SH}^a_*(V) \longrightarrow \mathrm{SH}^a_*(U).$$

And $h_\phi = \{h^a_\phi\}_{a>0}$ is a morphism of persistence modules

$$h_\phi : \mathbb{SH}_*(V) \longrightarrow \mathbb{SH}_*(U).$$

Namely, for $0 < a \leq b$, the diagram

$$\begin{array}{ccc} \mathrm{SH}^a_*(V) & \xrightarrow{\iota^V_{a,b}} & \mathrm{SH}^b_*(V) \\ {\scriptstyle h^a_\phi}\big\downarrow & & \big\downarrow{\scriptstyle h^b_\phi} \\ \mathrm{SH}^a_*(U) & \xrightarrow{\iota^U_{a,b}} & \mathrm{SH}^b_*(U) \end{array}$$

commutes.

(B2) The assignment $U \mapsto \mathbb{SH}_*(U)$ is contravariantly functorial. More precisely, if

$$U \xrightarrow{\phi} V \xrightarrow{\psi} W$$

are exact Liouville embeddings, then

$$h_{\psi\circ\phi} = h_\phi \circ h_\psi.$$

(B3) For every $C > 0$ there is a canonical scaling isomorphism

$$r^U_C(a) : \mathrm{SH}^a_*(U) \xrightarrow{\cong} \mathrm{SH}^{Ca}_*(CU).$$

And they define an isomorphism of persistence modules

$$r^U_C : \mathbb{SH}_*(U) \xrightarrow{\cong} \mathbb{SH}_*(CU)[\log C],$$

where the shift is understood in logarithmic parameter:

$$\log \mathrm{SH}_*^{t+\log C}(CU) \cong \log \mathrm{SH}_*^t(U).$$

Moreover, scaling is compatible with Viterbo restriction maps, that is, for every $a > 0$ *the diagram*

$$\begin{array}{ccc} \mathrm{SH}_*^a(V) & \xrightarrow{h_\phi^a} & \mathrm{SH}_*^a(U) \\ {\scriptstyle r_C^V(a)}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong r_C^U(a)} \\ \mathrm{SH}_*^{Ca}(CV) & \xrightarrow{h_{\phi(C)}^{Ca}} & \mathrm{SH}_*^{Ca}(CU) \end{array}$$

commutes.

(B4) Let $C \geq 1$*. For the standard inclusion* $i : U \hookrightarrow CU$*, the restriction map induced by* i *is the persistence comparison map after scaling. More precisely,*

$$h_i^{Ca} \circ r_C^U(a) = \iota_{a,Ca}^U.$$

In particular, for the standard inclusion $j : C^{-1}U \hookrightarrow CU$*, the restriction map*

$$h_j^a : \mathrm{SH}_*^a(CU) \longrightarrow \mathrm{SH}_*^a(C^{-1}U)$$

becomes exactly the persistence comparison map

$$\iota_{a/C,Ca}^U : \mathrm{SH}_*^{a/C}(U) \longrightarrow \mathrm{SH}_*^{Ca}(U),$$

after the scaling identifications $\mathrm{SH}_*^a(CU) \cong \mathrm{SH}_*^{a/C}(U), \mathrm{SH}_*^a(C^{-1}U) \cong \mathrm{SH}_*^{Ca}(U)$*.*

(B5) If $U \subset V$ *and* $\phi : U \hookrightarrow V$ *is strongly unknotted, i.e. Liouville-isotopic through exact Liouville embeddings to the standard inclusion* $i_U : U \hookrightarrow V$*, then*

$$h_\phi = h_{i_U}.$$

As in [PRSZ20, Section 9.6], the above properties implies Theorem 1.14, thus $d_{\mathrm{bottle,SH}}(\alpha_1, \alpha_2) \leq d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2})$. We will lay out the argument in Section 4.4.

Remark 1.16. Theorem 1.14 was stated in [PRSZ20, Theorem 9.4.9], where the notations of symplectic Banach-Mazur distance is different from ours. The key to their stability theorem is the functoriality result Theorem 9.5.1 therein(which is also different to our Theorem 1.15), which the authors did not present a complete proof of.

The inequality $d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2}) \leq d_{\mathrm{CBM}}(\alpha_1, \alpha_2)$ is obtained in [RZ21, Theorem 1.14]. Although the definition of the fine symplectic Banach-Mazur distance of Rosen-Zhang does not require the embeddings to be exact Liouville embeddings, the diffeomorphism constructed therein for the proof is actually Hamiltonian, so exact. Then their proof works in our setting and yields $d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2}) \leq d_{\mathrm{CBM}}(\alpha_1, \alpha_2)$.

Now the only remaining piece is $d_{\mathrm{CBM}}(\alpha_1, \alpha_2) \leq \delta(\alpha_1, \alpha_2)$. This is rather direct so we give a proof here.

Proof of $d_{CBM} \leq \delta$. Write $\alpha_1 = e^{f_1}\alpha_0$ for some $f_1 \in C^\infty(M, \mathbb{R})$. Since $\alpha_2 = f\alpha_1$, we have $\alpha_2 = e^{f_1 + \log f}\alpha_0$.

Set $\delta := \delta(\alpha_1, \alpha_2) = \|\log f\|_{C^0}$. By definition, $-\delta \leq \log f(x) \leq \delta$ for every $x \in M$. Hence

$$f_1(x) - \delta \leq f_1(x) + \log f(x) \leq f_1(x) + \delta \qquad \text{for every } x \in M.$$

By the definition of the order $\preceq$ on $\mathcal{O}_\xi(\alpha_0)$, this gives

$$e^{f_1 - \delta}\alpha_0 \preceq e^{f_1 + \log f}\alpha_0 \preceq e^{f_1 + \delta}\alpha_0.$$

Equivalently,

$$e^{-\delta}\alpha_1 \preceq \alpha_2 \preceq e^{\delta}\alpha_1.$$

Taking $\phi = \mathrm{id}_M \in \mathrm{Cont}_0(M, \xi)$ in the defining infimum of $d_{\mathrm{CBM}}(\alpha_1, \alpha_2)$, we obtain

$$d_{\mathrm{CBM}}(\alpha_1, \alpha_2) \leq \delta = \delta(\alpha_1, \alpha_2).$$

□

So the above ingredients would complete the proof of Theorem 1.5.

The paper is organized as follows.

In Section 2, we review the basics of topological data analysis, spelling out the necessary definitions and results in the persistent homology theory, including the isometry theorem connecting persistence modules and barcodes. In Section 3, we separately study the barcodes arising from symplectic homology and Cant's construction in detail, giving a proof of invariance result Theorem 1.13. In Section 4 the proof of Theorem 1.11 and Theorem 1.15 will be presented, thus concluding the proof of Theorem 1.5.

Acknowledgments.

2. Persistence module and barcode

In this section we provide basic definitions and results in the theory of persistence module. Our conventions follow [PRSZ20] and one can find more details in [BL15, UZ16, PSS17], etc. First we lay out the relevant definitions about persistence module.

Definition 2.1. ([PRSZ20, Definition 1.1]) Let $\mathbb{F}$ be a field.

(1) A **persistence module** (V, π) consists of a finite dimensional $\mathbb{F}$-vector space V_t associated to each $t \in \mathbb{R}$ with homomorphisms $\pi_{s,t}\colon V_s \to V_t$ whenever $s \leq t$ satisfying the functoriality properties that $\pi_{s,s} = I_{V_s}$, the identity map on module V_s, and $\pi_{s,u} = \pi_{t,u} \circ \pi_{s,t}$. We require that
   (a) For all but a finite number of points $t \in \mathbb{R}$ there exists a neighborhood U of t, such that $\pi_{s,r}$ is an isomorphism for any $s < r$ in U.
   (b) For any $t \in \mathbb{R}$ and any $s \leq t$ sufficiently close to t, the map $\pi_{s,t}$ is an isomorphism.

   We omit the symbol π from the notation if it causes no problem of ambiguity.

(2) Let V be a persistence module and $\delta \in \mathbb{R}$. The δ-shift of V is the persistence module $V[\delta]$ with $V[\delta]^s = V^{s+\delta}$ and $\pi[\delta]^{s,t} = \pi_{s+\delta,t+\delta}$.

(3) Let V and W be two persistence modules. A morphism from V to W is a collection of linear maps $\boldsymbol{f} = (f^s : V^s \to W^s)_{s\in\mathbb{R}}$ such that for all $s \leq t$ the following diagram commutes.

$$\begin{array}{ccc} V^s & \xrightarrow{f^s} & W^s \\ {\scriptstyle \pi^V_{s,t}}\downarrow & & \downarrow{\scriptstyle \pi^W_{s,t}} \\ V^t & \xrightarrow[f^t]{} & W^t \end{array}$$

   A morphism $A : V \to V'$ is an **isomorphism** if there is a morphism $B : V' \to V$ such that $A \circ B$ and $B \circ A$ are the identity morphisms on the corresponding persistence module.

Remark 2.2. There are many different definitions of persistence modules. The differences mainly take place in the condition imposed on the module.

(1) In some texts, for example [CB15, FLS26], the vector spaces V_s are not required to be finite dimensional in the definition of persistence module. However, since the finite dimensional condition is necessary in the structure theorem and all the persistence modules that we concern in this paper are all finite dimensional in a natural way, we omit the condition in the definition.

(2) The condition(a) in the definition is called *finite type condition* in [FLS26]. Since it enables us to count the number of intervals and this condition is met by the persistence module that we define through Floer-theoretic construction, we put this condition in our definition.

There are many examples of persistence modules that appear in applications.

Example 2.3. (1) Let X be a closed manifold (i.e. a smooth compact manifold without boundary) and let $f : X \to \mathbb{R}$ be a Morse function. Fix $0 \leq k \in \mathbb{Z}$ and put

$$V_t = H_k(\{f < t\}; \mathbb{F}).$$

Consider the natural inclusion $\{f < s\} \xhookrightarrow[i_{s,t}]{} \{f < t\}$ for $s \leq t$. It induces the map $\pi_{s,t} := (i_{s,t})_* : V_s \to V_t$ in homology, and one can verify that we get a persistence module.

(2) Every non-empty interval $I \subset \mathbb{R}$ defines a persistence module $\mathbb{F}I$ as follows:

$$\mathbb{F}I_t := \begin{cases} \mathbb{F} \text{ if } t \in I, \\ 0 \text{ otherwise.} \end{cases} \qquad \pi_{s,t} := \begin{cases} id_{\mathbb{F}} \text{ if } s, t \in I, \\ 0 \text{ otherwise.} \end{cases}$$

More generally,let $\{I_i | i \in \text{ an index set } J\}$ be a collection of intervals $I_i \subset \mathbb{R}$. Let $\{(\pi_i)_{s,t}\}_{s \leq t}$ be the collection of linear maps for the interval persistence module $\mathbb{F}I_i$ for all $i \in J$. Then, this collection defines a persistence module $\bigoplus_{i \in J} \mathbb{F}I_i$ as follows:

$$\left(\bigoplus_{i \in J} \mathbb{F}I_i \right)_t := \bigoplus_{i \in J} \left(\mathbb{F}I_i\right)_t, \pi_{s,t} := \bigoplus_{i \in J} \left(\pi_i\right)_{s,t}.$$

(3) For a persistence module (V, π) and $\delta \in \mathbb{R}$, define a persistence module $(V[\delta], \pi[\delta])$ by taking $(V[\delta])_t = V_{t+\delta}$ and $(\pi[\delta])_{s,t} = \pi_{s+\delta,t+\delta}$. This new persistence module is called a *δ-shift* of V. For $\delta > 0$, the map $\Phi^\delta : (V, \pi) \to (V[\delta], \pi[\delta])$ defined by $\Phi^\delta_t = \pi_{t,t+\delta}$ is a morphism of persistence modules (it will be referred to as *δ-shift morphism*). Also, if we have a morphism $F : V \to W$

between two persistence modules, let us denote by $F[\delta] : V[\delta] \to W[\delta]$ the corresponding morphism between their δ-shifts.

(4) Let (V, π) be a persistence module, and let T be a real number. Then, the **truncation of V at T**, denoted by $\boldsymbol{V^T}$, is a persistence module

$$V^T := \left(\{V_t^T\}_{t\in\mathbb{R}}, \{\pi_{a,b}^T\}_{a\leq b\in\mathbb{R}}\right),$$

defined as follows:

$$V_t^T = \begin{cases} V_t \text{ if } t < T \\ 0 \text{ otherwise} \end{cases}, \pi_{a,b}^T = \begin{cases} \pi_{a,b} \text{ if } b < T \\ 0 \text{ otherwise} \end{cases}.$$

In fact, any persistence module can be decomposed into the above defined interval modules. This is the structure theorem(also known as normal form theorem), which is fundamental in the persistence theory. To better describe the result of the theorem, we firstly introduce the definition of barcode.

Definition 2.4. A **barcode** $\mathcal{B}$ is a locally finite multiset of intervals, namely a collection $\{(I_j, m_j)\}_{j\in J}$ of intervals with multiplicities $m_j \in \mathbb{N}$, such that only finitely many intervals meet any compact subset of $\mathbb{R}$ in a nontrivial interval. The intervals in a barcode will be sometimes called **bars**.

Theorem 2.5 (Structure Theorem)**.** *Let (V, π) be a persistence module. Then there exists a locally finite collection $\{(I_i, m_i)\}_{i=1}^N$ of intervals I_i with their multiplicities m_i, where $I_i = [a_i, b_i)$ or $I_i = [a_i, \infty)$, $m_i \in \mathbb{N}$, $I_i \neq I_j$ for $i \neq j$, such that*

$$V = \bigoplus_{i=1}^N \mathbb{F}(I_i)^{m_i}.$$

By equality here we mean that they are isomorphic as persistence modules.
Moreover, this data is unique up to permutations, i.e., to any persistence module there corresponds a unique barcode $\mathcal{B}(V)$, which consists of the intervals I_i with multiplicity m_i. This barcode will be called the barcode of V.

Proof. See [Gab72, CB15] and [PRSZ20]. □

Remark 2.6. The definition of our barcode is different from the one in [FLS26] since we imposed the semicontinuity condition on the persistence module, which restricts the type of intervals appearing in the decomposition.

One can define a distance on isomorphism classes of persistence modules with the same vector space at $+\infty$.

Definition 2.7. ([PRSZ20, Definition 1.3.1]) Given a $\delta > 0$, we say that two persistence modules (V, π) and (W, θ) are *δ-interleaved* if there exist two morphisms $F : V \to W[\delta]$ and $G : W \to V[\delta]$, such that the following diagrams commute:

$$V \xrightarrow{F} W[\delta] \xrightarrow{G[\delta]} V[2\delta], \quad \varphi_V^{2\delta} : V \to V[2\delta] \qquad\qquad W \xrightarrow{G} V[\delta] \xrightarrow{F[\delta]} W[2\delta], \quad \varphi_W^{2\delta} : W \to W[2\delta]$$

where $\varphi_V^{2\delta}$ and $\varphi_W^{2\delta}$ are the shift morphisms. We will also refer to such a pair of morphisms F and G as *δ-interleaving morphisms*.

For two persistence modules (V, π) and (W, θ), define the interleaving distance between them to be

$$d_{int}(V, W) = \inf \{\delta > 0 \mid (V, \pi) \text{ and } (W, \theta) \text{ are } \delta\text{-interleaved}\}.$$

Now we move on to show that this distance actually defines a metric, i.e. it is non-degenerate. The structure theorem connects persistence module and barcode, in fact one can define a distance on the space of barcodes to upgrade this connection to the metric level.

Given an interval $I = [a, b)$, denote by $I^{-\delta} = [a - \delta, b + \delta)$ the interval obtained from I by expanding by δ on both sides. Let $\mathcal{B}$ be a barcode. For $\varepsilon > 0$, denote by $\mathcal{B}_\varepsilon$ the set of all bars from $\mathcal{B}$ of length greater than ε. (That is, by considering $\mathcal{B}_\varepsilon$ we neglect "short bars".)

A *matching* between two finite multi-sets X, Y is a bijection $\mu : X' \to Y'$, where $X' \subset X$, $Y' \subset Y$. In this case, $X' = \text{coim}\mu$, $Y' = \text{im}\mu$, and we say that elements of X' and Y' are *matched*. If an element appears in the multi-set several times, we treat its different copies separately, e.g. it could happen that only part of its copies are matched.

Definition 2.8. A *δ-matching* between two barcodes $\mathcal{B}$ and $\mathcal{C}$ is a matching $\mu : \mathcal{B} \to \mathcal{C}$, such that:

(1) $\mathcal{B}_{2\delta} \subset \text{coim}\mu$,
(2) $\mathcal{C}_{2\delta} \subset \text{im}\mu$,
(3) If $\mu(I) = J$, then $I \subset J^{-\delta}$, $J \subset I^{-\delta}$.

Definition 2.9. The *bottleneck distance*, $d_{bot}(\mathcal{B}, \mathcal{C})$, between two barcodes $\mathcal{B}, \mathcal{C}$ is defined to be the infimum over all δ for which there is a δ-matching between $\mathcal{B}$ and $\mathcal{C}$.

The following isometry theorem shows that this is the desired metric on the space of barcodes that corresponds to the interleaving distance of persistence modules.

Theorem 2.10 (Isometry Theorem)**.** *The map $V \mapsto \mathcal{B}(V)$ is an isometry, i.e. for any two persistence modules V, W, we have $d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W))$.*

Proof. See [BL15, PRSZ20]. □

It is straightforward to check that $d_{bot}(\mathcal{B}, \mathcal{C}) = 0$ if and only if $\mathcal{B} = \mathcal{C}$, so the isometry theorem gives

Corollary 2.11. $d_{int}(V, W) = 0$ *if and only if V and W are isomorphic.*

Remark 2.12. If one drops the semicontinuity condition(c) in the definition of persistence module, then the interleaving distance is only a pseudometric on the isomorphism class of persistence modules. For example for a ground field $\mathbb{K}$, $\mathbb{K}[0, 1]$ and $\mathbb{K}(0, 1)$ have vanishing interleaving distance, but they are not isomorphic.

3. Barcodes from Floer theory

3.1. **Floer cohomology of contact isotopy.** In this section we follow [Can23] to define a Floer cohomology with respect to a contact isotopy. We always consider a family $\phi_t \in \mathrm{Cont}_0(M, \xi), t \in [0, 1]$ with $\phi_0 = \mathrm{id}, \phi_1 \in \mathrm{Cont}_0^*(M, \xi)$, the subset of contactomorphism group consisting of elements lacking discriminant point. We might also consider a smooth extension to $t \in \mathbb{R}$ by $\phi_{t+1} = \phi_t \phi_1$. A *contact-at-infinity system* is a symplectic isotopy Ψ that commutes with the Liouville flow outside of a compact set, and the generator is a time-dependent and 1-periodic Hamiltonian vector field. The generating Hamiltonian functions is always assumed to be normalized in the sense of [AAC25]. We fix our ground field to be $\mathbb{Z}_2$.

Remark 3.1. If a contactomorphism ϕ has no discriminant points, then any Hamiltonian lift of a contact-at-infinity system with ideal restriction ϕ has no fixed points escaping to the cylindrical end. Consequently the relevant fixed-point Floer complex is generated by finitely many points in a compact region after a generic perturbation.

Almost complex structures. Suppose that the ideal restriction of ψ_t is the chosen contact isotopy ϕ_t. To define Floer cohomology, we choose a time-dependent almost complex structure $J_t (t \in \mathbb{R})$ satisfying the following conditions.

(1) Each J_t is ω-tame.
(2) On the cylindrical end of $\widehat{W}$, the family J_t is Liouville-equivariant; see [BC24].
(3) The family is twisted-periodic with respect to the time-one map ψ_1, namely

$$J_{t+1}(z) = d\psi_1^{-1}\, J_t(\psi_1(z))\, d\psi_1.$$

(4) On a compact subset containing all relevant fixed points, J_t is chosen generically so that the Floer moduli spaces appearing below are regular.

Floer cohomology. A pair (ψ_t, J_t) satisfying these conditions will be called an admissible Floer datum if the fixed points of ψ_1 are non-degenerate and all moduli spaces used to define the differential are cut out transversely. For such a datum, define $\mathrm{CF}(\psi_t, J_t)$ to be the $\mathbb{Z}_2$-vector space generated by the fixed points of ψ_1. The finiteness of this generating set follows from the contact-at-infinity condition and from the assumption that the ideal restriction of ψ_1 avoids the discriminant locus.

The differential is defined by counting finite-energy twisted holomorphic cylinders

$$w : \mathbb{R}_s \times \mathbb{R}_t \longrightarrow \widehat{W}$$

satisfying

$$\psi_1(w(s, t+1)) = w(s,t), \qquad \partial_s w + J_t(w)\partial_t w = 0.$$

The asymptotic limits of w as $s \to \pm\infty$ are fixed points of ψ_1. Equivalently, after the change of variables

$$u(s,t) = \psi_t(w(s,t)),$$

one obtains the usual Hamiltonian Floer equation for the system ψ_t. The twisted formulation is useful because the equation is written directly in terms of the time-one map ψ_1.

Over $\mathbb{Z}_2$, the coefficient of a generator x_+ in dx_- is the modulo-two count of rigid such cylinders from x_- to x_+. We denote the resulting cohomology by $\mathrm{HF}(\psi_t, J_t)$. The priori energy bound [MU19] and maximum principle for contact-at-infinity systems [Can23, Section 2.2.6] ensures that the relevant solutions stay in a compact subset of $\widehat{W}$, so the differential is well-defined and satisfies $d^2 = 0$.

The group $\mathrm{HF}(\psi_t, J_t)$ is independent of the auxiliary choices in the following sense. If (ψ_t^0, J_t^0) and (ψ_t^1, J_t^1) are two admissible Floer data with the same ideal restriction ϕ_t, then they can be connected by compactly supported continuation data. The associated continuation map is an isomorphism, and these isomorphisms are compatible with composition. Hence one defines $\mathrm{HF}(\phi_t)$ to be the canonically defined Floer cohomology group associated to the contact isotopy ϕ_t. Its isomorphism class depends only on the element represented by ϕ_t in the universal cover of $\mathrm{Cont}_0(M, \xi)$.

3.1.1. *Persistence module.* We next recall the continuation maps needed for the persistence module. Let $\phi_{\tau,t}$, $\tau \in [0,1]$, be a smooth family of contact isotopies such that the path of time-one maps $\tau \mapsto \phi_{\tau,1}$ is non-negative. Choose a contact-at-infinity lift $\psi_{\tau,t}$ of this family, together with a compatible family of almost complex structures. The corresponding s-dependent Floer equation defines a continuation map

$$\mathrm{HF}(\phi_{0,t}) \longrightarrow \mathrm{HF}(\phi_{1,t}).$$

The non-negativity assumption is precisely what gives the required energy bound near infinity. Standard continuation arguments imply that this map is independent of the chosen lift and almost complex structures, up to the canonical identifications above. Moreover, continuation maps are functorial under concatenation of non-negative homotopies [CHK23, Section 2.2].

Now fix a contact form $\alpha \in \mathcal{O}_\xi(\alpha_0)$, and let R_s^α denote its Reeb flow. For every regular parameter s, define

$$V_{\alpha,s}(\phi) := \mathrm{HF}\big(\phi_t^{-1} \circ R_{st}^\alpha\big).$$

Here "regular" means that the time-one map of the contact isotopy $\phi_t^{-1} \circ R_{st}^\alpha$ lies in the admissible locus, so that the above Floer cohomology group is defined.

For $s \leq s'$, choose a smooth cut-off function

$$\beta : \mathbb{R} \longrightarrow [0,1]$$

such that $\beta(\tau) = 0$ for $\tau \ll 0$, $\beta(\tau) = 1$ for $\tau \gg 0$, and $\beta'(\tau) \geq 0$. Consider the family of contact isotopies

$$\Phi_{\tau,t} := \phi_t^{-1} \circ R^\alpha_{\big((1-\beta(\tau))s + \beta(\tau)s'\big)t}.$$

Since the Reeb time is increased monotonically, this family gives a non-negative continuation datum. Therefore it induces a continuation map

$$c^{\alpha,\phi}_{s,s'} : V_{\alpha,s}(\phi) \longrightarrow V_{\alpha,s'}(\phi).$$

This map is independent of the choice of β. Furthermore, the usual homotopy-invariance and gluing properties of continuation maps imply

$$c^{\alpha,\phi}_{s,s} = \mathrm{id}, \quad c^{\alpha,\phi}_{s,s''} = c^{\alpha,\phi}_{s',s''} \circ c^{\alpha,\phi}_{s,s'}$$

whenever $s \leq s' \leq s''$ are regular parameters.

So Cant persistence module associated to (α, ϕ) is

$$\mathbb{V}_\alpha(\phi) := \Big(\{V_{\alpha,s}(\phi)\}_s, \{c^{\alpha,\phi}_{s,s'}\}_{s\leq s'}\Big),$$

where s ranges over regular parameters. The structure maps are the continuation maps defined above.

In other words, $V_\alpha(\phi)$ records how the Floer cohomology of $\phi_t^{-1} \circ R^\alpha_{st}$changes as the Reeb-time parameter s increases. Cant proves that this persistence module is locally constant on intervals containing no exceptional parameters [Can23, Lemma 1.2]. Therefore, after applying the structure theorem for locally finite persistence modules, one obtains a barcode $\mathcal{B}_\alpha(\phi)$. The bars are intervals of the form

$$[a, b), \qquad -\infty \leq a < b \leq +\infty.$$

Strictly speaking, Cant first defines $V_{\alpha,s}(\phi)$ for $s \notin \mathrm{Spec}_\alpha(\phi)$. Throughout the rest of this paper we use the standard extension of a tame persistence module across its spectrum, so that $V_{\alpha,s}(\phi)$ is defined for all $s > 0$ and is locally constant on each component of the complement of the spectrum. All continuation maps below are first constructed for regular parameters and then extended by this convention.

The logarithmic version of the persistence module and barcode is defined precisely as in the introduction.

Now we prove the contact invariance of Cant persistence module, Theorem 1.13, to conclude this section.

Proof of Theorem 1.13. We first compare the Reeb flows of $\beta = \chi^*\alpha$ and α. It is direct to check that

$$R^\beta_t = \chi^{-1} \circ R^\alpha_t \circ \chi.$$

Equivalently, we claim that $\chi_* R^\beta = R^\alpha$.

Now consider the Cant system $\phi_t^{-1} \circ R_{st}^{\beta}$ defining $V_{\beta,s}(\phi)$. Using the Reeb-flow identity above, we compute

$$\phi_t^{-1} \circ R_{st}^{\beta} = \phi_t^{-1} \circ \chi^{-1} \circ R_{st}^{\alpha} \circ \chi.$$

Conjugating this system by χ, we obtain

$$\chi \circ \big(\phi_t^{-1} \circ \chi^{-1} \circ R_{st}^{\alpha} \circ \chi\big) \circ \chi^{-1} = \chi \circ \phi_t^{-1} \circ \chi^{-1} \circ R_{st}^{\alpha} = (\chi \phi_t \chi^{-1})^{-1} \circ R_{st}^{\alpha}..$$

This is precisely the Cant system defining $V_{\alpha,s}(\chi\phi\chi^{-1})$.

It remains to explain why conjugate contact-at-infinity systems have canonically isomorphic Floer cohomology in Cant's framework, and why these isomorphisms commute with the persistence structure maps.

Since $\chi \in \mathrm{Cont}_0(M, \xi)$, choose a contact isotopy χ_τ, $\tau \in [0,1]$, such that $\chi_0 = \mathrm{id}$ and $\chi_1 = \chi$. By the path-lifting property for contact-at-infinity Floer data in Cant's setup, we may choose a contact-at-infinity exact symplectic isotopy Ξ_τ of W whose ideal restriction is χ_τ. Let $\Xi := \Xi_1$.

Choose admissible Floer data (Ψ_t, J_t) whose ideal restriction is $\phi_t^{-1} \circ R_{st}^{\beta}$. Then the conjugated data

$$(\widetilde{\Psi}_t, \widetilde{J}_t) := (\Xi \circ \Psi_t \circ \Xi^{-1},\ \Xi_* J_t)$$

has ideal restriction

$$(\chi \phi_t \chi^{-1})^{-1} \circ R_{st}^{\alpha}.$$

Admissibility is preserved under conjugation: non-degenerate fixed points are sent bijectively to non-degenerate fixed points, transversality of the Floer moduli spaces is preserved because the relevant Floer equations are identified by Ξ, and the contact-at-infinity conditions are preserved because Ξ is contact-at-infinity.

The fixed points of Ψ_1 are in bijection with the fixed points of $\widetilde{\Psi}_1$ by $x \mapsto \Xi(x)$. This gives a vector-space identification of the corresponding Floer chain groups.

If u is a Floer cylinder for (Ψ_t, J_t), then $\Xi \circ u$ is a Floer cylinder for $(\widetilde{\Psi}_t, \widetilde{J}_t)$. Conversely, every Floer cylinder for the conjugated data arises uniquely in this way. Thus conjugation gives a bijection between the zero-dimensional moduli spaces counted by the differential. Hence the above identification is a chain isomorphism.

Passing to Floer cohomology gives an isomorphism

$$\mathrm{HF}(\phi_t^{-1} \circ R_{st}^{\beta}) \cong \mathrm{HF}\big((\chi \phi_t \chi^{-1})^{-1} \circ R_{st}^{\alpha}\big),$$

or equivalently

$$\Theta_s : V_{\beta,s}(\phi) \xrightarrow{\cong} V_{\alpha,s}(\chi\phi\chi^{-1}).$$

We now verify compatibility with the persistence structure maps. Let $0 < s \le s'$. The structure map

$$c^{\beta,\phi}_{s,s'} : V_{\beta,s}(\phi) \longrightarrow V_{\beta,s'}(\phi)$$

is defined by Cant's continuation map associated to the continuation path

$$\phi_t^{-1} \circ R^{\beta}_{\lambda(\tau)t},$$

where $\lambda(\tau)$ is a non-decreasing function satisfying

$$\lambda(-\infty) = s, \qquad \lambda(+\infty) = s'.$$

Conjugating this whole continuation datum by χ gives

$$(\chi\phi_t\chi^{-1})^{-1} \circ R^{\alpha}_{\lambda(\tau)t}.$$

This is exactly the continuation datum defining the structure map

$$c^{\alpha,\chi\phi\chi^{-1}}_{s,s'} : V_{\alpha,s}(\chi\phi\chi^{-1}) \longrightarrow V_{\alpha,s'}(\chi\phi\chi^{-1}).$$

The same conjugation argument identifies the continuation cylinders on the two sides. Therefore

$$\Theta_{s'} \circ c^{\beta,\phi}_{s,s'} = c^{\alpha,\chi\phi\chi^{-1}}_{s,s'} \circ \Theta_s.$$

Hence $\{\Theta_s\}_{s>0}$ is an isomorphism of persistence modules.

Consequently the associated barcodes agree:

$$\mathcal{B}^+_\beta(\phi) = \mathcal{B}^+_\alpha(\chi\phi\chi^{-1}).$$

Applying the change of parameter $s \mapsto \log s$ gives

$$\log \mathcal{B}^+_\beta(\phi) = \log \mathcal{B}^+_\alpha(\chi\phi\chi^{-1}).$$

This proves the lemma. $\square$

3.2. **Filtered symplectic homology for the graph domains.** We now define the symplectic homology persistence module associated to the Liouville domain $W_\alpha \subseteq \widehat{W}$. The construction is the standard filtered symplectic homology construction for Liouville domains, using an inverse limit over compactly supported Hamiltonians; see [SZ21, Section 3.1] for construction in cotangent bundles and [CO18] for the general Liouville-domain framework. The point of the present subsection is to spell out the same construction for the graph domains arising from contact forms in the Liouville filling.

Again we work over the field $\mathbb{Z}_2$. All Floer homology groups, symplectic homology groups, persistence modules, and barcodes below are taken over $\mathbb{Z}_2$. Thus $\mathrm{HF}^{[a,b)}(H)$, $\mathrm{SH}^a(U_\alpha)$, and $S^t(U_\alpha)$ below denote ungraded $\mathbb{Z}_2$-vector spaces.

Remark 3.2. Since the stability statement we need is purely a statement about persistence modules and bottleneck distance, we do not use a grading. If one fixes an additional grading datum(for example a trivialization of the canonical bundle or a vanishing condition on $2c_1(T\hat{W})$), the same construction may be refined degree by degree as in [SZ21].

Recall that $(M, \xi = \ker \alpha_0)$ is a closed co-oriented contact manifold which admits a Liouville filling (W, ω, X). We write

$$\lambda := \iota_X \omega, \qquad \lambda|_{\partial W} = \alpha_0.$$

Let $(\widehat{W}, \widehat{\lambda})$ be the completion of W. On the positive symplectization end we use the coordinate

$$(u, x) \in \mathbb{R}_{>0} \times M, \qquad \widehat{\lambda} = u\alpha_0.$$

Thus $W = \{u \leq 1\}$ and $\partial W = \{u = 1\}$. For a contact form $\alpha = e^f \alpha_0 \in \mathcal{O}_\xi(\alpha_0)$ we consider the domain

$$W_\alpha = \{(u, x) \in \widehat{W} \mid u < e^{f(x)}\}.$$

Its boundary is $\partial W_\alpha = \{u = e^{f(x)}\}$ and the boundary contact form on Y_α is precisely α. In particular, the Reeb dynamics of ∂W^α are the Reeb dynamics of the contact form α.

The completion of W_α is canonically identified with $\widehat{W}$. Indeed, the cylindrical end of W_α is

$$[1, \infty)_r \times \partial W_\alpha, \qquad \widehat{\lambda}_\alpha = r\alpha,$$

and the map

$$[1,\infty)\times\partial W_\alpha \longrightarrow \{u\geq e^{f(x)}\}\subset \widehat{W},\qquad (r,(e^{f(x)},x))\longmapsto (re^{f(x)},x)$$

identifies $r\alpha$ with $u\alpha_0$. We shall therefore freely regard Floer data for W_α as Floer data on $\widehat{W}$, cylindrical with respect to the hypersurface ∂W_α.

Definition 3.3. Let R_α be the Reeb vector field of α, we say that W_α is non-degenerate if every closed Reeb orbit of $(\partial W_\alpha,\alpha)$ is non-degenerate. Its Reeb period spectrum is

$$\mathrm{Spec}(W_\alpha)=\mathrm{Spec}(\alpha):=\{T>0\mid \text{there exists a closed Reeb orbit of } R_\alpha \text{ of period } T\}.$$

For a non-degenerate contact form this spectrum is discrete, and below any fixed action level there are only finitely many Reeb orbits.

Hamiltonians and action filtration. Let $U=W_\alpha$. Denote by $\widehat{U}$ the completion of U, which we identify with $\widehat{W}$. For a smooth loop $x:S^1\to\widehat{U}$, define the symplectic action functional by

$$\mathcal{A}_H(x)=\int_0^1 H_t(x(t))\,dt-\int_{S^1}x^*\widehat{\lambda}.$$

Let $\mathcal{P}(H)$ be the set of all one-periodic Hamiltonian orbits of H, and define

$$\mathrm{Spec}(H):=\{\mathcal{A}_H(x)\mid x\in\mathcal{P}(H)\}.$$

For

$$\mathcal{H}_U:=\left\{H\in C^\infty(S^1\times\widehat{U})\;\middle|\;\operatorname{supp}H\subset S^1\times\mathrm{Int}(U)\right\}$$

and real numbers $a<b$, where $a,b\in\mathbb{R}\cup\{\pm\infty\}$, define

$$\mathcal{H}_{U,a,b}:=\{H\in\mathcal{H}_U\mid a,b\notin\mathrm{Spec}(H)\}.$$

When $b=+\infty$, write $\mathcal{H}_{U,a}:=\mathcal{H}_{U,a,+\infty}$.

In the construction of symplectic homology below we shall only use $[a,\infty)$-windows with $a>0$. Thus the constant orbits of action zero, which occur outside the support of a compactly supported Hamiltonian, do not enter the chain groups.

Almost complex structures. Let $\mathcal{J}_U$ be the class of S^1-families $J = \{J_t\}_{t\in S^1}$ of ω-compatible almost complex structures on $\widehat{U}$ satisfying the following conditions.

(1) Each J_t is ω-compatible.
(2) On the cylindrical end

$$[1,\infty)_r \times \partial W_\alpha, \qquad \widehat{\lambda} = r\alpha,$$

J_t is of contact type:

$$J_t(r\partial_r) = R_\alpha, \qquad J_t(R_\alpha) = -r\partial_r,$$

and

$$J_t(\xi) = \xi, \qquad d\alpha(\cdot, J_t\cdot)|_\xi \text{ is positive definite.}$$

(3) On the cylindrical end, J_t is independent of the radial coordinate r.
(4) On a compact subset containing the supports of the Hamiltonians under consideration, J_t is chosen generically so that all relevant Floer moduli spaces are regular.

For continuation maps one uses s-dependent families $J_{s,t}$ with the same cylindrical-end behavior and with fixed limiting almost complex structures as $s \to \pm\infty$. Because all Hamiltonians used below are compactly supported in $\mathrm{Int}(U)$, the usual maximum principle for contact-type almost complex structures prevents Floer cylinders with asymptotics in a compact set from escaping to the positive cylindrical end.

Filtered Floer homology. Let $H \in \mathcal{H}_{U,a,b}$ be non-degenerate in the action window $[a,b)$. Define $\mathrm{CF}^{[a,b)}(H)$ to be the $\mathbb{Z}_2$-vector space generated by all one-periodic Hamiltonian orbits $x \in \mathcal{P}(H)$ satisfying

$$\mathcal{A}_H(x) \in [a,b).$$

The Floer differential is defined by counting rigid solutions

$$u : \mathbb{R}_s \times S^1_t \longrightarrow \widehat{U}$$

of the Floer equation

$$\partial_s u + J_t(u)\big(\partial_t u - X_{H_t}(u)\big) = 0$$

with asymptotic conditions

$$\lim_{s\to-\infty} u(s,\cdot) = x_-, \qquad \lim_{s\to+\infty} u(s,\cdot) = x_+.$$

Over $\mathbb{Z}_2$, the coefficient of x_+ in ∂x_- is the modulo-two count of such rigid Floer cylinders. We denote the resulting filtered Floer homology by

$$\mathrm{HF}^{[a,b)}(H) := H\big(\mathrm{CF}^{[a,b)}(H), \partial\big).$$

If H is degenerate, $\mathrm{HF}^{[a,b)}(H)$ is defined by a sufficiently small non-degenerate perturbation whose action spectrum still avoids a and b. This gives a canonically well-defined group.

The Hamiltonian partial order and continuation maps. Define a standard partial order on $\mathcal{H}_{U,a,b}$ by

$$H_1 \preceq H_2 \quad \Longleftrightarrow \quad H_1(t,z) \geq H_2(t,z) \quad \text{for all } (t,z) \in S^1 \times U.$$

If $H_1 \preceq H_2$, choose a smooth homotopy H_s such that

$$H_s = H_1 \quad \text{for } s \ll 0, \qquad H_s = H_2 \quad \text{for } s \gg 0,$$

and

$$\partial_s H_s \leq 0.$$

Such a homotopy is called monotone. Counting solutions of the continuation equation

$$\partial_s u + J_{s,t}(u)\big(\partial_t u - X_{H_{s,t}}(u)\big) = 0$$

defines a continuation map

$$\sigma_{12} : \mathrm{HF}^{[a,b)}(H_1) \longrightarrow \mathrm{HF}^{[a,b)}(H_2).$$

This map is independent of the chosen monotone homotopy by a standard Floer continuation argument. Moreover, if $H_1 \preceq H_2 \preceq H_3$, then a similar argument to [BPS03] gives

$$\sigma_{13} = \sigma_{23} \circ \sigma_{12}.$$

The partially ordered set $(\mathcal{H}_{U,a,b}, \preceq)$ is downward directed: for any $H_2, H_3 \in \mathcal{H}_{U,a,b}$, there exists $H_1 \in \mathcal{H}_{U,a,b}$ such that

$$H_1 \preceq H_2, \qquad H_1 \preceq H_3.$$

Consequently,

$$\big\{\mathrm{HF}^{[a,b)}(H), \sigma_{HH'}\big\}_{H \in \mathcal{H}_{U,a,b}}$$

is an inverse system of $\mathbb{Z}_2$-vector spaces.

Explicitly, the inverse limit is

$$\varprojlim_{H\in\mathcal{H}_{U,a,b}} HF^{[a,b)}(H) = \left\{ (x_H)_H \in \prod_{H\in\mathcal{H}_{U,a,b}} \mathrm{HF}^{[a,b)}(H) \;\middle|\; \sigma_{HH'}(x_H) = x_{H'} \text{ whenever } H \preceq H' \right\}.$$

Definition 3.4 (Filtered symplectic homology)**.** Assume that $U = W_\alpha$ is non-degenerate. For $a > 0$ with

$$a \notin \mathrm{Spec}(U) = \mathrm{Spec}(\alpha),$$

define

$$\mathrm{SH}^a(U) := \varprojlim_{H\in\mathcal{H}_{U,a}} \mathrm{HF}^{[a,\infty)}(H).$$

Thus an element of $SH^a(U)$ is a compatible family

$$(x_H)_{H\in\mathcal{H}_{U,a}}, \qquad x_H \in \mathrm{HF}^{[a,\infty)}(H),$$

such that

$$\sigma_{HH'}(x_H) = x_{H'} \quad \text{whenever } H \preceq H'.$$

Remark 3.5. Stojisavljević–Zhang define $\mathrm{SH}^a(U;\mathfrak{c})$ for a fixed free homotopy class $\mathfrak{c}$, and their class-wise functoriality requires the Liouville embeddings to preserve $\mathfrak{c}$. In the present paper we use the total symplectic homology

$$\mathrm{SH}^a(U) = \bigoplus_{\mathfrak{c}} \mathrm{SH}^a(U;\mathfrak{c}),$$

whenever this decomposition is invoked. The total Viterbo transfer map is defined for ordinary exact Liouville embeddings and does not require choosing or preserving a particular free homotopy class. Therefore the class-triviality hypothesis is unnecessary for the total persistence module. This is the version naturally compatible with the comparison

$$d_{\mathrm{SBM}}(W_{\alpha_1}, W_{\alpha_2}) \le d_{\mathrm{CBM}}(\alpha_1, \alpha_2),$$

where no free homotopy class is specified.

Persistence maps. Let $0 < a \le b, a, b \notin \mathrm{Spec}(U)$. For every $H \in \mathcal{H}_{U,a} \cap \mathcal{H}_{U,b}$, there is a natural map

$$\iota^H_{a,b} : \mathrm{HF}^{[a,\infty)}(H) \longrightarrow \mathrm{HF}^{[b,\infty)}(H).$$

induced by the quotient map

$$\mathrm{CF}^{[a,\infty)}(H) := \mathrm{CF}^{(-\infty,\infty)}(H)/\,\mathrm{CF}^{(-\infty,a)}(H) \longrightarrow \mathrm{CF}^{[b,\infty)}(H) = \mathrm{CF}^{(-\infty,\infty)}(H)/\,\mathrm{CF}^{(-\infty,b)}(H).$$

and commute with continuation maps. So there is a well-defined map on inverse limits:

$$\iota_{a,b} : \mathrm{SH}^a(U) \longrightarrow \mathrm{SH}^b(U).$$

by $\iota_{a,b}\big((x_H)_H\big) = \big(\iota^H_{a,b}(x_H)\big)_H$. The maps satisfy

$$\iota_{a,a} = \mathrm{id}, \qquad \iota_{a,c} = \iota_{b,c} \circ \iota_{a,b} \quad \text{for } 0 < a \leq b \leq c.$$

If $a \in \mathrm{Spec}(U)$, choose $\epsilon > 0$ such that $(a, a+\epsilon]$ contains no point of $\mathrm{Spec}(U)$ other than possibly a itself, and define

$$\mathrm{SH}^a(U) := \varprojlim_{t \in (a, a+\epsilon]} \mathrm{SH}^t(U).$$

Equivalently, we define $\mathrm{SH}^a(U)$ by requiring that

$$\iota_{a,a+\epsilon} : \mathrm{SH}^a(U) \longrightarrow \mathrm{SH}^{a+\epsilon}(U)$$

be an isomorphism.

Definition 3.6 (Symplectic persistence module)**.** The total symplectic persistence module of $U = W_\alpha$ is

$$\mathbb{SH}_*(U) := \left(\{\mathrm{SH}^a(U)\}_{a>0}, \{\iota_{a,b} : \mathrm{SH}^a(U) \to \mathrm{SH}^b(U)\}_{0<a\leq b}\right).$$

Recall that for any $t \in \mathbb{R}$, define

$$\log \mathrm{SH}^t_*(U) := \log SH^{e^t}_*(U).$$

For $s \leq t$, let

$$\iota^{\log}_{s,t} := \iota^{\mathrm{SH}}_{e^s, e^t} : \log \mathrm{SH}^s_*(U) \longrightarrow \log \mathrm{SH}^t_*(U).$$

The logarithmic symplectic persistence module is

$$\log \mathbb{SH}_*(U) := \left(\{\log \mathrm{SH}^t_*(U)\}_{t\in\mathbb{R}}, \{\iota^{\log}_{s,t}\}_{s\leq t}\right).$$

Since U is non-degenerate, the module $\log \mathbb{SH}_*(U)$ is pointwise finite-dimensional under the standard finiteness assumption for non-degenerate Liouville domains. Its spectrum is

$$\log \mathrm{Spec}(U) = \{\log T \mid T \in \mathrm{Spec}(U)\}.$$

Let $\log \mathcal{B}_*(U)$ be the corresponding barcodes, then the bars are of the form

$$[p, q), \qquad p < q \leq +\infty.$$

4. Proof of main results

4.1. **Proof of Theorem 1.11.** We use the following formal properties of Cant's continuation-map construction. By standard arguments [Flo89, SZ92], these continuation maps are invariant under homotopies of continuation data with fixed endpoints, up to chain homotopy and hence on cohomology. Finally, we use the standard gluing property that continuation maps compose under concatenation of continuation data; Cant invokes this explicitly in [Can23, Section 2.3.1].

We first record the elementary contact-Hamiltonian calculation underlying the order morphisms. Suppose $\alpha_2 = f\alpha_1$, where $f > 0$. We claim that R^{α_2} is the α_1-contact Hamiltonian vector field with Hamiltonian $1/f$. Indeed, $\alpha_2(R^{\alpha_2}) = 1$ gives $\alpha_1(R^{\alpha_2}) = 1/f$. For $v \in \xi = \ker \alpha_1$, using $d\alpha_2 = d(f\alpha_1) = df \wedge \alpha_1 + f\, d\alpha_1$, we compute

$$0 = d\alpha_2(R^{\alpha_2}, v) = (df \wedge \alpha_1)(R^{\alpha_2}, v) + f\, d\alpha_1(R^{\alpha_2}, v).$$

Since $\alpha_1(v) = 0$ and $\alpha_1(R^{\alpha_2}) = 1/f$, the first term is

$$(df \wedge \alpha_1)(R^{\alpha_2}, v) = -df(v)\alpha_1(R^{\alpha_2}) = -\frac{df(v)}{f}.$$

Hence $0 = -df(v)/f + f\, d\alpha_1(R^{\alpha_2}, v)$, so $d\alpha_1(R^{\alpha_2}, v) = df(v)/f^2$. Since $d(1/f)(v) = -df(v)/f^2$, we have $d\alpha_1(R^{\alpha_2}, v) = -d(1/f)(v)$ for every $v \in \xi$. Together with $\alpha_1(R^{\alpha_2}) = 1/f$, this is precisely the defining equation for the α_1-contact Hamiltonian vector field with Hamiltonian $1/f$. Therefore the Reeb system $t \mapsto R^{\alpha_2}_{st}$ is generated, with respect to α_1, by the autonomous contact Hamiltonian s/f. Similarly, $t \mapsto R^{\alpha_1}_{st}$ is generated by the constant Hamiltonian s.

We now prove (A1). Assume $\alpha_1 \preceq \alpha_2$, so $\alpha_2 = f\alpha_1$ with $f \geq 1$. For fixed $s > 0$, the two Reeb systems are generated, with respect to α_1, by s/f and s. Because $f \geq 1$, one has $s/f \leq s$. Choose the linear homotopy of contact Hamiltonians

$$H^\tau_s = (1-\tau)\frac{s}{f} + \tau s, \qquad 0 \leq \tau \leq 1.$$

Then $\partial_\tau H^\tau_s = s - s/f \geq 0$. Let $\rho_{\tau,t}$ be the contact isotopy generated by H^τ_s. The condition $\partial_\tau H^\tau_s \geq 0$ implies that $\tau \mapsto \rho_{\tau,1}$ is a non-negative path. Indeed, if K_τ is the contact Hamiltonian of the variation vector field $\partial_\tau \rho_{\tau,1} \circ \rho^{-1}_{\tau,1}$, then the standard contact variation formula expresses K_τ as an integral of $\partial_\tau H^\tau_s$ multiplied by positive conformal factors; hence $K_\tau \geq 0$.

Now consider the path of systems $\Psi_{\tau,t} := \phi_t^{-1}\circ\rho_{\tau,t}$. Its time-one path is $\Psi_{\tau,1} = \phi_1^{-1}\circ\rho_{\tau,1}$. Left multiplication by the fixed contactomorphism ϕ_1^{-1} preserves non-negativity: if ρ_τ has contact Hamiltonian $K_\tau \geq 0$, then $\phi_1^{-1}\circ\rho_\tau$ has contact Hamiltonian equal to K_τ multiplied by the positive conformal factor of ϕ_1^{-1}. Thus $\tau \mapsto \Psi_{\tau,1}$ is non-negative. Cant's continuation construction therefore gives a map

$$h^{\phi,s}_{\alpha_2,\alpha_1} : \mathrm{HF}(\phi_t^{-1}\circ R^{\alpha_2}_{st}) \longrightarrow \mathrm{HF}(\phi_t^{-1}\circ R^{\alpha_1}_{st}),$$

i.e. a map $h^{\phi,s}_{\alpha_2,\alpha_1} : V_{\alpha_2,s}(\phi) \to V_{\alpha_1,s}(\phi)$. By homotopy invariance of continuation maps, this map is independent of the chosen monotone interpolation from s/f to s; hence it is canonical.

It remains to prove that the maps $h^{\phi,s}_{\alpha_2,\alpha_1}$ form a persistence morphism. Let $0 < s \leq s'$. We need to prove

$$c^{\alpha_1,\phi}_{s,s'} \circ h^{\phi,s}_{\alpha_2,\alpha_1} = h^{\phi,s'}_{\alpha_2,\alpha_1} \circ c^{\alpha_2,\phi}_{s,s'}.$$

In α_1-Hamiltonian notation, the left-hand side is represented by the concatenated monotone path $s/f \to s \to s'$, while the right-hand side is represented by the concatenated monotone path $s/f \to s'/f \to s'$.

Both are monotone paths from the same initial Hamiltonian s/f to the same final Hamiltonian s'. After smoothing the corners of the concatenated paths, they lie in the convex space of monotone Hamiltonian homotopies from s/f to s'. Indeed, if H_0^τ and H_1^τ are two such smoothed monotone homotopies with fixed endpoints, then $(1-r)H_0^\tau + rH_1^\tau$, $r \in [0,1]$, is again a monotone homotopy with the same endpoints. After composing on the left by ϕ_t^{-1}, this gives a homotopy through Cant-admissible non-negative continuation data with fixed endpoints. Therefore, by homotopy invariance of Cant's continuation maps, the two concatenated continuation maps agree. This proves that

$$c^{\alpha_1,\phi}_{s,s'} \circ h^{\phi,s}_{\alpha_2,\alpha_1} = h^{\phi,s'}_{\alpha_2,\alpha_1} \circ c^{\alpha_2,\phi}_{s,s'}.$$

Thus the required square commutes.

We next prove (A2). Assume $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$. Write $\alpha_2 = f\alpha_1$ and $\alpha_3 = g\alpha_1$. The order assumptions imply $1 \leq f \leq g$. At level s, the three Reeb systems are generated, with respect to α_1, by s, s/f, and s/g, respectively. Since $1 \leq f \leq g$, we have $s/g \leq s/f \leq s$. The map $h^{\phi,s}_{\alpha_3,\alpha_2}$ is the continuation map from s/g to s/f, and $h^{\phi,s}_{\alpha_2,\alpha_1}$ is the continuation map from s/f to s. Their composition is the continuation map associated to the concatenated monotone path $s/g \to s/f \to s$. By functoriality under concatenation, this equals the direct continuation map from s/g to s, namely $h^{\phi,s}_{\alpha_3,\alpha_1}$. Thus

$$h^{\phi,s}_{\alpha_3,\alpha_1} = h^{\phi,s}_{\alpha_2,\alpha_1} \circ h^{\phi,s}_{\alpha_3,\alpha_2}$$

for every $s > 0$. Since all maps involved are persistence morphisms by (A1), this equality holds as an equality of persistence morphisms. Taking $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ gives $h^{\phi}_{\alpha,\alpha} = \mathrm{id}_{V^+_\alpha(\phi)}$. This proves (A2).

We now prove (A3). Let $A > 0$. The Reeb vector field rescales as $R^{A\alpha} = A^{-1}R^{\alpha}$. Therefore $R^{A\alpha}_{Ast} = R^{\alpha}_{st}$ for every $s > 0$ and every t. Consequently,

$$\phi_t^{-1} \circ R^{A\alpha}_{Ast} = \phi_t^{-1} \circ R^{\alpha}_{st}.$$

Thus $V_{\alpha,s}(\phi) = \mathrm{HF}(\phi_t^{-1} \circ R^{\alpha}_{st})$ and $V_{A\alpha,As}(\phi) = \mathrm{HF}(\phi_t^{-1} \circ R^{A\alpha}_{Ast})$ are defined by the same contact-at-infinity system. We define $\rho^{\alpha,\phi}_{A,s} : V_{\alpha,s}(\phi) \to V_{A\alpha,As}(\phi)$ to be this canonical identification. Replacing A by $1/A$ immediately gives $(\rho^{\alpha,\phi}_{A})^{-1} = \rho^{A\alpha,\phi}_{1/A}$.

Now let $0 < s \le s'$. The structure map $c^{\alpha,\phi}_{s,s'}$ is induced by the continuation path

$$\phi_t^{-1} \circ R^{\alpha}_{\lambda(\tau)t}, \qquad \lambda(\tau) = (1-\beta(\tau))s + \beta(\tau)s',$$

where β is a non-decreasing cutoff function. The structure map $c^{A\alpha,\phi}_{As,As'}$ is induced by $\phi_t^{-1} \circ R^{A\alpha}_{A\lambda(\tau)t}$. Since $R^{A\alpha}_{A\lambda(\tau)t} = R^{\alpha}_{\lambda(\tau)t}$, the two continuation data are identical under the identifications $\rho^{\alpha,\phi}_{A,s}$ and $\rho^{\alpha,\phi}_{A,s'}$. Therefore

$$c^{A\alpha,\phi}_{As,As'} \circ \rho^{\alpha,\phi}_{A,s} = \rho^{\alpha,\phi}_{A,s'} \circ c^{\alpha,\phi}_{s,s'}.$$

This proves (A3).

Finally, we prove (A4). Assume $\alpha_1 \preceq \alpha_2$. Write $\alpha_2 = f\alpha_1$ with $f \ge 1$. Then $A\alpha_2 = f(A\alpha_1)$, so $A\alpha_1 \preceq A\alpha_2$. At level s, the map $h^{\phi,s}_{\alpha_2,\alpha_1}$ is represented, in α_1-Hamiltonian notation, by increasing the Hamiltonian s/f to s. After applying the rescaling identifications $\rho^{\alpha_2,\phi}_{A,s}$ and $\rho^{\alpha_1,\phi}_{A,s}$, the source and target become the systems $\phi_t^{-1} \circ R^{A\alpha_2}_{Ast}$ and $\phi_t^{-1} \circ R^{A\alpha_1}_{Ast}$. With respect to the contact form $A\alpha_1$, the first of these is generated by s/f, and the second by s. Thus the order morphism $h^{\phi,As}_{A\alpha_2,A\alpha_1}$ is represented by the same monotone continuation data as $h^{\phi,s}_{\alpha_2,\alpha_1}$ under the canonical rescaling identifications. Consequently,

$$h^{\phi,As}_{A\alpha_2,A\alpha_1} \circ \rho^{\alpha_2,\phi}_{A,s} = \rho^{\alpha_1,\phi}_{A,s} \circ h^{\phi,s}_{\alpha_2,\alpha_1}.$$

This proves (A4), and hence the theorem.

4.2. **Proof of Theorem 1.10.** Now we use Theorem 1.11 to prove Theorem 1.10.

Since $C^{-1}\alpha \preceq \beta \preceq C\alpha$, (A1) gives order morphisms

$$h^{\phi}_{C\alpha,\beta} : V^{+}_{C\alpha}(\phi) \longrightarrow V^{+}_{\beta}(\phi)$$

and

$$h^{\phi}_{\beta,C^{-1}\alpha} : V^{+}_{\beta}(\phi) \longrightarrow V^{+}_{C^{-1}\alpha}(\phi).$$

(A3) also gives rescaling isomorphisms

$$\rho^{\alpha,\phi}_{C} : V^{+}_{\alpha}(\phi) \xrightarrow{\cong} (m_C)^{*}V^{+}_{C\alpha}(\phi),$$

and

$$\rho^{\alpha,\phi}_{1/C} : V^{+}_{\alpha}(\phi) \xrightarrow{\cong} (m_{1/C})^{*}V^{+}_{C^{-1}\alpha}(\phi).$$

For every $s > 0$, define

$$F_s : V_{\alpha,s}(\phi) \longrightarrow V_{\beta,Cs}(\phi)$$

by

$$F_s := h^{\phi,Cs}_{C\alpha,\beta} \circ \rho^{\alpha,\phi}_{C,s}.$$

Similarly, define

$$G_s : V_{\beta,s}(\phi) \longrightarrow V_{\alpha,Cs}(\phi)$$

by

$$G_s := \left(\rho^{\alpha,\phi}_{1/C,Cs}\right)^{-1} \circ h^{\phi,s}_{\beta,C^{-1}\alpha}.$$

We first show that F is a multiplicative-shift persistence morphism. Let $0 < s \leq s'$. We need to prove

$$c^{\beta,\phi}_{Cs,Cs'} \circ F_s = F_{s'} \circ c^{\alpha,\phi}_{s,s'}.$$

Using the definition of F_s, the left-hand side is

$$c^{\beta,\phi}_{Cs,Cs'} \circ h^{\phi,Cs}_{C\alpha,\beta} \circ \rho^{\alpha,\phi}_{C,s}.$$

Since $h^{\phi}_{C\alpha,\beta}$ is a persistence morphism, there is

$$c^{\beta,\phi}_{Cs,Cs'} \circ h^{\phi,Cs}_{C\alpha,\beta} = h^{\phi,Cs'}_{C\alpha,\beta} \circ c^{C\alpha,\phi}_{Cs,Cs'}.$$

Therefore

$$c^{\beta,\phi}_{Cs,Cs'} \circ F_s = h^{\phi,Cs'}_{C\alpha,\beta} \circ c^{C\alpha,\phi}_{Cs,Cs'} \circ \rho^{\alpha,\phi}_{C,s}.$$

By (A3),

$$c^{C\alpha,\phi}_{Cs,Cs'} \circ \rho^{\alpha,\phi}_{C,s} = \rho^{\alpha,\phi}_{C,s'} \circ c^{\alpha,\phi}_{s,s'}.$$

Hence

$$c^{\beta,\phi}_{Cs,Cs'} \circ F_s = h^{\phi,Cs'}_{C\alpha,\beta} \circ \rho^{\alpha,\phi}_{C,s'} \circ c^{\alpha,\phi}_{s,s'}.$$

But

$$F_{s'} = h^{\phi,Cs'}_{C\alpha,\beta} \circ \rho^{\alpha,\phi}_{C,s'}.$$

Thus

$$c^{\beta,\phi}_{Cs,Cs'} \circ F_s = F_{s'} \circ c^{\alpha,\phi}_{s,s'}.$$

So F is a multiplicative-shift persistence morphism.

The proof that G is a multiplicative-shift persistence morphism is the same. Indeed, it follows from the persistence-morphism property of $h^{\phi}_{\beta,C^{-1}\alpha}$ and from the compatibility of the rescaling maps $\rho^{\alpha,\phi}_{1/C}$ with the persistence structure maps.

We now verify the two interleaving identities. First, we prove

$$G_{Cs} \circ F_s = c^{\alpha,\phi}_{s,C^2 s}.$$

By definition,

$$G_{Cs} \circ F_s = \left(\rho^{\alpha,\phi}_{1/C,C^2 s}\right)^{-1} \circ h^{\phi,Cs}_{\beta,C^{-1}\alpha} \circ h^{\phi,Cs}_{C\alpha,\beta} \circ \rho^{\alpha,\phi}_{C,s}.$$

Since

$$C^{-1}\alpha \preceq \beta \preceq C\alpha,$$

the functoriality part (A2) gives

$$h^{\phi,Cs}_{\beta,C^{-1}\alpha} \circ h^{\phi,Cs}_{C\alpha,\beta} = h^{\phi,Cs}_{C\alpha,C^{-1}\alpha}.$$

Hence

$$G_{Cs} \circ F_s = \left(\rho^{\alpha,\phi}_{1/C,C^2 s}\right)^{-1} \circ h^{\phi,Cs}_{C\alpha,C^{-1}\alpha} \circ \rho^{\alpha,\phi}_{C,s}.$$

We now identify this map. The rescaling map $\rho^{\alpha,\phi}_{C,s}$identifies $\phi_t^{-1} \circ R^{\alpha}_{st}$ with $\phi_t^{-1} \circ R^{C\alpha}_{Cst}$. The map $h^{\phi,Cs}_{C\alpha,C^{-1}\alpha}$ is the order continuation map from $\phi_t^{-1} \circ R^{C\alpha}_{Cst}$ to $\phi_t^{-1} \circ R^{C^{-1}\alpha}_{Cst}$. But

$$R^{C^{-1}\alpha}_{Cst} = R^{\alpha}_{C^2 st}.$$

Finally, $\left(\rho^{\alpha,\phi}_{1/C,C^2 s}\right)^{-1}$ identifies $V_{C^{-1}\alpha,Cs}(\phi)$ with $V_{\alpha,C^2 s}(\phi)$. Thus the entire composition is exactly the Cant continuation map

$$V_{\alpha,s}(\phi) \longrightarrow V_{\alpha,C^2 s}(\phi)$$

induced by increasing the Reeb parameter from s to $C^2 s$. That is,

$$G_{Cs} \circ F_s = c^{\alpha,\phi}_{s,C^2 s}.$$

Similarly there is $F_{Cs} \circ G_s = c^{\beta,\phi}_{s,C^2s}$. Then we have shown that F and G define a multiplicative C-interleaving between $V^+_\alpha(\phi)$ and $V^+_\beta(\phi)$.

Now pass to logarithmic parameters. For $t \in \mathbb{R}$, then $\widehat{F}_t = F_{e^t}$ and $\widehat{G}_t = G_{e^t}$ gives a $\log C$-interleaving between $\log V_\alpha(\phi)$ and $\log V_\beta(\phi)$. Then we obtain the desired result.

4.3. **Proof of Theorem 1.15.** Now we prove the properties of the symplectic homology barcode defined in the last section, which is, Theorem 1.15. We first prove (B1). Let

$$\phi : U \hookrightarrow V$$

be an exact Liouville embedding. Thus

$$\phi^*\lambda_V - \lambda_U = df$$

for some smooth function f on U.

In order to avoid artificial periodic orbits created by smoothing an extension by zero, we compute the inverse limit over the following cofinal subclass of Hamiltonians. We restrict to Hamiltonians $H \in \mathcal{H}_{U,a}$ such that H vanishes on a neighborhood of the boundary of its support. This subclass is cofinal for the inverse system in the following sense: for every $K \in \mathcal{H}_{U,a}$ there exists H in the subclass with $H \preceq K$, i.e. $H \geq K$, and with $a \notin \mathrm{Spec}(H)$. Therefore restricting the indexing set to this subclass does not change the inverse limit.

We use the above cofinal subclass of Hamiltonians $H \in \mathcal{H}_{U,a}$ which vanish on a neighborhood of ∂U. For such an H, define

$$(\phi_! H)(t,z) = \begin{cases} H(t, \phi^{-1}(z)), & z \in \phi(U), \\ 0, & z \notin \phi(U). \end{cases}$$

Since H vanishes near ∂U and ϕ is an embedding, this extension by zero is smooth and compactly supported in $\mathrm{Int}(V)$.

The Hamiltonian vector fields are related by

$$d\phi(X_H) = X_{\phi_! H} \circ \phi$$

on the support of H. The non-constant one-periodic orbits of $\phi_! H$ in the action window $[a, \infty)$ are precisely the images under ϕ of the one-periodic orbits of H in the same action window. The constant orbits outside the support of $\phi_! H$ have action

0, and hence do not enter the window because $a > 0$. Thus, in the relevant filtered complex, periodic orbits are identified by $x(t) \mapsto \phi(x(t))$.

We now check that the action filtration is preserved. For a loop $x : S^1 \to U$,

$$\begin{aligned}\mathcal{A}_{\phi_! H}(\phi \circ x) &= \int_0^1 (\phi_! H)_t(\phi(x(t)))\, dt - \int_{S^1} (\phi \circ x)^* \lambda_V \\ &= \int_0^1 H_t(x(t))\, dt - \int_{S^1} x^*(\phi^* \lambda_V) \\ &= \int_0^1 H_t(x(t))\, dt - \int_{S^1} x^*(\lambda_U + df) \\ &= \int_0^1 H_t(x(t))\, dt - \int_{S^1} x^* \lambda_U - \int_{S^1} d(f \circ x) \\ &= \mathcal{A}_H(x),\end{aligned}$$

because x is a closed loop. Therefore the orbit bijection induces an isomorphism of filtered Floer chain complexes

$$\Phi_H : \mathrm{CF}_*^{[a,\infty)}(H) \xrightarrow{\cong} \mathrm{CF}_*^{[a,\infty)}(\phi_! H),$$

and hence an isomorphism

$$\Phi_H : \mathrm{HF}_*^{[a,\infty)}(H) \xrightarrow{\cong} \mathrm{HF}_*^{[a,\infty)}(\phi_! H).$$

Let

$$p_K^V : \mathrm{SH}_*^a(V) \longrightarrow \mathrm{HF}_*^{[a,\infty)}(K)$$

be the canonical projection from the inverse limit, for $K \in \mathcal{H}_{V,a}$. We define h_ϕ^a by specifying its components in the inverse limit defining $\mathrm{SH}_*^a(U)$, that is, for every $H \in \mathcal{H}_{U,a}$, set

$$p_H^U \circ h_\phi^a := \Phi_H^{-1} \circ p_{\phi_! H}^V.$$

We must check that these components are compatible with the inverse system over $\mathcal{H}_{U,a}$. Suppose $H_1 \preceq H_2$, then $\phi_! H_1 \preceq \phi_! H_2$. Let

$$\sigma_{12}^U : \mathrm{HF}_*^{[a,\infty)}(H_1) \to \mathrm{HF}_*^{[a,\infty)}(H_2)$$

and

$$\sigma_{\phi_! H_1, \phi_! H_2}^V : \mathrm{HF}_*^{[a,\infty)}(\phi_! H_1) \to \mathrm{HF}_*^{[a,\infty)}(\phi_! H_2)$$

be the monotone continuation maps. The orbit-identification maps commute with continuation, because the monotone homotopy from $\phi_! H_1$ to $\phi_! H_2$ is the push-forward of the monotone homotopy from H_1 to H_2. Thus

$$\Phi_{H_2} \circ \sigma^U_{12} = \sigma^V_{\phi_! H_1, \phi_! H_2} \circ \Phi_{H_1}.$$

For $x \in \mathrm{SH}^a_*(V)$, the coherence of the inverse-limit projections gives

$$p^V_{\phi_! H_2}(x) = \sigma^V_{\phi_! H_1, \phi_! H_2}\big(p^V_{\phi_! H_1}(x)\big).$$

Therefore

$$\begin{aligned} \sigma^U_{12}\big(\Phi^{-1}_{H_1} p^V_{\phi_! H_1}(x)\big) &= \Phi^{-1}_{H_2} \sigma^V_{\phi_! H_1, \phi_! H_2}\big(p^V_{\phi_! H_1}(x)\big) \\ &= \Phi^{-1}_{H_2} p^V_{\phi_! H_2}(x). \end{aligned}$$

So the map

$$h^a_\phi = \Phi^{-1}_H \circ p^V_{\phi_! H} : \mathrm{SH}^a_*(V) \to \mathrm{SH}^a_*(U).$$

is well defined.

We now prove compatibility with the persistence structure maps. For $a \leq b$, the quotient map

$$\mathrm{CF}^{[a,\infty)}_*(H) \to \mathrm{CF}^{[b,\infty)}_*(H)$$

commutes with the orbit-identification isomorphism Φ_H, because Φ_H preserves action exactly. Hence

$$\Phi^{[b,\infty)}_H \circ q^U_{a,b,H} = q^V_{a,b,\phi_! H} \circ \Phi^{[a,\infty)}_H,$$

where $q_{a,b}$ denotes the filtered quotient map. Passing to homology and then to inverse limits gives

$$h^b_\phi \circ \iota^V_{a,b} = \iota^U_{a,b} \circ h^a_\phi.$$

This proves (B1).

We prove (B2). Let

$$U \xrightarrow{\phi} V \xrightarrow{\psi} W$$

be exact Liouville embeddings. For $H \in \mathcal{H}_{U,a}$, one has

$$(\psi \circ \phi)_! H = \psi_!(\phi_! H).$$

Moreover, the orbit-identification isomorphism for $\psi \circ \phi$ is the composition of the orbit-identification isomorphisms for ϕ and ψ:

$$\Phi^{\psi\circ\phi}_H = \Phi^\psi_{\phi_! H} \circ \Phi^\phi_H.$$

Using the componentwise definition of restriction maps, for every $x \in \mathrm{SH}^a_*(W)$ we compute the H-component of $h^a_\phi h^a_\psi(x)$:

$$\begin{aligned} p^U_H\big(h^a_\phi h^a_\psi(x)\big) &= (\Phi^\phi_H)^{-1} p^V_{\phi_! H}\big(h^a_\psi(x)\big) \\ &= (\Phi^\phi_H)^{-1}(\Phi^\psi_{\phi_! H})^{-1} p^W_{\psi_!(\phi_! H)}(x) \\ &= \big(\Phi^\psi_{\phi_! H} \circ \Phi^\phi_H\big)^{-1} p^W_{(\psi\circ\phi)_! H}(x) \\ &= (\Phi^{\psi\circ\phi}_H)^{-1} p^W_{(\psi\circ\phi)_! H}(x) \\ &= p^U_H\big(h^a_{\psi\circ\phi}(x)\big). \end{aligned}$$

Since this holds for every $H \in \mathcal{H}_{U,a}$, the two inverse-limit elements agree. Thus

$$h^a_{\psi\circ\phi} = h^a_\phi \circ h^a_\psi.$$

This proves (B2).

We prove (B3). Let $\delta_C : \widehat{W} \to \widehat{W}$ denote the Liouville scaling by factor C, so that $\delta_C(U) = CU$ and $\delta_C^*\lambda = C\lambda$. For $H \in \mathcal{H}_{U,a}$, define a Hamiltonian $H^{(C)} \in \mathcal{H}_{CU,Ca}$ by

$$H^{(C)}(t, \delta_C(z)) := C\, H(t,z).$$

The Hamiltonian orbits of H are in bijection with the Hamiltonian orbits of $H^{(C)}$ by

$$x(t) \longmapsto \delta_C(x(t)).$$

The action is multiplied by C:

$$\begin{aligned} \mathcal{A}_{H^{(C)}}(\delta_C \circ x) &= \int_0^1 C H_t(x(t))\, dt - \int_{S^1} (\delta_C \circ x)^*\lambda \\ &= C \int_0^1 H_t(x(t))\, dt - \int_{S^1} x^*(\delta_C^*\lambda) \\ &= C \int_0^1 H_t(x(t))\, dt - C \int_{S^1} x^*\lambda \\ &= C\mathcal{A}_H(x). \end{aligned}$$

Therefore there is a canonical chain isomorphism

$$\mathrm{CF}^{[a,\infty)}_*(H) \xrightarrow{\cong} \mathrm{CF}^{[Ca,\infty)}_*(H^{(C)}),$$

and hence an isomorphism

$$\mathrm{HF}^{[a,\infty)}_*(H) \xrightarrow{\cong} \mathrm{HF}^{[Ca,\infty)}_*(H^{(C)}).$$

This construction respects monotone continuation maps and hence passes to inverse limits, giving

$$r_C^U(a) : \mathrm{SH}_*^a(U) \xrightarrow{\cong} \mathrm{SH}_*^{Ca}(CU).$$

The compatibility with persistence maps follows from the same action-scaling identity. Namely, the quotient

$$\mathrm{CF}_*^{[a,\infty)}(H) \to \mathrm{CF}_*^{[b,\infty)}(H)$$

is carried to the quotient

$$\mathrm{CF}_*^{[Ca,\infty)}(H^{(C)}) \to \mathrm{CF}_*^{[Cb,\infty)}(H^{(C)}).$$

Therefore

$$r_C^U(b) \circ \iota_{a,b}^U = \iota_{Ca,Cb}^{CU} \circ r_C^U(a).$$

It remains to prove the compatibility with restriction maps. Let $\phi : U \hookrightarrow V$ be exact. The scaled embedding

$$\phi(C) : CU \hookrightarrow CV$$

satisfies

$$\phi(C)(\delta_C z) = \delta_C(\phi(z)).$$

For every $H \in \mathcal{H}_{U,a}$, one has the identity of Hamiltonians

$$(\phi(C))_!\big(H^{(C)}\big) = (\phi_! H)^{(C)}.$$

Moreover, the orbit-identification map for $\phi(C)$ applied to $H^{(C)}$ is the scaling of the orbit-identification map for ϕ applied to H. Thus the two ways of going around the square

$$\begin{array}{ccc} \mathrm{HF}_*^{[a,\infty)}(\phi_! H) & \longrightarrow & \mathrm{HF}_*^{[a,\infty)}(H) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HF}_*^{[Ca,\infty)}((\phi_! H)^{(C)}) & \longrightarrow & \mathrm{HF}_*^{[Ca,\infty)}(H^{(C)}) \end{array}$$

agree. Passing to inverse limits gives

$$r_C^U(a) \circ h_\phi^a = h_{\phi(C)}^{Ca} \circ r_C^V(a).$$

This proves (B3).

Now we prove (B4). Let $i : U \hookrightarrow CU$ be the standard inclusion. The identity

$$h_i^{Ca} \circ r_C^U(a) = \iota_{a,Ca}^U$$

is the standard compatibility between the Viterbo restriction map for the Liouville scaling inclusion and the persistence comparison map. Indeed, in the standard construction of filtered symplectic homology, the scaling map identifies the action window $[a, \infty)$ for U with the action window $[Ca, \infty)$ for CU, while the restriction map for i identifies the scaled complex back with the quotient of the original complex by generators of action below Ca. This gives the natural quotient map

$$\mathrm{CF}^{[a,\infty)}(H) \longrightarrow \mathrm{CF}^{[Ca,\infty)}(H),$$

and hence, after passing to homology and the inverse limit,

$$h_i^{Ca} \circ r_C^U(a) = \iota_{a,Ca}^U.$$

For the second assertion, let $j : C^{-1}U \hookrightarrow CU$ be the standard inclusion. Applying the first assertion first to the inclusion $C^{-1}U \hookrightarrow U$ and then to the inclusion $U \hookrightarrow CU$, and using functoriality of Viterbo restriction maps, we obtain that, after the scaling identifications

$$\mathrm{SH}^a(CU) \cong \mathrm{SH}^{a/C}(U), \qquad \mathrm{SH}^a(C^{-1}U) \cong \mathrm{SH}^{Ca}(U),$$

the map $h_j^a : \mathrm{SH}^a(CU) \to \mathrm{SH}^a(C^{-1}U)$ becomes

$$\iota_{a/C,Ca}^U : \mathrm{SH}^{a/C}(U) \longrightarrow \mathrm{SH}^{Ca}(U).$$

This proves the second assertion of (B4).

Finally, we prove (B5). We use the standard isotopy invariance of Viterbo restriction maps. Namely, if two exact Liouville embeddings $\phi_0, \phi_1 : U \hookrightarrow V$ are connected by a smooth isotopy $\phi_\tau : U \hookrightarrow V$ through exact Liouville embeddings, then the associated Viterbo restriction maps agree: $h_{\phi_0} = h_{\phi_1}$; see [CO18].

Indeed, after choosing primitives for $\phi_\tau^* \lambda_V - \lambda_U$, the exact embedding isotopy can be treated, on a collar of the image, as an exact Hamiltonian deformation. The naturality of filtered Floer complexes under exact Hamiltonian conjugation and the homotopy invariance of continuation maps imply that the component maps in the inverse-limit definition of Viterbo restriction are independent of τ. This argument is standard in the construction of Viterbo transfer maps.

Applying this to the isotopy from the standard inclusion $i_U : U \hookrightarrow V$ to ϕ, we obtain

$$h_\phi = h_{i_U}.$$

This proves (B5), and hence completes the proof of the theorem.

4.4. **Proof of Theorem 1.14.** Now we give a proof of Theorem 1.14.

Let

$$r > d_{\mathrm{SBM}}(U,V)$$

be arbitrary, and set $C = e^r$. By the definition of the symplectic Banach–Mazur distance, there exist exact Liouville embeddings

$$C^{-1}U \xrightarrow{\phi} V \xrightarrow{\psi} CU$$

with the corresponding strongly unknotted condition. We shall prove that the logarithmic persistence modules $\log \mathbb{SH}_*(U)$ and $\log \mathbb{SH}_*(V)$ are r-interleaved.

For any $t \in \mathbb{R}$, let $a = e^t$. Define a map

$$F_t : \log \mathrm{SH}_*^t(U) \longrightarrow \log \mathrm{SH}_*^{t+r}(V).$$

by

$$F_t := h_\psi^{Ca} \circ r_C^U(a).$$

Similarly, define

$$G_t : \log \mathrm{SH}_*^t(V) \longrightarrow \log \mathrm{SH}_*^{t+r}(U)$$

by

$$G_t := r_C^{C^{-1}U}(a) \circ h_\phi^a,$$

We first verify that $F = \{F_t\}_{t\in\mathbb{R}}$ is a morphism of persistence modules

$$F : \log \mathbb{SH}_*(U) \longrightarrow \log \mathbb{SH}_*(V)[r].$$

Let $t_0 \leq t_1$ and set $a_i = e^{t_i}$. We must prove

$$F_{t_1} \circ \iota^U_{a_0,a_1} = \iota^V_{Ca_0,Ca_1} \circ F_{t_0}.$$

Using the naturality of scaling with respect to the persistence maps (B3), we have

$$r_C^U(a_1) \circ \iota^U_{a_0,a_1} = \iota^{CU}_{Ca_0,Ca_1} \circ r_C^U(a_0).$$

Since h_ψ is a persistence morphism by (B1), we also have

$$h_\psi^{Ca_1} \circ \iota^{CU}_{Ca_0,Ca_1} = \iota^V_{Ca_0,Ca_1} \circ h_\psi^{Ca_0}.$$

Therefore

$$\begin{aligned} F_{t_1} \circ \iota^U_{a_0,a_1} &= h^{Ca_1}_\psi \circ r^U_C(a_1) \circ \iota^U_{a_0,a_1} \\ &= h^{Ca_1}_\psi \circ \iota^{CU}_{Ca_0,Ca_1} \circ r^U_C(a_0) \\ &= \iota^V_{Ca_0,Ca_1} \circ h^{Ca_0}_\psi \circ r^U_C(a_0) \\ &= \iota^V_{Ca_0,Ca_1} \circ F_{t_0}. \end{aligned}$$

Thus F is a persistence morphism. The same argument proves that G is also a persistence morphism.

We now compute the two interleaving composites. Fix $t \in \mathbb{R}$ and set $a = e^t$. Then the composite

$$G_{t+r} \circ F_t : \mathrm{SH}^a_*(U) \longrightarrow \mathrm{SH}^{C^2a}_*(U)$$

is

$$\begin{aligned} G_{t+r} \circ F_t &= \left(r^{C^{-1}U}_C(Ca) \circ h^{Ca}_\phi\right) \circ \left(h^{Ca}_\psi \circ r^U_C(a)\right) \\ &= r^{C^{-1}U}_C(Ca) \circ h^{Ca}_\phi \circ h^{Ca}_\psi \circ r^U_C(a). \end{aligned}$$

By (B2),

$$h^{Ca}_\phi \circ h^{Ca}_\psi = h^{Ca}_{\psi\circ\phi}.$$

Therefore

$$G_{t+r} \circ F_t = r^{C^{-1}U}_C(Ca) \circ h^{Ca}_{\psi\circ\phi} \circ r^U_C(a).$$

Also by (B5), one has

$$h^{Ca}_{\psi\circ\phi} = h^{Ca}_j.$$

Note that for $j : C^{-1}U \hookrightarrow CU$, by the observation in (B4), the map

$$r^{C^{-1}U}_C(Ca) \circ h^{Ca}_j \circ r^U_C(a)$$

is exactly the persistence comparison map

$$\iota^U_{a,C^2a} : \mathrm{SH}^a_*(U) \longrightarrow \mathrm{SH}^{C^2a}_*(U).$$

Thus

$$G_{t+r} \circ F_t = \iota^U_{a,C^2a}.$$

In logarithmic notation this says

$$G[r] \circ F = \iota^{\log \mathrm{SH}_*(U)}_{\bullet,\bullet+2r}.$$

The computation of the other composite is almost a word-by-word repetition of above so we omit here.

We have therefore constructed persistence morphisms

$$F : \log \mathbb{SH}_*(U) \to \log \mathbb{SH}_*(V)[r], \qquad G : \log \mathbb{SH}_*(V) \to \log \mathbb{SH}_*(U)[r],$$

whose shifted composites are the corresponding $2r$-structure maps. Hence $\log \mathbb{SH}_*(U)$ and $\log \mathbb{SH}_*(V)$ are r-interleaved. Since the modules are locally finite, the isometry theorem gives

$$d_{\text{bottle}}\big(\log \mathbb{SH}_*(U), \log \mathbb{SH}_*(V)\big) \leq r.$$

Since $r > d_{\text{SBM}}(U, V)$ was arbitrary, we conclude

$$d_{\text{bottle}}\big(\log \mathbb{SH}_*(U), \log \mathbb{SH}_*(V)\big) \leq d_{\text{SBM}}(U, V).$$

References

[AAC25] Habib Alizadeh, Marcelo S. Atallah, and Dylan Cant, *Lagrangian intersections and the spectral norm in convex-at-infinity symplectic manifolds*, Math. Z. **310** (2025), no. 1, Paper No. 2, 48. MR 4879080

[AM13] Peter Albers and Will J. Merry, *Translated points and Rabinowitz Floer homology*, J. Fixed Point Theory Appl. **13** (2013), no. 1, 201–214. MR 3071949

[AM19] Marcelo R. R. Alves and Matthias Meiwes, *Dynamically exotic contact spheres in dimensions* ≥ 7, Comment. Math. Helv. **94** (2019), no. 3, 569–622. MR 4014780

[BC24] Filip Broćić and Dylan Cant, *Bordism classes of loops and Floer's equation in cotangent bundles*, J. Fixed Point Theory Appl. **26** (2024), no. 3, Paper No. 26, 29. MR 4756196

[BHS21] Lev Buhovsky, Vincent Humilière, and Sobhan Seyfaddini, *The action spectrum and C^0 symplectic topology*, Math. Ann. **380** (2021), no. 1-2, 293–316. MR 4263685

[BL15] Ulrich Bauer and Michael Lesnick, *Induced matchings and the algebraic stability of persistence barcodes*, J. Comput. Geom. **6** (2015), no. 2, 162–191. MR 3333456

[BPS03] Paul Biran, Leonid Polterovich, and Dietmar Salamon, *Propagation in Hamiltonian dynamics and relative symplectic homology*, Duke Math. J. **119** (2003), no. 1, 65–118. MR 1991647

[Can23] Dylan Cant, *Shelukhin's Hofer distance and a symplectic cohomology barcode for contactomorphisms*, arXiv preprint 2309.00529 (2023).

[CB15] William Crawley-Boevey, *Decomposition of pointwise finite-dimensional persistence modules*, J. Algebra Appl. **14** (2015), no. 5, 1550066, 8. MR 3323327

[CGG24] Erman Cineli, Viktor L. Ginzburg, and Başak Z. Gürel, *Topological entropy of Hamiltonian diffeomorphisms: a persistence homology and Floer theory perspective*, Math. Z. **308** (2024), no. 4, Paper No. 73, 38. MR 4822772

[CGH23] Dan Cristofaro-Gardiner and Richard Hind, *On the large-scale geometry of domains in an exact symplectic 4-manifold*, arXiv preprint 2311.06421v2 (2023).

[CHK23] D. Cant, J. Hedicke, and E. Kilgore, *Extensible positive loops and vanishing of symplectic cohomology*, arXiv preprint 2311.18267 (2023).

[CO18] Kai Cieliebak and Alexandru Oancea, *Symplectic homology and the Eilenberg-Steenrod axioms*, Algebr. Geom. Topol. **18** (2018), no. 4, 1953–2130, Appendix written jointly with Peter Albers. MR 3797062

[Die25] Patricia Dietzsch, *Bounding the Lagrangian Hofer metric via barcodes*, J. Topol. Anal. **17** (2025), no. 5, 1289–1311. MR 4926607

[Fer24] Rafael. Fernandes, *Barcode entropy and wrapped Floer homology*, arXiv preprint 2410.05528 (2024).

[Flo89] Andreas Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611. MR 987770

[FLS26] Elijah Fender, Sangjin Lee, and Beomjun Sohn, *Barcode entropy for Reeb flows on contact manifolds with Liouville fillings*, Commun. Contemp. Math. **28** (2026), no. 3, Paper No. 2550044. MR 5023169

[FZ25] Qi Feng and Jun Zhang, *Spectrally-large scale geometry via set-heaviness*, arXiv preprint 2503.14961v2 (2025).

[Gab72] Peter Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. 6 (1972), 309. MR 332887

[GU19] Jean Gutt and Michael Usher, *Symplectically knotted codimension-zero embeddings of domains in $\mathbb{R}^4$*, Duke Math. J. **168** (2019), no. 12, 2299–2363. MR 3999447

[Hof90] H. Hofer, *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25–38. MR 1059642

[Kaw22] Yusuke Kawamoto, *On C^0-continuity of the spectral norm for symplectically non-aspherical manifolds*, Int. Math. Res. Not. IMRN (2022), no. 21, 17187–17230. MR 4504915

[LM95] Francois Lalonde and Dusa McDuff, *The geometry of symplectic energy*, Ann. of Math. (2) **141** (1995), no. 2, 349–371. MR 1324138

[MU19] Will J. Merry and Igor Uljarevic, *Maximum principles in symplectic homology*, Israel J. Math. **229** (2019), no. 1, 39–65. MR 3905596

[Pol01] Leonid Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001. MR 1826128

[PRSZ20] Leonid Polterovich, Daniel Rosen, Karina Samvelyan, and Jun Zhang, *Topological persistence in geometry and analysis*, University Lecture Series, vol. 74, American Mathematical Society, Providence, RI, [2020] ©2020. MR 4249570

[PS16] Leonid Polterovich and Egor Shelukhin, *Autonomous Hamiltonian flows, Hofer's geometry and persistence modules*, Selecta Math. (N.S.) **22** (2016), no. 1, 227–296. MR 3437837

[PSS17] Leonid Polterovich, Egor Shelukhin, and Vukasin Stojisavljević, *Persistence modules with operators in Morse and Floer theory*, Mosc. Math. J. **17** (2017), no. 4, 757–786. MR 3734662

[Rud00] M. Rudelson, *Distances between non-symmetric convex bodies and the MM^*-estimate*, Positivity **4** (2000), no. 2, 161–178. MR 1755679

[RZ21] Daniel Rosen and Jun Zhang, *Relative growth rate and contact Banach-Mazur distance*, Geom. Dedicata **215** (2021), 1–30. MR 4330331

[San12] Sheila Sandon, *On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$*, Internat. J. Math. **23** (2012), no. 2, 1250042, 14. MR 2890476

[Sch00] Matthias Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), no. 2, 419–461. MR 1755825

[Sch06] Felix Schlenk, *Applications of Hofer's geometry to Hamiltonian dynamics*, Comment. Math. Helv. **81** (2006), no. 1, 105–121. MR 2208800

[Sey12] Sobhan Seyfaddini, *Descent and C^0-rigidity of spectral invariants on monotone symplectic manifolds*, J. Topol. Anal. **4** (2012), no. 4, 481–498. MR 3021773

[She17] Egor Shelukhin, *The Hofer norm of a contactomorphism*, J. Symplectic Geom. **15** (2017), no. 4, 1173–1208. MR 3734612

[SZ92] Dietmar Salamon and Eduard Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45** (1992), no. 10, 1303–1360. MR 1181727

[SZ21] Vukasin Stojisavljević and Jun Zhang, *Persistence modules, symplectic Banach-Mazur distance and Riemannian metrics*, Internat. J. Math. **32** (2021), no. 7, Paper No. 2150040, 76. MR 4284596

[Ush13] Michael Usher, *Hofer's metrics and boundary depth*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 1, 57–128. MR 3087390

[Ush22] ———, *Symplectic Banach-Mazur distances between subsets of $\mathbb{C}^n$*, J. Topol. Anal. **14** (2022), no. 1, 231–286. MR 4411106

[UZ16] Michael Usher and Jun Zhang, *Persistent homology and Floer-Novikov theory*, Geom. Topol. **20** (2016), no. 6, 3333–3430. MR 3590354

[ZZ26] Jun Zhang and Antong Zhu, *Geometry and dynamics on Liouville domains in $T^*\mathbb{T}^2$*, arXiv preprint 2603.29253 (2026).

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