偏微分方程

Ref. ①《Elliptic PDE of second order》— Gilbarg, Trudinger

②麻讲义 ③《Elliptic PDE》— 韩.林 ④Evans

Rmk. 调和 ⟺ 任意球中有均值性质

一. 调和函数

1. 均值性质: $u\in C^2(\Omega)\cap C(\bar{\Omega})$ $\Delta u \overset{\ge}{=} 0$. 则 $\forall B=B_R(y)\subset\Omega$

$$u(y) \overset{\le}{\underset{\ge}{=}} \frac{1}{\omega_n R^n}\int_B u\,dx \quad ① \qquad u(y) \overset{\le}{=} \frac{1}{n\omega_n R^{n-1}}\int_{\partial B} u\,ds \quad ②$$

Pf. $B_\rho = B_\rho(y) \xrightarrow{Green} \int_{\partial B_\rho}\frac{\partial u}{\partial \nu}ds = \int_{B_\rho}\Delta u\,dx \overset{\ge}{=} 0$

极坐标(r,ω)

$$\int_{\partial B_\rho}\frac{\partial u}{\partial r}(y+\rho\omega)ds = \rho^{n-1}\int_{|\omega|=1}\frac{\partial u}{\partial r}(y+\rho\omega)d\omega$$

$$= \rho^{n-1}\frac{\partial}{\partial\rho}\int_{|\omega|=1}u(y+\rho\omega)d\omega$$

$$\lim_{\rho\to 0}\rho^{1-n}\int_{\partial B_\rho}u\,ds = n\omega_n u(y) \Rightarrow ②$$

积分 $\Downarrow$ ①

$$= \rho^{n-1}\frac{\partial}{\partial\rho}\left[\rho^{1-n}\int_{\partial B_\rho}u\,ds\right]$$

2. 极值原理 ①(弱) $u\in C^2(\Omega)\cap C(\bar{\Omega})$ $\Delta u = 0$. 则u最值取于$\partial\Omega$.

(强) ② ……, $\Delta u \ge 0$ ($\le$) 且$\exists y\in\Omega$ st $u(y)=\sup_{\Omega} u$ ($\inf$) 则u常值

Pf 1. $\Rightarrow$ ② $\Rightarrow$ ① 或者构造$v=u+\varepsilon x_i^2$ 证①

Cor ① $u\in C^2(\Omega)\cap C(\bar{\Omega})$. $\Delta u \overset{\le}{\ge} 0$ 则 $\inf_{\partial\Omega}u \le u(x) \le \sup_{\partial\Omega} u$

② u, v ……, $\Delta u = \Delta v = 0$ in Ω. $u=v$ on $\partial\Omega \Rightarrow u=v$.

3. Harnack不等式. $u\ge 0$调和. 则$\forall\Omega'\subset\Omega$有界 $\exists C=C(n,\Omega',\Omega)$

s.t. $\sup_{\Omega'} u \le C\inf_{\Omega'} u$.

Pf $y\in\Omega$. $B_{4R}(y)\subset\Omega$. 则$\forall x_1, x_2\in B_R(y)$

$$u(x_1) = \frac{1}{\omega_n R^n}\int_{B_R(x_1)} u \le \frac{1}{\omega_n R^n}\int_{B_{2R}(y)} u$$

$$u(x_2) = \frac{1}{\omega_n (3R)^n}\int_{B_{3R}(x_2)} u \ge \frac{1}{\omega_n R^n}\int_{B_{2R}(y)} u$$

$$\Rightarrow \sup_{B_R(y)} u \le 3^n \inf_{B_R(y)} u$$

取 $x_1, x_2 \in \bar{\Omega}$ s.t. $u(x_1) = \sup_{\bar{\Omega}} u \quad u(x_2) = \inf_{\bar{\Omega}} u$.

Γ: 连接 x_1, x_2. 取 $R < \frac{1}{4} d(\Gamma, \partial\Omega)$. 用 N 个 R-ball cove Γ

$$\Rightarrow u(x_1) \le 3^{Nn} u(x_2)$$

4. 整体梯度估计: $|Du|$ 最大值取于 $\partial\Omega$

pf $\varphi(x) = |Du(x)|^2 = \sum_i u_i^2 \Rightarrow \varphi_i(x) = 2\sum_j u_j u_{ij}$

$$\varphi_{ii}(x) = 2\sum_j u_{ij}^2 + u_j u_{iij}$$

$$\Rightarrow \Delta\varphi = 2\sum_{i,j} u_{ij}^2 + 2\underbrace{\sum_{i,j} u_j u_{iij}}_{0} \ge 0$$

5 梯度内估计: $u \in C^3(\Omega) \cap C(\bar{\Omega})$. $\Delta u = 0 \quad B_{x_0}(r) \subset\subset \Omega$

平移到0 ↓ 有 $\sup_{B_{x_0}(r)} |Du| \le \frac{C_n}{r} \max_{\bar{\Omega}} |u|$

Pf. $\xi = r^2 - x^2 \qquad \varphi = \xi^2 |Du|^2 + \alpha u^2 \qquad \xi_i = -2x_i \quad |D\xi|^2 = 4x^2$

$\Delta\xi = -2n$

$$\varphi_i = (\xi^2)_i |Du|^2 + \xi^2 (|Du|^2)_i + 2\alpha u u_i$$

$$\Delta\varphi = \Delta(\xi^2)|Du|^2 + \xi^2 \Delta(|Du|^2) + 2(\xi^2)_i (|Du|^2)_i + 2\alpha u \Delta u + 2\alpha |Du|^2$$

$$= (2\xi\Delta\xi + 2|D\xi|^2)|Du|^2 + \underbrace{\xi^2 \Delta(|Du|^2)}_{\ge 0 \text{ 由4}} + 8\xi\xi_i u_j u_{ij} + 2\alpha|Du|^2$$

$$\ge (8|x|^2 - 4n\xi)|Du|^2 - 2\xi^2 |D^2u|^2 - 8|D\xi|^2|Du|^2 + 2\alpha|Du|^2$$

$$= |Du|^2 (2\alpha - 24|x|^2 - 4n\xi)$$

取 $\alpha = 2(n+6)r^2 \quad \Rightarrow r^4 |Du(0)|^2 \le \sup_{B_r(0)} \varphi \le \alpha \sup_{B_r(0)} u^2$

$$\Rightarrow |Du(0)| \le \frac{\sqrt{2n+12}}{r} \sup_{\partial B_r(0)} |u|$$

6. 对数梯度估计: $u>0$. $u\in C^3(\Omega)\cap C(\bar{\Omega})$ $\Delta u=0$.

$B_r(0)\subset\subset\Omega$ 则 $\sup\limits_{B_{\frac{r}{2}}(0)} |D\log u| \leq \frac{C_n}{r}$ $\Rightarrow$ Harnack

Pf. $v=\log u$ $u_i=e^v v_i$ $u_{ii}=e^v v_{ii}+e^v (v_i)^2$

$\Rightarrow \Delta u = e^v(\Delta v + |Dv|^2) \Rightarrow \Delta v = -|Dv|^2$ $(\triangle)$

$\xi = r^2 - x^2$ $\varphi = \xi^2 |Dv|^2$ 设 φ 在 x_0 处最大. 则在 x_0 处

$\varphi_i = 0 \Rightarrow$ $\xi(|Dv|^2)_i = -2\xi_i |Dv|^2$ $(*)$

$$\Delta\varphi = \underline{\xi^2\Delta(|Dv|^2)} + \Delta(\xi^2)|Dv|^2 + \underbrace{2(\xi^2)_i(|Dv|^2)_i}_{=4\xi\xi_i(|Dv|^2)_i \overset{(*)}{=} -8|D\xi|^2|Dv|^2}$$

$$= \xi^2\left(\sum_i\Big(2\sum_j v_j v_{ji}\Big)_i\right) = \xi^2\Big(2\sum_{i,j} v_{ij}^2 + 2\sum_j v_j(\Delta v)_j\Big)$$

$$\overset{(\triangle)}{=} 2\xi^2\Big(\sum_{i,j} v_{ij}^2 - \sum_j v_j |Dv|^2_j\Big)$$

$$\overset{(*)}{=} 2\xi^2 \sum_{i,j} v_{ij}^2 + 4\sum_j v_j \xi\xi_j |Dv|^2$$

$$\Rightarrow \overset{\substack{0\\ \veebar}}{\Delta\varphi(x_0)} = 2\xi^2 \underbrace{\sum_{i,j} v_{ij}^2}_{\geq \sum_i v_{ii}^2 \geq \frac{|\Delta v|^2}{n} = \frac{|Dv|^4}{n}} + 4\sum_j \xi\xi_j v_j |Dv|^2 + \big(\Delta(\xi^2) - 8|D\xi|^2\big)|Dv|^2$$

$$\Rightarrow \frac{2}{n}\xi^2|Dv|^2 \leq -4\sum_j \xi\xi_j v_j + 8|D\xi|^2 - \Delta(\xi^2)$$

$$\leq \frac{1}{n}\xi^2|Dv|^2 + 4n|D\xi|^2 + 8|D\xi|^2 - \Delta(\xi^2)$$

$$\Rightarrow \Big(\frac{1}{n}\xi^2|Dv|^2\Big)(x_0) \leq (16n+24)r^2 \Rightarrow \varphi(x_0) \leq n(16n+24)r^2$$

又 $\sup\limits_{B_{\frac{r}{2}}(0)} \varphi = \sup\limits_{B_{\frac{r}{2}}(0)} (r^2-x^2)^2|Dv|^2 \geq \frac{9}{16}r^4 \sup\limits_{B_{\frac{r}{2}}(0)} |Dv|^2$

$$\Rightarrow \frac{9}{16}r^4 \sup_{B_{\frac{r}{2}}(0)} |Dv|^2 \leq \sup_{B_r(0)} \varphi = \varphi(x_0) \leq n(16n+24)r^2$$

$$\Rightarrow \sup_{B_{\frac{r}{2}}(0)} |D\log u| \leq \frac{C_n}{r}$$

二. 椭圆方程的估计

$Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u \qquad a^{ij} = a^{ji}$

椭圆: $[a^{ij}(x)]$ 正定. $\Rightarrow 0 < \lambda(x)|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2$

一致椭圆: $\frac{\Lambda}{\lambda}$ 有界

一般假设 $\frac{|b^i(x)|}{\lambda(x)} \le const < \infty$

1. 弱极值① Ω 有界 L 椭圆 $c=0$ $u \in C^2(\Omega) \cap C(\bar{\Omega})$

若 $Lu \underset{\le}{\ge} 0$ in Ω 则 $\sup_{\Omega} u = \sup_{\partial\Omega} u$ (inf)

Pf. $Lu>0$: 则对 x_0 在内部最大 $Du(x_0)=0$ $D^2u(x_0)$ 非正

则 $Lu(x_0) = a^{ij}(x_0) D_{ij} u(x_0) \le 0$. 矛盾!

$Lu \ge 0$: $\frac{|b^i|}{\lambda} \le b_0 = const$ $\quad a_{11} \ge \lambda$

$\Rightarrow \exists \gamma$ s.t. $Le^{\gamma x_1} = (\gamma^2 a^{11} + \gamma b^1) e^{\gamma x_1} \ge \lambda(\gamma^2 - \gamma b_0) e^{\gamma x_1} > 0$

则 $\forall \varepsilon$ $\sup_{\Omega}(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\gamma x_1})$ $\quad \varepsilon \to 0$ 即可.

2. 弱极值② Ω 有界 L 椭圆 $c \le 0$ $\quad Lu \overset{\le}{\ge} 0$ $\quad u \in C^2(\Omega) \cap C(\bar{\Omega})$

则 $\sup_{\Omega} u \le \sup_{\partial\Omega} u^+$ $\left(\inf_{\Omega} u \ge \inf_{\partial\Omega} u^-\right)$ 特别若 $Lu=0 \Rightarrow \sup_{\Omega}|u| = \sup_{\partial\Omega}|u|$

Pf. $Lu>0$: 同上有 $Lu(x_0) \le 0$ 矛盾!

$Lu \ge 0$ 取 $\gamma^2 - \gamma b_0 + c > 0$ 同上.

3. 强极值: L 一致椭圆 $Lu \overset{\le 0}{\ge 0}$ ① $c=0$ 若 u 在内部取最大(小)则 u 常值

② c 无条件 $Lu \ge 0$ $u \le 0$ 则 u 恒 <0 或恒 $=0$

Hopf引理 Ω有界内部球 $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ $Lu\geq 0$ $c\leq 0$

若$x_0\in\partial\Omega$ s.t. $u(x)<u(x_0)$ $(\forall x\in\bar{\Omega})$ $u(x_0)\leq 0$ 则$\frac{\partial u}{\partial n}(x_0)>0$

pf. $\varphi=u+\varepsilon h$ $h=e^{-\alpha|x|^2}-e^{-\alpha R^2}\geq 0$ 其中R为x_0处内切球的半径

$h_i=-2\alpha e^{-\alpha|x|^2}x_i$ $h_{ij}=-2\alpha e^{-\alpha|x|^2}\delta_{ij}+4\alpha^2e^{-\alpha|x|^2}x_ix_j$

$$=e^{-\alpha|x|^2}(4\alpha^2x_ix_j-2\alpha\delta_{ij})$$

$$\Rightarrow Lh=e^{-\alpha|x|^2}(4\alpha^2a^{ij}x_ix_j-2\alpha\Sigma a_{ii})-2\alpha e^{-\alpha|x|^2}b^ix_i+ch$$

$$\geq e^{-\alpha|x|^2}(4\alpha^2\lambda R^2-2\alpha n\Lambda-2\alpha|b|R+c)$$

取α足够大 则 $Lh>0$ on $A=B_R(y)-B_\rho(y)$

$g=u-u(x_0)+\varepsilon h$ 取ε s.t. $g\leq 0$ on $\partial B_\rho(y)$ $\Rightarrow g\leq 0$ on ∂A

则 $Lg\geq -cu(x_0)\geq 0$ in A

由弱极值原理 $g\leq 0$ in A $\Rightarrow \frac{\partial u}{\partial n}(x_0)\geq-\varepsilon\frac{\partial v}{\partial n}(x_0)$

$$=-\varepsilon v'(R)>0$$

Hopf引理→强极值:

① 若u在内部取最大$M\geq 0$ $\Sigma=\{x\in\Omega\,|\,u(x)=M\}$闭 若$\Sigma\neq\Omega$

取$B_R\subset\Omega\backslash\Sigma$ $x_2\in\partial B_R\cap\Sigma$ 则B_R上$u(x)\leq u(x_2)$

由Hopf $\frac{\partial u}{\partial n}\Big|_{x_2}>0$ 与x_0最大 $|Du(x_0)|=0$矛盾

② $c(x)=c^+(x)-c^-(x)$ $\tilde{L}u=Lu-c^+u$

$u\leq 0\Rightarrow\tilde{L}u\geq 0$ 由强极值 若$u(x_0)=0$ 则$u\equiv 0$ ✓

4.有界估计 $Lu \overset{(=)}{\geq} f$ Ω有界 L 椭圆 $c \leq 0$ $u \in C^2(\Omega) \cap C(\bar{\Omega})$

↓

§3.3 Gilbarg 则 $\sup_{\Omega} u(|u|) \leq \sup_{\partial\Omega} u^+(|u|) + C \sup_{\Omega} \frac{|f^-|}{\lambda}\left(\frac{|f|}{\lambda}\right)$

$C = C(\operatorname{diam}\Omega, \beta = \sup \frac{|b|}{\lambda})$ 若$\Omega \subset \{0 < x_1 < d\}$ 则可令 $C = e^{(\beta+1)d} - 1$

Pf 设$\Omega \subset \{0 < x_1 < d\}$ $L_0 = a^{ij} D_{ij} + b^i D_i$ 对 $\alpha \geq \beta + 1$

$L_0 e^{\alpha x_1} = (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta) e^{\alpha x_1} \geq \lambda$

令 $v = \sup_{\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f^-|}{\lambda}$ $\Rightarrow Lv = L_0 v + cv \leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda}$

⇓

$v - u \geq 0$ on $\partial\Omega$ $L(v-u) \leq -\lambda\left(\sup_{\Omega} \frac{|f^-|}{\lambda} + \frac{f}{\lambda}\right) \leq 0$

$v - u \geq 0$ in Ω

令 $C = e^{\alpha d} - 1$ 则 $\sup_{\Omega} u \leq \sup_{\Omega} v = \sup_{\Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda}$

§2.4 柳青

↑

5.整体梯度估计 $Lu = f$ in Ω有界 (不妨)$\Omega \subset \{x_1 > 0\}$) $c = 0$ $u \in C^3(\Omega) \cap C(\bar{\Omega})$

$\|u\|_{L^\infty} \leq M$ 则 $\sup_{\Omega} |Du|^2 \leq \sup_{\partial\Omega} |Du|^2 + C$ $C = C(\lambda, \|a\|_1, \|b\|_1, \operatorname{diam}\Omega, \|u\|_\infty, \|f\|_1)$

Pf $\varphi = |Du|^2 + \alpha u^2 + e^{\beta x_1}$ $\varphi_i = 2u_k u_{ki} + 2\alpha u u_i + \beta \delta_{1i} e^{\beta x_1}$

$\varphi_{ij} = 2u_k u_{kij} + 2u_{kj} u_{ki} + 2\alpha u_i u_j + 2\alpha u u_{ij} + \beta^2 \delta_{1i} \delta_{1j} e^{\beta x_1}$

$L\varphi = 2a^{ij} \underline{u_k u_{ijk}} + 2a^{ij} u_{kj} u_{ki} + 2\alpha a^{ij} u_i u_j + 2\alpha a^{ij} u u_{ij}$

$+ \beta^2 e^{\beta x_1} a^{11} + \underline{2b^i u_k u_{ki}} + 2\alpha b^i u u_i + \beta e^{\beta x_1} b^1$

又 $a^{ij} u_{ijk} + b^i u_{ik} + c u_k + a^{ij}_k u_{ij} + b^i_k u_i + c_k u = f_k$

$\Rightarrow L\varphi = (\underline{2u_k f_k - 2a^{ij}_k u_{ij} u_k - 2b^i_k u_i u_k} + \underbrace{2a^{ij} u_{kj} u_{ki}}_{\geq \lambda |D^2 u|^2})$

$+ \underbrace{2\alpha a^{ij} u_i u_j}_{\geq 2\lambda |Du|^2 \alpha} + \underbrace{(2\alpha a^{ij} u u_{ij} + 2\alpha b^i u u_i)}_{= 2\alpha u f} + \beta^2 e^{\beta x_1} a^{11} + \beta e^{\beta x_1} b^1$

$\geq -C|Du|^2 - C|Du| - C|D^2 u||Du|$

$\geq -C|Du|^2 - C|Du| - \varepsilon_0 |D^2 u|^2 - \frac{C^2}{\varepsilon_0} |Du|^2$

$$\Rightarrow L\varphi \geq (\lambda-\varepsilon_0)|D^2u|^2+\left(2\lambda\alpha-C-\frac{C^2}{\varepsilon_0}\right)|Du|^2-C|Du|$$

$$+\beta^2\lambda e^{\beta x_1}-C\beta e^{\beta x_1}$$ 取 α. β 充分大. ε_0 小 $L\varphi>0$ ✓

原证

§3.4 - Gilbarg 叙述.

↑

6. 梯度内估计

都有界 $Lu=f$ $u\in C^3(\Omega)$ $B_r(0)\subset\subset\Omega$. $c=0$ (设 $\Omega\subset\{0<x_1<d\}$)

则 $\sup\limits_{B_{\frac{r}{2}}(0)}|Du|\leq\frac{C}{r}(1+\sup\limits_{\Omega}u)$ C与 a^{ij}. b^i. d. f. n 有关.

Pf. $\varphi=\xi^2|Du|^2+\alpha u^2+e^{\beta x_1}$

$$L\varphi=\underline{a^{ij}|Du|^2(\xi^2)_{ij}}+\underbrace{a^{ij}\xi^2(|Du|^2)_{ij}}_{①}+\underbrace{2a^{ij}(\xi^2)_i(|Du|^2)_j}_{②}$$

$$+\underbrace{a^{ij}\alpha(u^2)_{ij}}_{③}+\underbrace{\beta^2a^{11}e^{\beta x_1}}_{\geq\lambda\beta^2e^{\beta x_1}}+\underbrace{|Du|^2b^i(\xi^2)_i}_{\geq-C|Du|^2}+\underbrace{b^i\xi^2(|Du|^2)_i}_{④}+\alpha b^i(u^2)_i+\underbrace{\beta e^{\beta x_1}b^1}_{\geq-C\beta e^{\beta x_1}}$$

$\geq-C|Du|^2$

$$\frac{①}{2}=\frac{1}{2}\xi^2a^{ij}(u_k^2)_{ij}=\underbrace{\xi^2a^{ij}u_{ki}u_{kj}}_{\geq\xi^2\lambda|D^2u|^2}+\underline{\xi^2a^{ij}u_ku_{kij}}$$

$$\downarrow$$

$$=\xi^2u_k(f_k-b^i_ku_k-b^iu_{ik}-a^{ij}_ku_{ij})$$

$$\geq\xi^2\lambda|D^2u|^2-\xi^2|Du||Df|-\xi^2|Db||Du|^2$$

C与 f a b 有关 $$-\xi^2|Db||D^2u||Du|-\xi^2|Da||D^2u||Du|$$

$$\geq\xi^2\lambda|D^2u|^2-C\xi^2|Du|-C\xi^2|Du|^2-C\xi^2|D^2u||Du|$$

$$\geq\xi^2\lambda|D^2u|^2-C\xi^2|Du|-C\xi^2|Du|^2-\varepsilon_0\xi^2|D^2u|^2-\xi^2\frac{C^2}{\varepsilon_0}|Du|^2$$

$$=\xi^2(\lambda-\varepsilon_0)|D^2u|^2-\frac{C^2}{\varepsilon_0}|Du|^2-C\xi^2|Du|-C\xi^2|Du|^2$$

$$②=8a^{ij}\xi\xi_iu_ku_{kj}\geq8\xi\lambda|D\xi||D^2u||Du|\geq$$

$$③=2\alpha a^{ij}u_iu_j+2\alpha a^{ij}uu_{ij}\geq2\alpha\lambda|Du|^2+2\alpha u(f-b^iu_i)$$

$$\geq2\alpha\lambda|Du|^2-2\alpha|u||f|-\alpha b^i(u^2)_i$$

$$\Rightarrow L\varphi\geq2\xi^2(\lambda-\varepsilon_0)|D^2u|^2+\left(2\alpha\lambda-C\xi^2-\frac{C^2}{\varepsilon_0}-2C\right)|Du|^2-C\xi^2|Du|$$

$$+\lambda\beta^2e^{\beta x_1}-C\beta e^{\beta x_1}-2\alpha|u||f|$$

取 α β ε_0 合适即可.

则 $\frac{9}{16}r^4 \sup\limits_{B_{\frac{r}{2}}(0)} |Du|^2 \le \sup\limits_{B_r(0)} \varphi \le \alpha \sup\limits_{\partial B_r(0)} u^2 + \sup\limits_{\partial B_r(0)} e^{\beta x_1}$

$$\le C + \alpha \sup_{\Omega} |u|^2$$

7. 对数梯度估计 $Lu=0$. L椭圆 $b_i = c = 0$ $a_{ij} \in C^2(\bar{\Omega})$

$B_1(0) \subset\subset \Omega$ 则 $\sup\limits_{B_{\frac{1}{2}}(0)} |D\log u| \le C_0 = C_0(n, |a|_{C^2})$

Pf. $v = \log u$. $u_i = e^v v_i$ $u_{ij} = e^v(v_{ij} + v_i v_j)$

$$\Rightarrow 0 = e^v(a^{ij}v_{ij} + a^{ij}v_iv_j) \qquad (*)$$

$\varphi = \xi^2 a^{ij} v_i v_j$ 设在内部x_0处最大 $w = a^{ij}v_iv_j$ 在x_0处

$\varphi = \xi^2 w$ $\varphi_i = (\xi^2)_i w + \xi^2 w_i$ $\varphi_{ij} = (\xi^2)_{ij} w + \xi^2 w_{ij} + (\xi^2)_i w_j + (\xi^2)_j w_i$

$\Downarrow (**)$

正定 半负定

$\xi w_i = -2\xi_i w$

$x_0\ \ 0 \ge a^{ij}\varphi_{ij} = w a^{ij}(\xi^2)_{ij} + \xi^2 a^{ij} w_{ij} + 2a^{ij}(\xi^2)_i w_j$

$$\overset{(**)}{=} w a^{ij}(\xi^2)_{ij} + \underbrace{\xi^2 a^{ij} w_{ij}}_{②} - 8a^{ij}w$$

$② = \xi^2 a^{ij}(a^{kl}v_kv_l)_{ij} = \xi^2 a^{ij}(a^{kl}_i v_kv_l + a^{kl}v_{ik}v_l + a^{kl}v_kv_{il})_j$

$= 2\xi^2 a^{ij}a^{kl}v_{ik}v_{jl} + 2\xi^2 a^{ij}a^{kl}v_l v_{kij}$

$+ 4\xi^2 a^{ij}a^{kl}_j v_{ik}v_l + \xi^2 a^{ij}a^{kl}_{i,j}v_kv_l$

$②(1) \ge 2\lambda^2\xi^2|Dv|^2$ $②(4) \ge -C\xi^2|Dv|^2$

$②(3) \ge -\varepsilon_0\xi^2|D^2v|^2 - \frac{C^2}{\varepsilon_0}|Dv|^2$

$(*) \Rightarrow a^{ij}_k v_{ij} + a^{ij}v_{ij,k} = -w_k$ $\Rightarrow ②(2) = 2\xi^2 a^{kl}v_l(-w_k - a^{ij}_k v_{ij})$

$$\overset{(**)}{=} \underline{4\xi a^{kl}v_l w\xi_k}$$

$\swarrow(*)$ $-2\underbrace{\xi^2 a^{ij}_k v_{ij} v_l}$

$=$ $\ge -\varepsilon_0\xi^2|D^2v|^2$

$-\frac{C^2}{\varepsilon_0}|Dv|^2$

$$\Rightarrow 0 \geq a^{ij}\varphi_{ij} = \omega a^{ij}(\xi^2)_{ij} - 8\omega a^{ij}\xi_i\xi_j + \xi^2|D^2v|^2(2\lambda^2 - 2\varepsilon_0)$$

$$\varepsilon_0 = \frac{\lambda^2}{2} \qquad - |Dv|^2\left(\frac{2C^2}{\varepsilon_0} + C\xi^2\right)$$

又 $\omega \leq \Lambda|Dv|^2 \Rightarrow \frac{\lambda^2}{\Lambda}\xi^2\omega|D^2v| \leq \lambda^2\xi^2|D^2v|^2 \leq 2C\omega + C\omega\xi|Dv| + C|Dv|^2$

$\omega \leq \Lambda|D^2v|$

$\Rightarrow \xi^2|D^2v| \leq C\xi|Dv| + C_1$

$\Rightarrow \xi^2|Dv|^2 \leq C_1\xi|Dv| + C_2 \leq C_3.$

三 Neumann与Dirichlet问题

1. Neumann有界估计

$$\begin{cases} Lu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + d(x)u = \varphi & \text{on } \partial\Omega \end{cases} \qquad \begin{matrix} d(x) \geq d_0 > 0 \\ c(x) \leq -c_0 < 0 \end{matrix}$$

$F = \|f\|_{C^0}$. $\Phi = \|\varphi\|_{C^0}$ $\qquad v = \frac{F}{c_0} + \frac{\Phi}{d_0} + u$

$\Rightarrow Lv = Lu + c\left(\frac{F}{c_0} + \frac{\Phi}{d_0}\right) = f + \frac{c}{c_0}F + \frac{c\Phi}{d_0} < 0$

① v的非正最小在$\partial\Omega$达到 设在x_0处. （则 u_{min}也在x_0处）

则 $0 \geq \frac{\partial v}{\partial n}\Big|_{x_0} = \frac{\partial u}{\partial n}\Big|_{x_0} = \varphi(x_0) - d(x_0)u(x_0)$

$\Rightarrow \min\limits_{\Omega} u = u(x_0) \geq \frac{\varphi(x_0)}{d(x_0)} \geq -\frac{\Phi}{d_0}$ $\qquad \Rightarrow \min\limits_{\Omega} u \geq -\left(\frac{F}{c_0} + \frac{\Phi}{d_0}\right)$

② v无非正最小. 则 $v > 0$ $\Rightarrow u \geq -\frac{F}{c_0} - \frac{\Phi}{d_0}$

类似 $\max\limits_{\Omega} u \leq \frac{F}{c_0} + \frac{\Phi}{d_0}$

$\Rightarrow \|u\|_{C^0} \leq \frac{F}{c_0} + \frac{\Phi}{d_0}$

2. Neumann梯度估计

$$\begin{cases} \Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = \varphi \text{ on } \partial\Omega \end{cases}$$

$\varphi \in C^3(\bar{\Omega})$ $\partial\Omega \in C^3$ $C(M, f, \Omega', \Omega)$

若 $|u|_{L^\infty} \leq M$ 且有梯度内估计 $\sup_{\Omega'} u \leq \overset{\shortparallel}{C}$

则 $\sup_{\Omega} |Du| \leq C(M_0, f, \Omega)$

(pf. Fact: $\partial\Omega \in C^2 \Rightarrow \exists d_0 > 0$ s.t. $d(x) = dist(x, \partial\Omega) \in C^2(d_0)$

$\Omega_{d_0} = \{x \in \Omega \mid d(x) < d_0\}$

$|Dd|^2 = 1$ $|D^2 d| \leq C_0$ $\frac{\partial d}{\partial n} = -1$

令 $w = u + \varphi d \Rightarrow w_n = u_n + \varphi_n d + \varphi d_n = 0$ $|\partial\Omega$

又 $\Omega \setminus \Omega_{d_0}$ 已有梯度内估计 则只须在 Ω_{d_0} 处理

$\Phi \triangleq \alpha d + h(u) + \log|Dw|^2$. $h(u) = -\log(1 + M_0 - u)$

①Φ 在 $\partial\Omega_\alpha$ 取最大，由梯度内估计知结果

②Φ 在 $x_0 \in \partial\Omega$ 最大 $\Rightarrow 0 \leq \frac{\partial \Phi}{\partial n}(x_0) = -\alpha + h'\varphi + \frac{(|Dw|^2)_n}{|Dw|^2}$ $(*)$

$(|Dw|^2)_n = (w_n^2 + |D'w|^2)_n \overset{w_n = 0}{\underset{\text{on } \partial\Omega}{=}} (|D'w|^2)_n$ on $\partial\Omega$

in Ω $|D'w|^2 = |Dw|^2 - (Dw \cdot Dd)^2 = (\delta_{ij} - d_i d_j) w_i w_j \triangleq c^{ij} w_i w_j$

$|D'w|^2_n = c^{ij}_n w_i w_j + 2c^{ij} w_{in} w_j$

又 $w_{in} = u_{in} - (\varphi d)_{in}$ $c^{ij}(\varphi d)_{in} \leq C_0$ $\nearrow$ $|D'w|^2_n \leq C_0|Dw|^2 + C_1|Dw|$

而 $c^{ij}(u_n - \varphi)_i = 0 \Rightarrow c^{ij} u_{ni} = c^{ij} \varphi_i$

$(*) \Rightarrow -\alpha + h'\varphi + \frac{C_0|Dw| + C_1|Dw|}{|Dw|^2} \geq 0$

又 $\frac{1}{1+2M} \leq h'(u) \leq 1$ 取 α 足够大 矛盾！ 故不在 $\partial\Omega$ 最大

③ Φ 在 $x_1 \in \Omega_{d_0}$ 最大. 则在 x_1 处 $\Phi_i = 0$ $\quad \Delta\Phi \le 0$

$$\left(\Delta(|Dw|^2) = 2(w_j w_{j\cdot})_i = 2w_{ij}^2 + 2w_j(\Delta w)_j\right) \qquad \overset{\Phi_i \,=\, 0}{\parallel} \quad \frac{(|Dw|^2)_i}{|Dw|^2} + h'u_i + \alpha d_i \quad (*)$$

$$\Delta\Phi = \underbrace{\frac{\Delta(|Dw|^2)}{|Dw|^2}}_{①} - \underbrace{\frac{|D|Dw|^2|^2}{|Dw|^4}}_{②} + h''|Du|^2 + h'\Delta u + \alpha\Delta d$$

$$① = \underbrace{\frac{2w_{ij}^2}{|Dw|^2}}_{①_1} + \underbrace{\frac{2(\Delta w)_j w_j}{|Dw|^2}}_{①_2} \qquad ①_2 = \frac{2\Sigma w_j f_j + 2\Sigma w_j(\Delta(\varphi d))_j}{|Dw|^2} \geq -C|Dw|$$

$$①_1 \downarrow \quad \frac{2\sum_i\sum_j w_{ij}^2 \sum_j w_j^2}{|Dw|^4} \geq \frac{2\sum_i(\sum_j w_{ij}w_j)^2}{|Dw|^4} = \frac{1}{2}②$$

$$①_1 - ② = -\frac{1}{2}\frac{|D|Dw|^2|}{|Dw|^4} \overset{(*)}{=} -\frac{1}{2}\Sigma(h'u_i + \alpha d_i)^2$$

$$\geq -\frac{3}{4}h'|Du|^2 - \frac{3}{2}\alpha^2$$

若 $|Dw|^2 > 1 \overset{否则}{\Rightarrow} |Du| \leq |Dw| + C \leq C + C'$

则 $0 \geq \Delta\Phi(x_1) \geq (h'' - \frac{3}{4}h')|Du|^2 - C_1|Du| + h'f - C_0\alpha - \frac{3}{2}\alpha \Rightarrow |Du| \leq C$)

3. Dirichlet 问题边界梯度估计

$$\begin{cases} Lu = f \\ u|_{\partial\Omega} = \varphi \end{cases} \qquad \begin{array}{l} \Omega \subset \{0 < x_1 < d\} \\ a^{ij}\ b^i\ c \in C^1(\Omega) \quad \varphi \in C^2(\bar\Omega) \end{array}$$

则 $\forall x \in \bar\Omega$. $x_0 \in \partial\Omega$. $\quad |u(x) - u(x_0)| \leq C|x - x_0|$ $\quad C$ 与 λ. $\|a^{ij}, b^i, c\|_{L^\infty}$. $\|f\|_{L^\infty}$ $\|\varphi\|_{C^2}$ $\|u\|_{L^\infty}$ 有关

(Pf. $\tilde{L}u = Lu - cu = f - cu = \tilde{f}$ $\qquad v = u - \varphi$

$$\Rightarrow \begin{cases} \tilde{L}v = \tilde{f} - \tilde{L}\varphi \quad \text{in } \Omega. \\ v|_{\partial\Omega} = 0 \end{cases}$$

只须考虑 $\begin{cases} \tilde{L}u = \tilde{f} \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$ 找 w s.t $w(x_0) = 0$ $w(x) \geq 0$ on $\partial\Omega$

$\tilde{L}w \geq -F$ $F = \|f\|_{C^{\alpha}(\bar{\Omega})}$

令 $v = w - u \Rightarrow \begin{cases} \tilde{L}v = \tilde{L}w - \tilde{L}u \leq 0 \\ v|_{\partial\Omega} \leq 0 \quad v(x_0) = 0 \end{cases}$ $\Rightarrow v_{min}$ 在边界达到 $v \geq 0$

$\Rightarrow u \leq w$ 同理 $u \geq -w$

故只须找 $|w(x) - w(x_0)| \leq C|x - x_0|$

令 $d(x) = |x-y| - R$. $w = \psi(d)$. s.t. $\begin{cases} \psi(0) = 0. \\ \psi'(d) > 0 \\ \psi''(d) < 0. \end{cases}$

$$\tilde{L}w = \psi'' a^{ij} d_i d_j + \psi' a^{ij} d_{ij} + \psi' b^i d_i$$

$$d_i = \frac{x_i - y_i}{|x-y|} \qquad d_{ij} = \frac{\delta_{ij}}{|x-y|} - \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3}$$

又 $a^{ij} d_i d_j \geq \lambda |Dd|^2 = \lambda$

$$a^{ij} d_{ij} = \frac{a^{ii}}{|x-y|} - \frac{a^{ij}(x-y)_i (x-y)_j}{|x-y|^3}$$

$$\leq \frac{n\Lambda}{|x-y|} - \frac{\lambda}{|x-y|} \leq \frac{n\Lambda - \lambda}{R}$$

$|b^i d_i| \leq \lambda$

$\Rightarrow$ 找 $\lambda\psi'' + \left(\frac{n\Lambda - \lambda}{R} + \lambda\right)\psi' \geq -F$

取 $\psi(d) = \frac{b}{a}\left[\frac{e^{aD}}{1 - e^{-ad}} - d\right]$ $D = \text{diam}\,\Omega$ 即可)

四、分部积分

$\int_\Omega \operatorname{div} \vec{X}\, dV = \int_{\partial\Omega} \vec{X}\, \vec{\nu}\, d\sigma$. 散度定理

1. 均值公式（之前讲过）

Ex. $\Delta u=0 \Rightarrow |Du(x_0)| \le \frac{n^{\frac{3}{2}}}{r} \sup_{\partial B_r(x_0)} |u|$ （梯度内估计）

Pf. $\Delta u_i=0 \Rightarrow u_i(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u_i\, dx = \frac{1}{|B_r(x_0)|}\int_{\partial B_r(x_0)} u\, \nu_i\, d\sigma$

$\Rightarrow |u_i(x_0)| \le \frac{|\partial B_r(x_0)|}{|B_r(x_0)|} \sup_{\partial B_r(x_0)} |u| = \frac{n}{r} \sup_{\partial B_r(x_0)} |u|$. ✓

2. 能量法：两边乘一个东西再积分

Ex ① $\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \subset \mathbb{R}^n \text{ 星形区域} \\ u|_{\partial\Omega} = 0 \end{cases}$ $u>0$ $p>\frac{n+2}{n-2} \Rightarrow$ 无解.

Pf 同乘 $u_i x_i \Rightarrow \int_\Omega \underbrace{x_i u_i \Delta u}_{①} + \underbrace{x_i u_i u^p}_{②}\, dx = 0$

$$① = \int_\Omega x_i u_i u_{jj}\, dx = \int_\Omega (x_i u_j u_i)_j - \delta_{ij} u_i u_j - x_i u_j u_{ij}\, dx$$

$$= \int_\Omega (x_i u_j u_i)_j - |Du|^2 - \frac{1}{2}\left((x_i|Du|^2)_i - n|Du|^2\right) dx$$

$$= \int_\Omega \frac{n-2}{2}|Du|^2 + (x_i u_j u_i)_j\, dx - \frac{1}{2}\int_{\partial\Omega} x_i n_i |Du|^2$$

（$u|_{\partial\Omega}=0$
$\Rightarrow Du$ 切向为0
只须考虑法向）

$$= \int_{\partial\Omega} x_i u_j u_i n_j\, d\sigma - \frac{1}{2}\int_{\partial\Omega} x_i n_i |Du|^2 d\sigma + \frac{n-2}{2}\int_\Omega |Du|^2 dx$$

$$= \frac{1}{2}\int_{\partial\Omega} (\vec{x}\cdot\vec{n})|Du|^2 d\sigma + \frac{n-2}{2}\int_\Omega |Du|^2 dx$$

$$② = \int_\Omega x_i u_i u^p dx = \int_\Omega x_i \frac{(u^{p+1})_i}{p+1} dx = \frac{1}{p+1}\int_{\partial\Omega} \underset{\overset{\downarrow}{0}}{u^{p+1}}\, \vec{x}\cdot\vec{n}\, dx - \frac{n}{p+1}\int_\Omega u^{p+1} dx$$

故 $\frac{n-2}{2}\int_\Omega |Du|^2 dx + \frac{1}{2}\int_{\partial\Omega} |Du|^2 (\vec{x}\cdot\vec{n}) d\sigma = \frac{n}{p+1}\int_\Omega u^{p+1} dx$

再同乘u $\Rightarrow -\int_\Omega u\Delta u\, dx = \int_\Omega |Du|^2 dx = \int_\Omega u^{p+1} dx$

$\Rightarrow 0 \le \frac{1}{2}\int_{\partial\Omega} |Du|^2 (\vec{x}\cdot\vec{n}) d\sigma = \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_\Omega |Du|^2 dx$

若 $p > \frac{n+2}{n-2} \Rightarrow \frac{n}{p+1} - \frac{n-2}{2} < 0 \quad \Rightarrow u = 0$

② $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \begin{matrix} u \in C^2(\Omega) \cap C(\bar\Omega) \\ f \in L^2(\Omega) \end{matrix}$

$\Rightarrow \int_\Omega |Du|^2 = \int_\Omega fu \le \left(\int_\Omega f^2\right)^{\frac12}\left(\int_\Omega u^2\right)^{\frac12}$

$\overset{\text{Fredrich}}{\le} c\left(\int_\Omega |Du|^2\right)^{\frac12}\left(\int_\Omega f^2\right)^{\frac12}$

$\Rightarrow \int_\Omega |Du|^2 \le c^2 \int f^2$

③ $a^{ij}u_{ij} + b^i u_i + cu = f$

$\Rightarrow -\int a^{ij}u_{ij}u - \int b^i u_i u - \int cu^2 = -\int fu$

$ua^{ij}u_{ij} = (ua^{ij}u_j)_i - a^{ij}_i u u_j - a^{ij}u_i u_j$

① $= \int_\Omega a^{ij}u_i u_j + \underline{a^{ij}_i u u_j}$

$\le c_0 \int_\Omega u|Du| \le \varepsilon_0 \int_\Omega |Du|^2 + \frac{c_0^2}{4\varepsilon_0}\int_\Omega u^2$

$\ge \frac{\lambda}{2}\int_\Omega |Du|^2 - \frac{c_0^2}{2\lambda}\int_\Omega u^2$

② $\ge -\frac{\lambda}{4}\int_\Omega |Du|^2 - \frac{c_1^2}{\lambda}\int_\Omega u^2$

④ $\le \varepsilon_0 \int_\Omega u^2 + \frac{1}{\varepsilon_0}\int_\Omega f^2$

$\Rightarrow \exists c_2 \text{ s.t } \int_\Omega |Du|^2 + u^2 \le c_2 \int f^2.$

3. 截断函数

Lem $\forall \Omega' \subset\subset \Omega$ $\exists \xi \in C_c^\infty(\Omega)$ s.t. $\begin{cases} \xi = 1 & \text{on } \Omega' \\ 0 \le \xi \le 1 & \text{on } \Omega \\ \|D^\alpha(\xi)\|_{C^\alpha(\Omega)} \le \dfrac{C_n}{(d(\Omega', \partial\Omega))^{|\alpha|}} \end{cases}$

Ex ① 调和函数梯度内估计. $\Delta u = 0$ $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\Omega' \subset\subset \Omega$.

则 $\sup_{\Omega'} |Du| \le \dfrac{C_n}{d(\Omega', \partial\Omega)} \sup_{x \in \partial\Omega} |u|$.

Pf. $\varphi = \xi |Du|^2 + C_0 u^2$

$$\Delta\varphi = \Delta(\xi^2)|Du|^2 + 2(\xi^2)_i (|Du|^2)_i + \xi^2 \Delta(|Du|^2) + C_0 \Delta(u^2)$$

$$\overset{\Delta u = 0}{=} \Delta(\xi^2)|Du|^2 + 8\xi\xi_i u_j u_{ji} + 2\xi^2 u_{ij}^2 + 2C_0|Du|^2$$

$$\ge \Delta(\xi^2)|Du|^2 - 2\xi^2 u_{ij}^2 - 8n^2|Du|^2|D\xi|^2 + 2\xi^2 u_{ij}^2 + 2C_0|Du|^2$$

$$= (2C_0 + \Delta(\xi^2) - 8n^2|D\xi|^2)|Du|^2$$

又 $|D\xi|^2 + |D^2\xi| \le \dfrac{c}{d^2}$ $\Rightarrow$ 取 $C_0 = \dfrac{(4n^2+1)C}{d^2}$ $\Rightarrow \Delta\varphi \geq 0$

$$\Rightarrow \underset{\substack{\vee\!| \\ \sup_{\Omega'} |Du|^2}}{\sup_{\Omega'} \varphi} \le \sup_{\partial\Omega} \varphi \le C_0 \sup_{\partial\Omega} u^2 \quad \Rightarrow \sup_{\Omega'} |Du| \le \frac{\sqrt{(4n^2+1)C}}{d} \sup_{\partial\Omega} u$$

② 能量估计: $\Delta u = f$ $f \in C(\Omega)$ $\Rightarrow \int_\Omega \xi^2 u \Delta u = \int_\Omega f u \xi^2$

$$\text{LHS} = \int_\Omega (\xi^2 u u_i)_i - (\xi^2 u)_i u_i = -\int_\Omega (\xi^2)_i u u_i - \xi^2 |Du|^2$$

$$\Rightarrow \int_\Omega \xi^2 |Du|^2 = -\int_\Omega \xi^2 u f - 2\int_\Omega \xi\xi_i u u_i$$

$$\le \frac{1}{2}\int_\Omega \xi^2 f^2 + \frac{1}{2}\int_\Omega \xi^2 u^2 + \int_\Omega \xi^2 |Du|^2 + u^2 |D\xi|^2$$

$$\Rightarrow \int_{\Omega'} |Du|^2 \le C\left(\int_\Omega u^2 + f^2\right)$$

③高阶梯度内估计 $\int_\Omega \xi^2 u_{i_0i_0}\Delta u=\int_\Omega \xi^2 u_{i_0i_0}f$

$$LHS=\int_\Omega (\xi^2 u_{i_0i_0}u_j)_j-(\xi^2 u_{i_0i_0})_j u_j$$

$$=\int_\Omega -2\xi\xi_j u_j u_{i_0i_0}-\underline{\xi^2 u_{i_0i_0j}}\,u_j \longrightarrow (\xi^2 u_{i_0j}u_j)_{i_0}-(\xi^2 u_j)_{i_0}u_{i_0j}$$

$$=-2\int_\Omega \xi\xi_j u_j u_{i_0i_0}+2\int_\Omega \xi\xi_{i_0}u_j u_{i_0j}+\int_\Omega \xi^2 u_{i_0j}^2$$

$$\Rightarrow \int_\Omega \xi^2 u_{i_0j}^2=\int_\Omega \xi^2 u_{i_0i_0}f-2\int_\Omega \xi\xi_{i_0}u_j u_{i_0j}+2\int_\Omega \xi\xi_j u_j u_{i_0i_0}$$

$$\le \frac{1}{4}\int_\Omega \xi^2 u_{i_0i_0}^2+4\int_\Omega \xi^2 f^2+\frac{1}{4}\int_\Omega \xi^2 u_{i_0j}^2+4\int_\Omega |Du|^2|D\xi|^2$$

$$+\frac{1}{4}\int_\Omega \xi^2 u_{i_0i_0}^2+4\int_\Omega |Du|^2|D\xi|^2$$

$$\Rightarrow \int_{\Omega'} u_{ij}^2\le C\int_\Omega \xi^2 u_{i_0j}^2\le C_1\left(\int_{\Omega'} |Du|^2+|f|^2\right)$$

$$\overset{②}{\le} C_1\left(\int_\Omega u^2+\int_\Omega f^2\right)$$

④Liouville：$u\in L^2(\mathbb{R}^n)$ $\Delta u=0\Rightarrow u=0$

取 $\xi\in C_c^\infty(B_R)$ $\xi=1$ in $B_{\frac{R}{2}}$

$$\Rightarrow 0=\int_{\mathbb{R}^n}\xi^p u\Delta u=\int_{\mathbb{R}^n}\xi^p\left((uu_i)_i-|Du|^2\right)$$

$$=-\int_{\mathbb{R}^n}p\xi^{p-1}\xi_i u u_i-\int_{\mathbb{R}^n}\xi^p|Du|^2$$

$$\Rightarrow \int_{\mathbb{R}}\xi^p|Du|^2\le p\int_{\mathbb{R}^n}\xi^{p-1}uu_i\xi_i$$

$$\le \frac{1}{2}\int_{\mathbb{R}^n}\underbrace{\xi^{2p-2}}_{\le\xi^p}|Du|^2+\frac{p^2}{2R^2}\int_{\mathbb{R}^n}u^2$$

令 $R\to\infty$ 有 $u=0$.

⑤ $\Delta u+u^{\alpha}=0$ in $\mathbb{R}^n$ $u>0$ $1<\alpha<\frac{n}{n-2}$ $\Rightarrow$ 无解

取 $\xi\in C_c^{\infty}(B_{2R})$ $\begin{cases}\xi=1 \text{ in } B_R\\ 0\le\xi\le 1\\ |D^{\alpha}\xi|\le\frac{C_n}{R^{|\alpha|}}\end{cases}$

第二Green公式：$\int_{\Omega}u\Delta v-v\Delta u$
$=\int_{\partial\Omega}u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}$

$$\Rightarrow\int_{\mathbb{R}^n}u^{\alpha}\xi^{p}dx=-\int_{\mathbb{R}^n}\Delta u\,\xi^{p}dx=-\int_{\mathbb{R}^n}u\,\Delta\xi^{p}\,dx$$
$$\le\frac{C_n}{R^2}\int_{\mathbb{R}^n}u\,\xi^{p-2}dx$$
$$\le\frac{1}{2}\int_{\mathbb{R}^n}u^{\alpha}\xi^{p}dx+CR^{n-\frac{2\alpha}{\alpha-1}}$$

则 $n-\frac{2\alpha}{\alpha-1}<0$. $R\to\infty$ 可知 $u=0$.

Rmk. 事实上 对 $\alpha\in(1,\frac{n+2}{n-2})$ 均成立

证明复杂 略去

五 Sobolev空间

1. Hölder空间

Def. ① $U\subset\mathbb{R}^n$ 开集 $0<\alpha\le1$. 定义 $[u]_{C^{0,\alpha}(\bar{U})}\triangleq\sup\limits_{x\neq y\in U}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$: 半范

$\|u\|_{C^{k,\alpha}(\bar{U})}=\sum\limits_{|\alpha|\le k}\|D^{\alpha}u\|_{C(\bar{U})}+\sum\limits_{|\alpha|=k}[D^{\alpha}u]_{C^{0,\alpha}(\bar{U})}$: Hölder范数

② Hölder空间 $C^{k,\nu}(\bar{U})$: $u\in C^{k}(\bar{U})$. $\|u\|_{C^{k,\nu}(\bar{U})}<+\infty$

Thm. $C^{k,\nu}(\bar{U})$ 为Banach空间

(易证为范数 完备: $\{u_n\}$ 为 $C^{k,\nu}(\bar{U})$ 中Cauchy

$\Rightarrow\|u_n-u_m\|_{C^k(\bar{U})}=\max\limits_{|\alpha|\le k}\max\limits_{x\in\bar{U}}|D^{\alpha}u_m-D^{\alpha}u_n|$ $\overset{0}{\uparrow}$

$\le\max\limits_{|\alpha|\le k}\|D^{\alpha}u_m-D^{\alpha}u_n\|_{C^0(\bar{U})}\le\sum\limits_{|\alpha|\le k}\|D^{\alpha}(u_m-u_n)\|_{C^0(\bar{U})}$

$C^k(\bar{U})$ 完备
$\Rightarrow u_n\xrightarrow{C^k(\bar{U})}u$. 再证 $u\in C^{k,\nu}(\bar{U})$ 即可.)

2 Sobolev空间

Def $u.v\in L^1_{loc}(U)$ 称 v 为 u 的 α 阶弱导数. 若 $\forall\phi\in C_0^\infty(U)$

$$\int_U uD^\alpha\phi=(-1)^{|\alpha|}\int_U v\phi \qquad 记\ D^\alpha u=v.$$

Lem 若 α 阶弱导存在 则 a.e. 意义下唯一

Pf 即证若 $\forall\phi\in C_0^\infty(U)$. $\int_U v\phi=0 \Rightarrow v=0$ a.e.

若不然 v 在正测集 E 上正 $\Rightarrow$ 取 $\varphi_k\in C_c(U)$ s.t. $\begin{cases}\int_R|\chi_E(x)-\varphi_k(x)|\to 0\\ |\varphi_k(x)|\le 1\\ \lim\limits_{k\to\infty}\varphi_k(x)=\chi_E(x)\end{cases}$

余集

$\Rightarrow 0<\int_E v(x)dx=\lim\limits_{k\to\infty}\int_{R^n}v(x)\varphi_k(x)dx=0$. 矛盾.

Def $W^{k,p}(U)=\{u\in L^1_{loc}\ U\to R$. $D^\alpha u$(弱)存在 且 $D^\alpha u\in L^p(U)$ $(\forall|\alpha|\le k)\}$

$$\|u\|_{W^{k,p}(U)}=\begin{cases}\left(\sum\limits_{|\alpha|\le k}\int_U|D^\alpha u|^p dx\right)^{\frac{1}{p}} & 1\le p<+\infty\\ \sum\limits_{|\alpha|\le k}\operatorname{ess\,sup}\limits_{U}|D^\alpha u| & p=+\infty\end{cases}$$

记 $W_0^{k,p}(U)$ 为 $C_0^\infty(U)$ 在 $W^{k,p}(U)$ 范数下的完备化.

例1. $u(x)=|x|^{-\alpha}$ $\alpha>0$ $u\in W^{1,p}(B_1(0))$

$$u_{x_i}=\frac{-\alpha x_i}{|x|^{\alpha+2}}\Rightarrow|Du|=\frac{|\alpha|}{|x|^{\alpha+1}}$$

弱导存在: $\Leftrightarrow \int_{B_1(0)}u\phi_i=-\int_{B_1(0)}v\phi$

$$\begin{aligned}&\overset{\Vert}{\lim_{\varepsilon\to0^+}}\int_{B_1(0)-B_\varepsilon(0)}u\phi_i\\ &=\lim_{\varepsilon\to0^+}\int_{\partial B_\varepsilon(0)}u\phi\nu^i-\int_{B_1(0)}u_i\phi\\ &\le\lim_{\varepsilon\to0^+}-\int_{B_1(0)}u_i\phi+C\varepsilon^{n-1-\alpha}\ \Rightarrow\alpha<n-1\end{aligned}$$

$$|Du|\in L^p(U)\Leftrightarrow\int_{B_1(0)}\frac{\alpha^p}{|x|^{(\alpha+1)p}}<+\infty$$

$$\Updownarrow$$

$$\int_0^1 r^{n-1-(\alpha+1)p}dr<\infty$$

$$\Updownarrow$$

$$\alpha<\frac{n-p}{p}$$

$$\downarrow$$

$$\nearrow\ \alpha<\frac{n}{p}-1.$$

Prop $u, v \in W^{k,p}(U)$ $|\alpha| \le k$. 则

(1) $D^\alpha u \in W^{k-|\alpha|,p}(U)$. $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ (若 $|\alpha|+|\beta| \le k$)

(2) $W^{k,p}(U)$ 线性空间 且若 $V \subset U$ 开. 则 $W^{k,p}(U) \subset W^{k,p}(V)$

(3) $\xi \in C_0^\infty(U)$ 则 $\xi u \in C_0^\infty(U)$ 且 $D^\alpha(\xi u) = \sum_{\beta \le \alpha} C_{|\alpha|}^{|\beta|} D^\beta \xi D^{\alpha-\beta} u$.

$\downarrow$

单位分解

Thm. $W^{k,p}(U)$ 为 Banach 空间 $1 \le p \le +\infty$

3. 逼近.

假设 $k \in \mathbb{N}$. $1 \le p < +\infty$. $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$

Thm (内部光滑) $u \in W^{k,p}(U)$. $u^\varepsilon = \eta_\varepsilon * u$ in U_ε 则 $\begin{cases} u^\varepsilon \in C^\infty(U_\varepsilon) \\ u^\varepsilon \to u \text{ in } W^{k,p}_{loc}(U) \end{cases}$

(pf. 先算 $D^\alpha u^\varepsilon$: $D^\alpha u^\varepsilon(x) = D^\alpha \int_U \eta_\varepsilon(x-y) u(y) dy$

$$= \int_U D_x^\alpha \eta_\varepsilon(x-y) \, u(y) dy$$

$$= (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\varepsilon(x-y) u(y) dy$$

令 $\phi(y) = \eta_\varepsilon(x-y) \in C_0^\infty(U) \Rightarrow \int_U D_y^\alpha \eta_\varepsilon(x-y) \, u(y) dy = (-1)^{|\alpha|} \int_U \eta_\varepsilon(x-y) D^\alpha u(y) dy$

$$\Rightarrow D^\alpha u^\varepsilon(x) = (\eta_\varepsilon * D^\alpha u)(x)$$

则取 $V \subset\subset U$ $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(V)$

$$\Rightarrow \|u^\varepsilon - u\|_{W^{k,p}(V)}^p = \sum_{|\alpha| \le k} \|D^\alpha u^\varepsilon - D^\alpha u\|_{L^p(V)}^p \to 0$$)

Thm (整体光滑①) U有界 $u\in W^{k,p}(U)$ $\exists u_m\in \underbrace{C^\infty(U)\cap W^{k,p}(U)}_{\text{not } C^\infty(\bar{U})!}$ s.t. $u_m\xrightarrow{W^{k,p}(U)}u$

(pf ① $U=\bigcup_{i=1}^{\infty}U_i$ $U_i=\{x\in U \mid d(x,\partial U)>\frac{1}{i}\}$ $V_i=U_{i+3}-\bar{U}_{i+1}$

任取 $V_0\subset\subset U$ s.t. $U=\bigcup_{i=0}^{\infty}V_i$ 取 $\{\zeta_i\}_{i=0}^{\infty}$ P.O.U to $\{V_i\}$

则 $\zeta_i u\in W^{k,p}(U)$ 且 $\mathrm{supp}(\zeta_i u)\subset V_i$

② 固定 $\delta>0$ 取 $\varepsilon_i>0$ s.t. $u^i=\eta_{\varepsilon_i}*(\zeta_i u)$ $\begin{cases}\|u^i-\zeta_i u\|\le\frac{\delta}{2^{i+1}} & (i\ge0)\\ \mathrm{supp}\,u^i\subset W_i & (i\ge1)\end{cases}$

其中 $W_i=U_{i+4}-\bar{U}_i\supset V_i$

③ $v\triangleq\sum_{i=0}^{\infty}u^i\in C^\infty(U)\Rightarrow\|v-u\|_{W^{k,p}(V)}\le\sum_{i=0}^{\infty}\|u^i-\zeta_i u\|_{W^{k,p}(U)}\le\delta$

对 $V\subset\subset U$ 取sup $\Rightarrow\|v-u\|_{W^{k,p}(U)}\le\delta$)

Thm (整体光滑②) U有界 $\underline{\partial U\ C^1}$ 则 $\exists u_m\in C^\infty(\bar{U})$ s.t. $u_m\xrightarrow{W^{k,p}(U)}u$

(pf ① 固定 $x^0\in\partial U$ 则 $\exists r>0$ $\gamma:\mathbb{R}^{n-1}\to\mathbb{R}$ 使 $U\cap B(x^0,r)=\{x\in B(x^0,r)\mid x_n>\gamma(x_1,\cdots,x_{n-1})\}$

令 $V=U\cap B(x^0,\frac{r}{2})$ $x^\varepsilon=x+\lambda\varepsilon e_n$ 则对固定的大 $\lambda>0$. $B(x^\varepsilon,\varepsilon)\subseteq U\cap B(x^0,r)$ ($\forall x\in V$, ε小)

则定义 $u^\varepsilon=u(x^\varepsilon)$ $v^\varepsilon=\eta_\varepsilon*u_\varepsilon\in C^\infty(\bar{V})$

② $\|D^\alpha v^\varepsilon-D^\alpha u\|_{L^p(V)}\le\underbrace{\|D^\alpha v^\varepsilon-D^\alpha u_\varepsilon\|_{L^p(V)}}_{\downarrow 0}+\underbrace{\|D^\alpha u_\varepsilon-D^\alpha u\|_{L^p(V)}}_{\downarrow 0}$

$\Rightarrow v^\varepsilon\to u$ in $W^{k,p}(V)$

③ $\forall\delta>0$. ∂U紧 $\Rightarrow$ 取有限个 $x_i^0\in\partial U$ r_i V_i 如上 $v_i\in C^\infty(\bar{V}_i)$

s.t. $\partial U\subset\bigcup_{i=1}^{N}B(x_i^0,\frac{r_i}{2})$ 且 $\|v_i-u\|_{W^{k,p}(V_i)}\le\delta$

再令 $V_0\subset\subset U$ s.t. $U\subset\bigcup_{i=0}^{N}V_i$ 由局部逼近 $\exists v_0\in C^\infty(\bar{V}_0)$. $\|v_0-u\|_{W^{k,p}(V_0)}\le\delta$

取 $\{\zeta_i\}_{i=0}^{N}$ 为 P.O.U. to $\{V_i\}_{i=0}^{N}$ $v=\sum_{i=0}^{N}\zeta_i v_i\in C^\infty(\bar{U})$

$\|D^\alpha v-D^\alpha u\|_{L^p(U)}\le\sum_{i=0}^{N}\|D^\alpha(\zeta_i v_i)-D^\alpha(\zeta_i u)\|_{L^p(V_i)}\le CN\delta$)

4. 延拓

Thm. U有界 $\partial U \in C^1$ 有界开集V s.t. $U \subset\subset V$ 则$\exists$有界线性算子

$E: W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ s.t. (1) $Eu = u$ a.e in U

(2) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C(n,U)\|u\|_{W^{1,p}(U)}$

(3) $\operatorname{supp}(Eu) \subset V$

此时记Eu为u到$\mathbb{R}^n$的延拓

Pf ①边界平直. $x_0 \in \partial U$处. 设$\partial U \subset \{x_n=0\}$ $B^+ = B(x_0,r) \cap \{x_n \ge 0\} \subset \bar{U}$

$B^- = \cdots\cdots < 0\} \subset \mathbb{R}^n - U$

若$u \in C^1(B^+)$ 令 $\bar{u}(x) = \begin{cases} u(x). & x \in B^+ \\ -3u(x_1,\cdots x_{n-1},-x_n) + 4u(x_1,\cdots x_{n-1},-\frac{x_n}{2}) & x \in B^- \end{cases}$

Claim: $u \in C^1(B)$ 连续性显然. 令 $u^+ = \bar{u}|_{B^+}$ $u^- = \bar{u}|_{B^-}$

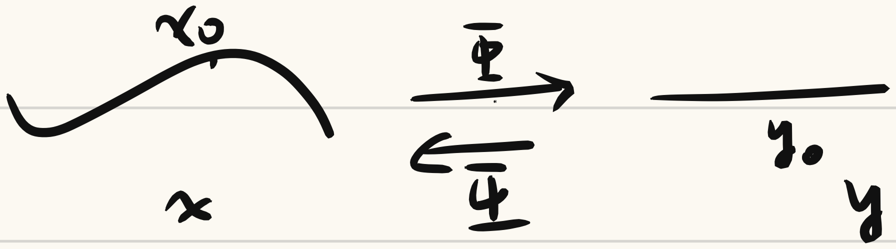
$\frac{\partial u^-}{\partial x_n} = 3\frac{\partial u}{\partial x_n}(x_1,\cdots x_{n-1},-x_n) - 2\frac{\partial u}{\partial x_n}(x_1,\cdots x_{n-1},-\frac{x_n}{2})$

$\Rightarrow \frac{\partial u^-}{\partial x_n} = \frac{\partial u^+}{\partial x_n}$ on $\{x_n=0\}$ 又 $u^+ = u^-$ on $\{x_n=0\} \Rightarrow \frac{\partial u^+}{\partial x_i} = \frac{\partial u^-}{\partial x_i}\Big|_{\{x_n=0\}}$

$\Rightarrow D^\alpha u^+ = D^\alpha u^-$ on $\{x_n=0\}$ $(|\alpha| \le 1)$ ✓

由构造可验证 $\|\bar{u}\|_{W^{1,p}(B)} \le C_0\|u\|_{W^{1,p}(B^+)}$

②不平直:



设用Φ拉直

$u'(y) = u(\Psi(y))$ 在y_0处取B'. B'^+ $\bar{u}'$如上 $\xrightarrow{拉回}$ x_0处 B. B^+. $\bar{u}$

由Φ. Ψ的Jacobi有界 $\Rightarrow \|\bar{u}\|_{W^{1,p}(B)} \le C_0\|u\|_{W^{1,p}(B^+)}$

③P.O.U: 用有限个B_i覆盖∂U 取B_0开 s.t. $U \subset \bigcup_{i=0}^{N} B_i$

取$\{\zeta_i\}_{i=0}^N$ 为关于$\{B_i\}_{i=0}^N$ P.O.U $\bar{u}(x) = \sum_{i=0}^{N} \zeta_i(x)\bar{u}_i(x)$

$(\bar{u}_0(x) = u(x))$

则 $\bar{u}(x)=u(x)$ on U 且 $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le \sum_{i=0}^{N} \|\bar{u}\|_{W^{1,p}(V_i)} \le C\|u\|_{W^{1,p}(U)}$

同时由 $\zeta_i \in C_0^\infty(V_i)$ $\exists V \subset \bigcup_{i=0}^{N} V_i$ s.t. $\mathrm{supp}(\bar{u}) \subset V$

则定义 $Eu=\bar{u}$ 即可

④ 若 $u \notin C^1(U)$ 取 $u_m \in C^1(\bar{u})$ s.t. $u_m \xrightarrow{W^{1,p}(U)} u$ $\quad Eu_m \triangleq \bar{u}_m$

则 $\|Eu_n - Eu_m\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u_n - u_m\|_{W^{1,p}(U)} \to 0$

由完备性 $Eu_n \xrightarrow{W^{1,p}(\mathbb{R}^n)} Eu$ 验证 Eu 满足条件

(i) $Eu=u$ a.e. in U: $\|Eu-u\|_{L^p(U)} \le \|Eu-Eu_m\|_{L^p} + \|Eu_m-u_m\|_{L^p(U)}$
$+\|u_m-u\|_{L^p(U)}$

$\to 0.$

(ii) $\forall m$ $\exists$ 有界开集 V_m' s.t. $\mathrm{supp}(\bar{u}_m) \subset V_m' \subset \bigcup_{i=0}^{N} V_i \Rightarrow \mathrm{supp}(Eu) \subset \bigcup_{i=0}^{N} V_i$

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le \|Eu-Eu_m\|_{W^{1,p}(\mathbb{R}^n)} + \|Eu_m\|_{W^{1,p}(\mathbb{R}^n)}$

$\le \|Eu-Eu_m\|_{W^{1,p}(\mathbb{R}^n)} + C\|u_m-u\|_{W^{1,p}(U)}$

$+ C\|u\|_{W^{1,p}(U)}$

令 $m\to\infty$ 即可

还需验证 Eu 不依赖 $u_m \to u$ 选取

设 $u_m \to u$, $v_n \to u$ in $W^{1,p}(U)$ 则 $\|u_m - v_n\|_{W^{1,p}(U)} \to 0$ $(m,n\to\infty)$

$\overset{\text{有界}}{\Rightarrow} \|Eu_m - Ev_n\|_{W^{1,p}(\mathbb{R}^n)} \to 0$

$\overset{m,n\to\infty}{\Rightarrow}$ 两个极限相等 a.e ✓

5. Trace

Thm. $U\subset\mathbb{R}^n$ 有界 C^1. 则 ∃ 线性算子 $T: W^{1,p}(U)\to L^p(\partial U)$

s.t. (1) $Tu = u|_{\partial\Omega}$ 若 $u\in W^{1,p}\cap C(\bar{U})$

(2) $\|Tu\|_{L^p(\partial U)} \le C(p,U)\,\|u\|_{W^{1,p}(\bar{U})}$ 此时称 Tu 为 u 的迹

Pf. 和上定理思路一致. 故只须限制在拉直的边界上考虑

$x_0\in\partial U$ $\quad \partial U\cap B(x_0,r)\subset\{x_n=0\}$ $\quad B^+=B(x_0,r)\cap\{x_n\ge 0\}$

$\Gamma=B(x_0,\frac{r}{2})\cap\partial U$ 并取 $\zeta\in C_0^\infty(B(x_0,r))$. $\zeta\equiv 1$ on $B(x_0,\frac{r}{2})$

$$\Rightarrow \int_\Gamma |u|^p dx \le \int_{B(x_0,r)\cup\{x_n=0\}} \zeta|u|^p dx = \int_{B^+}(\zeta|u|^p)_{x_n}dx$$

$$\le \int_{B^+}|\zeta_{x_n}||u|^p + p|u|^{p-1}|u_{x_n}|\,dx$$

$$\le C\left(\int_{B^+}|u|^p+|Du|^p dx\right)$$

再用 C^1 的 u_m 逼近一般的 u 定义即可

Thm U 同上 $u\in W^{1,p}(U)$. 则 $u\in W_0^{1,p}(U) \Leftrightarrow Tu=0$ a.e on ∂U

Pf. $\Rightarrow$ $u_m \xrightarrow{W^{1,p}(U)} u$ $\quad u_m\in C_0^\infty(U)$ $\quad$ 又 $Tu_m=0$ on ∂U $\Rightarrow Tu=0$

$\Leftarrow$. 拉直 + p.o.u $\Rightarrow$ 不妨设 $\begin{cases} u\in W^{1,p}(\mathbb{R}^n_+) \\ Tu=0 \text{ on } \mathbb{R}^{n-1}\end{cases}$

则 $\exists u_m\in C^1(\overline{\mathbb{R}^n_+})$ $\quad u_m\xrightarrow{W^{1,p}} u$ $\quad$ 且 $Tu_m = u_m|_{\mathbb{R}^{n-1}} \xrightarrow{L^p(\mathbb{R}^{n-1})} 0$

对 $x'\in\mathbb{R}^{n-1}$. $x_n\ge 0$ $\Rightarrow |u_m(x',x_n)|\le|u_m(x',0)|+\int_0^{x_n}|u_{m,x_n}(x',t)|dt$

$$\Rightarrow \int_{\mathbb{R}^{n-1}}|u_m(x',x_n)|^p dx \le C\left(\int_{\mathbb{R}^{n-1}}|u_m(x',0)|^p dx' + x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}|Du_m(x',t)|^p\right)$$

$$\to C x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}|Du|^p dx'dt$$

现取 $\zeta\in C^\infty(\mathbb{R})$ s.t. $\begin{cases} 0\le\zeta\le 1 \\ \zeta=1 \text{ on } [0,1] \\ \zeta=0 \text{ on } \mathbb{R}\setminus[0,2]\end{cases}$

$\zeta_m(x) \triangleq \zeta(mx_n)$. $\qquad$ $w_m \triangleq u(x)(1-\zeta_m)$

则 $w_{m,x_n} = u_{x_n}(1-\zeta_m) - mu\zeta'$ $\qquad$ $D_{x'}w_m = D_{x'}u(1-\zeta_m)$

$$\Rightarrow \int_{\mathbb{R}^n_+} |Dw_m - Du|^p dx \le C\int_{\mathbb{R}^n_+} |\zeta_m|^p |Du|^p dx + Cm^p \int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |u|^p dx' dt$$

$$\triangleq A+B$$

$\nearrow Dw_m \to Du$ in $L^p(\mathbb{R}^n_+)$

$m\to\infty$时 $A\to 0$.

又 $B \le Cm^p \left(\int_0^{\frac{2}{m}} t^{p'}dt\right)\left(\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |Du|^p dx' dx^n\right) \to 0$. $(m\to\infty)$

且显然 $w_m \xrightarrow{L^p(\mathbb{R}^n)} u$ $\quad \Rightarrow w_m \xrightarrow{W^{1,p}} u$ $\quad$ 再磨光 w_m 得 $u_m \in C_0^\infty(\mathbb{R}^n_+)$ 且 $u_m \xrightarrow{W^{1,p}} u$ ✓

6. Sobolev不等式

(1) Gigliardo-Nirenberg-Sobolev不等式

$(p^* = \frac{np}{n-p})$

$1\le p<n$ 则 $\exists C = C(n,p)$ s.t. $\forall u \in C_0^1(\mathbb{R}^n)$ $\quad \|u\|_{L^{p^*}} \le C\|Du\|_{L^p}$

pf. $p=1$: $\quad u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \cdots x_{i-1}, y_i, x_{i+1}, \cdots x_n) dy_i$

$$\Rightarrow |u|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1,\cdots x_{i-1}, y_i, x_{i+1}\cdots x_n)| dy_i\right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \le \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du| dy_i\right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |Du| dy_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^n \left(\int_{-\infty}^{\infty} |Du| dy_i\right)^{\frac{1}{n-1}} dx_1$$

$$\underset{\text{Hölder}}{\overset{\text{General}}{\le}} \left(\int_{-\infty}^{\infty} |Du| dy_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dy_i dx_1\right)^{\frac{1}{n-1}}$$

重复操作 $\Rightarrow \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \le \left(\int_{\mathbb{R}^n} |Du| dx\right)^{\frac{n}{n-1}}$

$1<p<n$: $\quad$ 令 $v = |u|^\gamma \Rightarrow \int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} \le c\int_{\mathbb{R}^n} D(|u|^\gamma)$

$$= C\gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} Du$$

$$\le C'\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\right)^{\frac{1}{p}}$$

取 $\gamma = \frac{p(n-1)}{n-p}$ 即可.

Thm. U有界 C^1. $u\in W^{1,p}(U)$ $1\le p<n$ 则∃ $C=C(n,p)$

s.t. $\|u\|_{L^{p^*}(U)}\le C\|u\|_{W^{1,p}(U)}$ ⇒ $u\in L^{p^*}(U)$

pf. 取 $u_m\in C_0^\infty(\mathbb{R}^n)$ $u_m\xrightarrow{W^{1,p}(\mathbb{R}^n)}\underset{\underset{W^{1,p}(\mathbb{R}^n)}{\cap}}{\bar u}$ ⇒ $\|u_m-u_l\|_{L^{p^*}}\to 0$

则 $u_m\to\bar u$ in $L^{p^*}(\mathbb{R}^n)$

$\|\bar u\|_{L^{p^*}}\le\|u_m-\bar u\|_{L^{p^*}}+\|u_m\|_{L^{p^*}}$

$\vee|$

$\|u\|_{L^{p^*}(U)}\le\|u_m-\bar u\|_{L^{p^*}(\mathbb{R}^n)}+C\|Du_m-D\bar u\|_{L^p(\mathbb{R}^n)}+C\|D\bar u\|_{L^p(\mathbb{R}^n)}$

$\underset{\text{延拓(3)}}{\overset{m\to\infty}{\le}} C\|u\|_{W^{1,p}(U)}$

Thm. (Poincaré) $U\subset\mathbb{R}^n$ 有界 $1\le p<n$ $u\in W_0^{1,p}(U)$

则∃ $C=C(n,p,q,U)$ s.t. $\|u\|_{L^q(U)}\le C\|Du\|_{L^p(U)}$ $\forall 1\le q\le p^*$

特别 $\|u\|_{L^p(U)}\le C\|Du\|_{L^p(U)}$

pf. $u_m\in C_0^\infty(U)$ $u_m\xrightarrow{W^{1,p}(U)}u$ 设 $u_m=0$ in $\mathbb{R}^n\setminus\bar U$ ⇒ $u_m\in C_0^\infty(\mathbb{R}^n)$

⇒ $\|u_m\|_{L^{p^*}(\mathbb{R}^n)}\le C\|Du_m\|_{L^p(\mathbb{R}^n)}$

$\underset{m\to\infty}{\overset{\text{上面Thm}}{\Longrightarrow}}\|u\|_{L^{p^*}(U)}\le C\|Du\|_{L^p(U)}$

则 $\forall 1\le q\le p^*$ $\|u\|^q_{L^q(U)}\le\int_U|u|^q dx$

$\le\|u\|^q_{L^{\frac{p^*}{q}}(U)}\|1\|_{L^{\frac{p^*}{p^*-q}}(U)}$

$\le\|u\|^q_{L^{p^*}(U)}|U|^{\frac{p^*-q}{p^*}}$

$\le C\|Du\|^q_{L^p(U)}$

(2) Morrey 不等式

Lemma $u\in C^1(\mathbb{R}^n)$ 则 $\fint_{B(x,r)} |u(z)-u(x)|\,dz \le C_n \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$

pf $\int_{B(x,r)} |u(z)-u(x)|\,dz = \int_0^r \int_{\partial B(x,s)} |u(z)-u(x)|\,d\sigma_z\,ds$

$$= \int_0^r \int_{\partial B(0,1)} |u(x+s\omega)-u(x)|\, s^{n-1}\,d\sigma_\omega\,ds$$

$$\le \int_0^r \int_{\partial B(0,1)} \left(\int_0^s |Du(x+t\omega)|\,dt\right) d\sigma_\omega\, s^{n-1}\,ds$$

$$= \int_0^r \int_0^s \int_{\partial B(x,t)} |Du(y)|\,d\sigma\, \frac{1}{t^{n-1}}\,dt\, s^{n-1}\,ds$$

$$= \int_0^r \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy\, s^{n-1}\,ds \le r^n \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

Thm (Morrey) $n\le p\le +\infty$ $u\in C^1(\mathbb{R}^n)$. $\exists \overset{C(p,n)}{\overset{\shortparallel}{C}}$ s.t $\|u\|_{C^{0,\nu}} \le C\|u\|_{W^{1,p}}$ $(\nu = 1-\frac{n}{p})$

pf $\|u\|_{C^{0,\nu}} = \|u\|_{C^0(\mathbb{R}^n)} + [u]_{C^\nu(\mathbb{R}^n)}$

① $|u(x)| \le \fint_{B(x,1)} |u(x)-u(y)|\,dy + \fint_{B(x,1)} |u(y)|\,dy$

$$\overset{\text{Holder \& Lemma}}{\le} C\left(\int_{B(x,1)} |Du(y)|^p\,dy\right)^{\frac{1}{p}} \left(\int_{B(x,1)} |x-y|^{(1-n)\frac{p}{p-1}}\,dy\right)^{\frac{p-1}{p}} + \fint_{B(x,1)} |u(y)|\,dy$$

$$\overset{\text{Holder}}{\le} C\,\|Du\|_{L^p(B(x,1))} \left(\int_0^1 r^{n-1-(n-1)\frac{p}{p-1}}\,dr\right)^{\frac{p-1}{p}} + \|u\|_{L^p(B(x,1))}\, |B(x,1)|^{1-\frac{1}{p}}$$

$$\le C\|u\|_{W^{1,p}(B(x,1))} \qquad \Rightarrow \|u\|_{C^0(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

② x y. $r=|x-y|$ $W = B(x,r)\cap B(y,r)$

$$\fint_W |u(x)-u(z)|\,dz \le C\fint_{B(x,r)} |u(x)-u(z)|\,dz \underset{\text{Lemma}}{\overset{\text{Holder}}{\le}} C\, r^{1-\frac{n}{p}}\, \|Du\|_{L^p(B(x,r))}$$

$$\Rightarrow |u(x)-u(y)| \le \fint_W |u(x)-u(z)|\,dz + \fint_W |u(y)-u(z)|\,dz \le C r^{\nu}\, \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\le C r^{\nu}\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\Rightarrow \frac{|u(x)-u(y)|}{|x-y|^{\nu}} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$ ✓

Def. u^* 是 u 的一个 version if $u=u^*$ a.e.

Thm. U 有界 C^1. $n<p\le+\infty$ 则对 $u\in W^{1,p}(U)$ $\exists$ version $u^*\in C^{0,\gamma}(\bar{U})$

s.t. $\|u^*\|_{C^{0,\gamma}(U)}\le C\|u\|_{W^{1,p}(U)}$

pf 取 $\bar{u}\in W^{1,p}(\mathbb{R}^n)$ 为延拓 $u_m\to\bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

则 $\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)}\le C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$ $\Rightarrow$ $\{u_m\}$ cauchy in $C^{0,\gamma}(\mathbb{R}^n)$

故设 $u_m\to u^*$ in $C^{0,\gamma}(\mathbb{R}^n)$

则 $\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)}\le C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}\le C'\|u\|_{W^{1,p}(U)}$

7. 紧性

Def X,Y Banach空间 $X\subset Y$ 称 X 紧嵌入到 Y 记 $X\to\to Y$

若 ① $\|x\|_Y\le C\|x\|_X$ ② X 中有界列在 Y 中列紧

Thm (Rellich-Kondrachov紧性定理) U 有界 C^1. $1\le p<n$ 则 $W^{1,p}(U)\to\to L^q(U)$ $(\forall 1\le q<p^*)$

Lemma ① $\forall\varepsilon>0$ $\{u_m\}\subseteq W^{1,p}(U)$ $\|u_m\|_{W^{1,p}(U)}\le C_0$ $u_m^\varepsilon=\eta_\varepsilon * u_m$

则 $\{u_m^\varepsilon\}$ 一致有界且等度连续

② $1\le s\le r\le t\le\infty$ $u\in L^s(U)\cap L^t(U)$ 则 $u\in L^r(U)$

且 $\|u\|_{L^r(U)}\le\|u\|_{L^s(U)}^{\theta}\|u\|_{L^t(U)}^{1-\theta}$ 其中 $\frac{\theta}{s}+\frac{1-\theta}{t}=\frac{1}{r}$

③ $\{u_m\}\subseteq W^{1,p}(V)$ 一致有界 u_m^ε $(\varepsilon\ll 1)$ s.t. $\operatorname{supp}(u_m^\varepsilon)\subset\subset V$

则 $u_m^\varepsilon\to u_m$ in $L^1(V)$ (关于任意 m 一致).

Pf of 紧性 Thm. 由延拓 Thm. 不妨设 $U=\mathbb{R}^n$ 和有界开集 V s.t. $\operatorname{supp}(u_m)\subset V$

设 $\sup_m \|u_m\|_{W^{1,p}(V)} < +\infty$ $\quad u_m^\varepsilon = \eta_\varepsilon * u_m$. 设 $\operatorname{supp}(u_m^\varepsilon) \subset V$

则由② $u_m^\varepsilon \to u_m$ in $L^1(V)$ (对m一致成立)

$$\Rightarrow \|u_m^\varepsilon - u_m\|_{L^q(V)} \overset{②}{\le} \|u_m^\varepsilon - u_m\|_{L^1(V)}^{\theta} \Big(\underbrace{\|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}}_{\le C\|u_m^\varepsilon - u_m\|_{W^{1,p}(V)}}\Big)^{1-\theta} \to 0$$

又①+A-A ⇒ $\forall \varepsilon>0$. $\{u_m^\varepsilon\}$在$C^0(V)$中列紧

则∃子列$\{u_{m_l}^\varepsilon\}$ s.t $\|u_{m_i}^\varepsilon - u_{m_j}^\varepsilon\|_{C^0(V)} \to 0 \Rightarrow \|u_{m_i}^\varepsilon - u_{m_j}^\varepsilon\|_{L^q(V)} \to 0$

固定$\delta>0$. 则∃ε小 s.t $\|u_m^\varepsilon - u_m\|_{L^q(V)} < \frac{\delta}{3}$ (m大)

则 i j 大时 $\|u_{m_i} - u_{m_j}\|_{L^q(V)} \le \|u_{m_i} - u_{m_i}^\varepsilon\|_{L^q(V)} + \|u_{m_i}^\varepsilon - u_{m_j}^\varepsilon\|_{L^q(V)} + \|u_{m_j}^\varepsilon - u_{m_j}\|_{L^q(V)}$ $< \delta$

由对角线论证. ∃$\{u_{m_l}\}$ s.t. $\limsup\limits_{k,l\to\infty} \|u_{m_k} - u_{m_l}\|_{L^q(V)} = 0$

则$\{u_{m_l}\}$为$L^q(V)$中收敛子列 ✓

EX.① $\begin{cases}\Delta u = f(x) & x\in U\\ u|_{\partial U} = 0\end{cases}$ $\quad u \triangleq Kf = (\Delta^{-1})(f)$

$K: H_0^1(U) \subset L^2(U) \to L^2(U)$ 线性

$$\Rightarrow \left|-\int_U |Du|^2\right| = \left|\int_U u\Delta u\right|$$
$$= \left|\int fu\right| \le \|u\|_{L^2(U)} \|f\|_{L^2(U)}$$
$$\le C\|Du\|_{L^2(U)} \|f\|_{L^2(U)}$$
$$\le \frac{1}{2}\|Du\|_{L^2(U)} + C_0\|f\|_{L^2(U)}$$

则 $\|u\|_{H_0^1(U)} \le C\|f\|_{L^2(U)}$

再由紧性Thm K为紧算子.

② (Poincaré 不等式)

Thm. U 有界 C^1 $1\le p\le +\infty$ $\forall u\in W^{1,p}(U)$ $\exists \overset{c(n,p,U)}{\overset{\shortparallel}{C}}$ s.t $\|u-(u)_U\|_{L^p(U)}\le C\|Du\|_{L^p(U)}$

其中 $(u)_U=\frac{1}{|U|}\int_U u\,dx$.

Pf. 若不然. $\exists u_k\in W^{1,p}(U)$ $\|u_k-(u_k)_U\|_{L^p(U)}\ge k\|Du_k\|_{L^p(U)}$

令 $v_k=\dfrac{u_k-(u_k)_U}{\|u_k-(u_k)_U\|_{L^p(U)}}$ 则 $\|v_k\|_{L^p(U)}=1$ $\int_U v_k\,dx=0$

由紧性 Thm $\exists v_{k_j}$ s.t $v_{k_j}\to v$ in $L^q(U)$ $(\forall 1\le q\le p^*)$

又 $\left|\int_U v_k-v\,dx\right|\le |U|^{\frac{p-1}{p}}\|v_k-v\|_{L^p(U)}\to 0$ 则 $\int_U v\,dx=0$. $\|v\|_{L^q(U)}=1$

$\downarrow$ $\nearrow$ 矛盾!

$v=0$

$\uparrow$

又 $\forall\phi\in C_0^\infty(U)$. $\int_U v\phi_{x_i}\,dx=\lim_{j\to\infty}\int_U v_{k_j}\phi_{x_i}\,dx$

$=-\lim_{j\to\infty}\int_U (v_{k_j})_{x_i}\phi\,dx$

$\le \lim_{j\to\infty}\underbrace{\|Dv_{k_j}\|_{L^p(U)}}_{\le\frac{1}{k}}\|\phi\|_{L^q(U)}=0$ $\Rightarrow Dv=0$

§ 差商

Def. $D_i^h u=\dfrac{u(x+he_i)-u(x)}{h}$ $(x\in V.\ h\in\mathbb{R}\ 0<|h|<d(V,\partial U))$

$D^h u=(D_1^h u,\cdots D_n^h u)$

Thm. (1) $1\le p<+\infty$. $u\in W^{1,p}(U)$. 则对 $V\subset\subset U$ $0<h<\frac{1}{2}d(V,\partial U)$

有 $\|D^h u\|_{L^p(V)}\le C\|Du\|_{L^p(U)}$

(2) $1<p<+\infty$. $u\in L^p(V)$. 且 $\|D^h u\|_{L^p(V)}\le C$ $(\forall\, 0<|h|<\frac{1}{2}d(V,\partial U))$

则 $u\in W^{1,p}(V)$

pf. $\int_V u(x) D_i^h \varphi dx = \int_V u(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx$

$$= \int_V u(y-he_i) \frac{\varphi(y)}{h} dy - \int_V u(y-he_i) \frac{\varphi(y-he_i)}{h} dy$$

$$= \int_V \frac{u(x-he_i) - u(x)}{h} \varphi(x) dx = - \int_V (D_i^{-h} u(x)) \varphi(x) dx$$

(1) 由逼近不妨 u 光滑

$$|u(x+he_i) - u(x)| = \left|\int_0^1 u_{x_i}(x+the_i) he_i dt\right| \leq h \int_0^1 |Du(x+the_i)| dt$$

$$\Rightarrow \|D^h u\|_{L^p(V)} = \left(\int_V |D^h u|^p dx\right)^{\frac{1}{p}} = \left(\int_V \left(\sum_{i=1}^n |D_i^h u|^2\right)^{\frac{2}{p}} dx\right)^{\frac{1}{p}}$$

$$\overset{\text{Minkowski}}{\leq} \sum_{i=1}^n \left(\int_V |D_i^h u(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq \sum_{i=1}^n \int_0^1 \left(\int_V \left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^p dx\right)^{\frac{1}{p}} dt$$

$$\overset{\text{Minkowski}}{\leq} C\|Du\|_{L^p(V)} \leq C\|Du\|_{L^p(U)}$$

(2) $\forall\, 0<h<\frac{1}{2} d(V, \partial U)$ $\varphi \in C_0^\infty(V)$

则 $\sup_h \|D_i^{-h} u\|_{L^p(V)} < \infty$. 又 L^p 自反. 故有界 $\Rightarrow$ 弱列紧.

即 $\exists v_i \in L^p(V)$. $h_k \to 0$ s.t. $D_i^{-h_k} u \rightharpoonup v_i$ in $L^p(V)$

则 $\int_V u \varphi_{x_i} dx = \int_U u \varphi_{x_i} dx = \lim_{h_k \to 0} \int_U u D_i^{h_k} \varphi dx$

$$= -\lim_{h_k \to 0} \int_V D_i^{-h_k} u \,\varphi dx$$

$$= -\int_V v_i \varphi dx = -\int_U v_i \varphi dx$$

则 $v_i = u_{x_i} \in L^p(V) \Rightarrow Du \in L^p(V)$ $\quad u \in W^{1,p}(V)$

(且有 $\|Du\|_{L^p(V)} \leq C$)

六. 散度型二阶一致椭圆方程

1. 解的存在唯一性.

Def. $u\in H_0^1(U)$ 为方程 $\begin{cases} -\Delta u=f & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 的弱解 若

$\forall v\in H_0^1(U)$ $\int_U \nabla u\nabla v\,dx=\int_U fv\,dx$

Thm. 若 $f\in L^2(U)$. 则弱解存在唯一

pf. 定义 $J(v)=\frac{1}{2}\int_U |Dv|^2dx-\int_U fv\,dx$

由 $\left|\int_U fv\,dx\right| \le \|f\|_{L^2(U)}\|v\|_{L^2(U)} \le C\|f\|_{L^2(U)}\|Dv\|_{L^2(U)}$

$\le \frac{1}{4}\int_U |Dv|^2dx+C'^2\|f\|_{L^2(U)}$

$\Rightarrow J(v)\ge \frac{1}{4}\int_U |Dv|^2dx - C'^2\|f\|_{L^2(U)}$ 有下界

设 $J_0=\inf\limits_{v\in H_0^1(U)} J(v)$ $\Rightarrow \exists v_k\in H_0^1(U)$ s.t $J_0\le J(v_k)<J_0+\frac{1}{k}$

则 $\|D(v_k-v_l)\|^2_{L^2(U)}=\int_U 2(|Dv_k|^2+|Dv_l|^2)-(Dv_k+Dv_l)^2dx$

$=4J(v_k)+4J(v_l)-2J(v_k+v_l)+2\int_U f(v_k+v_l)dx$

$=4J(v_k)+4J(v_l)-8J(\frac{v_k+v_l}{2})$

$\le 4(J_0+\frac{1}{k})+4(J_0+\frac{1}{l})-8J_0 \rightarrow 0$

$\Rightarrow \{Dv_k\}$ Cauchy in $L^2(U)$

由Poincaré $\{v_k\}$ Cauchy in $L^2(U)$

则 $v_k\rightarrow v_0$ in $H_0^1(U)$. $J(v_0)=J_0$

再证 v_0 确实为弱解

$\forall \varphi \in H_0^1(U) \quad h(t) \triangleq J(v_0 + t\varphi)$

则 $h'(t)|_{t=0} = \frac{d}{dt}\Big|_{t=0} \left(\frac{1}{2}\int_U |D(v_0+t\varphi)|^2 - \int_U f(v_0+t\varphi)dx \right)$

$\overset{\|}{0}$

$= \int_U Dv_0 D\varphi - f\varphi dx$ ✓

唯一性. 若$u.v \in H_0^1(U)$弱解

则$\forall \varphi \in H_0^1(U)$. $\int_U Du D\varphi dx = \int_U f\varphi = \int_U Dv D\varphi$ 取$\varphi = u-v \Rightarrow u=v$ ✓

定义 $B[u.v] = \int_U a^{ij} u_i v_j + b^i u_i v + cuv\, dx$

Thm (能量估计) $\exists \alpha. \beta > 0 \quad \nu \geq 0$ s.t. $\forall u.v \in H_0^1(U)$

$$|B[u.v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \qquad ①$$

$$\beta \|u\|^2_{H_0^1(U)} \leq B[u.u] + \nu \|u\|^2_{L^2(U)} \qquad ②$$

pf. 式①. $|B[u.v]| \leq \|a^{ij}\|_{L^\infty(U)} \int_U |Du||Dv| dx + \|b^i\|_{L^\infty(U)} \int_U |Du||v| dx$

$+ \|c\|_{L^\infty(U)} \int_U |u||v| dx$

$\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

式②: $\lambda \|Du\|^2_{L^2(U)} \leq \int_U a^{ij} u_i u_j dx = B[u.u] - \int_U b^i u_i u - cu^2 dx$

$\leq B[u.u] + \|b^i\|_{L^\infty(U)} \left(\varepsilon \int_U |Du|^2 + \frac{1}{4\varepsilon} \int_U |u|^2 \right)$

$+ \|c\|_{L^\infty(U)} \int_U u^2 dx$

取 $\varepsilon \|b^i\|_{L^\infty(U)} < \frac{\lambda}{2}$

$\Rightarrow \frac{\lambda}{2} \|Du\|^2_{L^2(U)} \leq B[u.u] + C\|u\|^2_{L^2(U)}$

$\overset{\text{Poincaré}}{\leq} B[u.u] + C\|Du\|^2_{L^2(U)}$

$\Rightarrow \beta \|u\|^2_{H_0^1(U)} \leq B[u.u] + \nu \|u\|^2_{L^2(U)}$ ✓

Thm.(弱解第一存在定理) $\exists \nu \geq 0$ s.t. $\forall \mu \geq \nu$ $f \in L^2(U)$

方程 $\begin{cases} Lu+\mu u=f & \text{in } U \\ u|\partial U=0 \end{cases}$ 有唯一弱解

$H^{-1}(U)$
$\cup$
$(\forall u.v \in H_0^1(U))$

Pf. 取ν同能量估计中所示　定义 $B_\mu[u,v]=B[u,v]+\mu(u,v)$

$|B_\mu[u,v]| \leq (\alpha+\mu)\|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

$\beta\|u\|^2_{H_0^1(U)} \leq B[u,u]+\nu\|u\|^2_{L^2(U)} \leq B_\mu[u,u]$

由Lax-Milgram $\forall f \in L^2(U) \subset H^{-1}(U)$ $\exists!\, u \in H_0^1(U)$

s.t. $B_\mu[u,v] = \langle f,v \rangle$ $(\forall v \in H_0^1(U))$

$= (f,v)_{L^2(U)}$ $\Rightarrow$ u为弱解　✓

Thm (Fredholm二择一) H: Hilbert空间　$K: H \to H$ 紧算子

则 (1) $N(I-K)$有限维　(2) $R(I-K)$闭

(3) $R(I-K)=N(I-K^*)^\perp$　(4) $N(I-K)=\{0\} \Leftrightarrow R(I-K)=H$

(5) $\dim N(I-K)=\dim N(I-K^*)$.

Pf. 略 泛函分析中学过.

Def (1) L的形式伴随 $L^*v = -(a^{ij}v_j)_i + (c-b^i_i)v$

(2) $B^*: H_0^1(U) \times H_0^1(U) \to \mathbb{R}$.　$B^*[v,u]=B[u,v]$

(3) $v \in H_0^1(U)$ 为 $\begin{cases} L^*v=f & \text{in } U \\ v|\partial U=0 \end{cases}$ 的弱解. 若 $B^*[v,u]=(f,u)$ $(\forall u \in H_0^1(U))$

Thm (弱解第二存在定理) ① 下面两种只有一种发生

(i) $\forall f\in L^2(U)$ $\begin{cases} Lu=f & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 弱解存在唯一 (边值问题)

(ii) $\begin{cases} Lu=0 & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 存在弱解 $u\neq 0$ (齐次问题)

② 若(ii)成立 解空间 $N\subset H_0^1(U)$ 有限维 且与 N^* 同维数

其中N^*为 $\begin{cases} L^*v=0 & \text{in } U \\ v|_{\partial U}=0 \end{cases}$ 的解空间

③ (i)有解 $\Leftrightarrow (f,v)=0$ $(\forall v\in N^*)$

Pf. 取 $\mu=\gamma$ $L_\gamma u=Lu+\gamma u$ 则由第一存在性. $\forall g\in L^2(U)$

$\exists!\, u\in H_0^1(U)$ s.t. $B_\gamma[u,v]=(g,v)_{L^2(U)}$ $(\forall v\in H_0^1(U))$ 记 $u=L_\gamma^{-1}g$

而u为边值问题弱解 $\Leftrightarrow B_\gamma[u,v]=(\gamma u+f,v)$

$$\Leftrightarrow u=L_\gamma^{-1}(\gamma u+f) \underset{h=L_\gamma^{-1}f}{\overset{K=\gamma L_\gamma^{-1}}{\Leftrightarrow}} (I-K)u=h$$

$\forall g\in L^2(U)$ $u=L_\gamma^{-1}g$ 则 $\beta\|u\|^2_{H_0^1(U)}\le B_\gamma[u,u]=(g,u)$

$$\le\|g\|_{L^2(U)}\|u\|_{H_0^1(U)}$$

即 $\|Kg\|_{H_0^1(U)}\le\frac{\gamma}{\beta}\|g\|_{L^2(U)}$

再由紧性定理知 K为紧算子

则: 要么 (i) $\forall h\in L^2(U)$ $(I-K)u=h$ 有唯一解 $u\in L^2(U)$

(ii) $(I-K)u=0$ 有非零解 且解空间有限维 $\Rightarrow$ ①② ✓

③: (i) $\Rightarrow h\in N(I-K^*)^\perp$ $\forall v\in N^*$

$0=(h,v)=(L_\gamma^{-1}f,v)=\frac{1}{\gamma}(f,K^*v)=\frac{1}{\gamma}(f,v)$. 反之显然 ✓

Ex. ① 要么 $\begin{cases} x''(t)+\lambda x(t) = f(t) & t\in[0,1] \\ x(0)=x(1)=0 \end{cases}$ 存在唯一解

要么 $\begin{cases} x''(t)+\lambda x(t)=0 & t\in[0,1] \\ x(0)=x(1)=0 \end{cases}$ 有非零解 此时必有 $\lambda=(k\pi)^2$

边值问题有解 $\Leftrightarrow \int_U f(t)\sin k\pi t\, dt=0$

② $\begin{cases} \Delta u+2u=f & \text{in } U=[0,1]^2 \\ u|_{\partial U}=0 \end{cases}$ 有解 $\Leftrightarrow \int_0^1\int_0^1 f(x,y)\sin \pi x \sin \pi y\, dxdy = 0$

Thm (弱解第三存在定理) (1) 存在至多可数集 $\Sigma\subset\mathbb{R}$ s.t.

$\forall f\in L^2(U)$, $\begin{cases} Lu=\lambda u+f & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 存在唯一弱解 $\Leftrightarrow \lambda\notin\Sigma$

(2) 若 Σ 无限, 则 $\lambda_k\to+\infty$ $(k\to+\infty)$.

Pf. 不妨设能量估计中 $\nu>0$ 且设 $\lambda>\nu$

则边值问题有唯一弱解 $\Leftrightarrow \begin{cases} Lu=\lambda u & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 只有零解

$\Leftrightarrow \begin{cases} L_\nu u=(\lambda+\nu)u & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 只有零解

$L_\nu u=(\lambda+\nu)u \Leftrightarrow u=\frac{\nu+\lambda}{\nu}Ku$

$\Leftrightarrow \frac{\nu}{\nu+\lambda}$ 不为 K 的特征值

不妨 $\sigma(K)=\{\mu_i\}_{i=1}^{+\infty}$ $\lambda_i=\frac{\nu}{\mu_i}-\nu\to+\infty$

(则由K紧 $\mu_i\to0$)

令 $\Sigma=\{\lambda_i\}$ 即可 ✓

Thm (逆的有界性)

若$\lambda \notin \Sigma$ 则$\exists C$ s.t. $\|u\|_{L^2(U)} \le C\|f\|_{L^2(U)}$

其中$f \in L^2(U)$ $u \in H_0^1(U)$且$\begin{cases} Lu=(\lambda+\nu)u+f & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ $C=C(\lambda,U,L)$

Pf. 取$\{f_k\} \subset L^2(U)$ $\{u_k\} \subset H_0^1(U)$ s.t. $\begin{cases} Lu_k=\lambda u_k+f_k & \text{in } U \\ u_k|_{\partial U}=0 \end{cases}$

且$\|u_k\|_{L^2(U)}=1$ $\Rightarrow \|f_k\|_{L^2(U)} \to 0$

由能量估计 $\{u_k\}$在$H_0^1(U)$中有界 $\Rightarrow u_{k_j} \rightharpoonup u$ in $H_0^1(U)$

又由紧性 $H_0^1(U) \subset\subset L^2(U)$. $\exists u_{k_{j_i}} \to u$ in $L^2(U)$

则$\begin{cases} Lu=\lambda u & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ $\overset{\lambda\notin\Sigma}{\Longrightarrow} u=0$ 矛盾!

2. 正则性

(1) 内正则性

先回忆在差商一节中的结果:

(1) $1 \le p < \infty$ $u \in W^{1,p}(U)$ 则$\forall V \subset\subset U$ $0<h<\frac{1}{2}d(V,\partial U)$

$\exists C$ s.t. $\|D^h u\|_{L^p(V)} \le \|Du\|_{L^p(V)}$ $(*)$

(2) $1<p\le\infty$ $u \in L^p(V)$. 且对h一致 $\|D^h u\|_{L^p(V)} \le C$ 则 $u \in W^{1,p}(V)$ $\|u\|_{L^p(V)} \le C$ $(**)$

Thm (内H^2正则) $a^{ij} \in C^1(U)$. $b^i, c \in L^\infty(U)$ $f \in L^2(U)$ 设$u \in H^1(U)$ 为$Lu=f$的弱解. 则 $u \in H^2_{loc}(U)$ 且 $\forall V \subset\subset U$

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)}+\|u\|_{L^2(U)}) \quad C=C(V,U,L)$$

pf. 取 $V \subset\subset W \subset\subset U$ 和 $\xi \in C_0^\infty(W)$ s.t. $\begin{cases} \xi = 1 \text{ on } V \\ \xi = 0 \text{ on } W^c \\ 0 \le \xi \le 1 \\ |\nabla \xi| \le C \end{cases}$

$Lu=f \Rightarrow \overset{\overset{A}{\shortparallel}}{\int_U a^{ij} u_i v_j dx} = \overset{\overset{B}{\shortparallel}}{\int_U \tilde{f} v dx}.$ $\tilde{f} = f - b^i u_i - cu$ $v \in H_0^1(U)$

取 $|h|$ 小. $v = -D_k^{-h}(\xi^2 D_k u)$

$$A = -\int_U a^{ij} u_i [D_k^{-h}(\xi^2 D_k u)]_j = -\int_U a^{ij} u_i D_k^{-h}\left((\xi^2 D_k^h u)_j\right)$$

$$= \int_U D_k^h(a^{ij} u_i)(\xi^2 D_k^h u)_j = \int_U a_h^{ij}(D_k^h u_i)(\xi^2 D_k^h u)_j + (D_k^h a^{ij}) u_i (\xi^2 D_k^h u)_j$$

$a_h^{ij}(x) \triangleq a^{ij}(x+he_k)$

$$= \int_U a_h^{ij}(D_k^h u_i)(D_k^h u_j)\xi^2 + \int_U 2a_h^{ij}\xi\xi_j(D_k^h u_i)(D_k^h u)$$
$$+ (D_k^h a^{ij}) u_i (D_k^h u_j)\xi^2 + 2\xi\xi_j(D_k^h a^{ij}) u_i D_k^h u$$

$\triangleq A_1 + A_2$

$A_1 \le \theta \int_U \xi^2 |D_k^h Du|^2$

$|A_2| \le C_1 \int_U \xi |D_k^h Du| |D_k^h u| + \xi |D_k^h Du| |Du| + \xi |D_k^h u| |Du|$

$\le \varepsilon \int_U \xi^2 |D_k^h Du|^2 + \frac{C_2}{\varepsilon} \int_W |D_k^h u|^2 + |Du|^2$

$\varepsilon = \frac{\theta}{2}$ $\overset{(*)}{\le} \frac{\theta}{2} \int_U \xi^2 |D_k^h Du|^2 + C_3 \int_U |Du|^2$

故 $A \ge \frac{\theta}{2} \int_U \xi^2 |D_k^h Du|^2 - C_3 \int_U |Du|^2$

又 $|B| \le C_1' \int_U (|f| + |Du| + |u|)|v| dx$

其中 $\int_U |v|^2 dx \overset{(*)}{\le} C_2' \int_U |D(\xi^2 D_k^h u)|^2 dx \le C_3' \int_W |D_k^h u|^2 + \xi^2 |D_k^h Du|^2$

$\overset{(*)}{\le} C_4' \int_U |Du|^2 + \xi^2 |D_k^h Du|^2 dx$

故 $|B| \le \varepsilon \int_U \xi^2 |D_k^h Du|^2 + \frac{c}{\varepsilon}\int_U f^2 + \frac{c}{\varepsilon}\int u^2 + \frac{c}{\varepsilon}\int_U |Du|^2$

$$\overset{\varepsilon=\frac{\theta}{4}}{\le} \frac{\theta}{4}\int_U \xi^2 |D_k^h Du|^2 + C\left(\int_U f^2+u^2+|Du|^2 dx\right)$$

又 $A=B \Rightarrow \int_V |D_k^h Du|^2 \le \int_U \xi^2 |D_k^h Du|^2 \le C\int_U f^2+u^2+|Du|^2 dx$

由(**)知 $Du \in H^1_{loc}(U) \Rightarrow u \in H^2_{loc}(U)$

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

Thm(高阶内正则) $m\in\mathbb{Z}^+$ $a^{ij}, b^i, c \in C^{m+1}(U)$ $f\in H^m(U)$ $u\in H^1(U)$ 弱解

则 $u\in H^{m+2}_{loc}(U)$ 且 $\forall V\subset\subset U$ $\|u\|_{H^{m+2}(V)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$

pf 略 见Evans 6.3 Thm 2

Rmk $m\to\infty$ 可得关于光滑性的条件

(2) 边界正则性

Thm(边界→整体正则) $a^{ij}\in C^1(\bar{U})$ $b^i, c\in L^\infty(U)$ $f\in L^2(U)$

若 $u\in H^1_0(U)$ 且为 $\begin{cases} Lu=f & \text{in } U \\ u|_{\partial U}=0 \end{cases}$ 弱解. $\partial U\in C^2$

则 $u\in H^2(U)$ 且 $\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$ $C=C(U,L)$

pf. 仍然采用 拉直 + 局部 → 整体的论证

①: $U = B^0(0,1)\cap\mathbb{R}^n_+$ $V = B^0(0,\frac{1}{2})\cap\mathbb{R}^n_+$

$\xi\in C_0^\infty(U)$ s.t. $\begin{cases} \xi=1 & \text{on } B^0(0,\frac{1}{2})\cap U \\ \xi=0 & \text{on } \mathbb{R}^n\setminus B^0(0,1) \\ 0\le\xi\le 1 \\ |D\xi|\le C \end{cases}$

则由u为弱解 $\forall v\in H_0^1(U)$ $\int_U a^{ij}u_iv_j\,dx=\int_U \tilde{f}v$ $\tilde{f}=f-b^iu_i-cu$

(i) 取$h>0$小 $1\le k\le n-1$ 令 $v=-D_k^{-h}(\zeta^2D_k^hu)\in H_0^1(U)$

(k=n时 则无法定义v)

则用同内H^2正则的方法可知 $u_k\in H^1(V)$ 且

$$\sum_{\substack{i,j=1\\ i+j<2n}}^n \|u_{ij}\|_{L^2(V)}\le C(\|f\|_{L^2(U)}+\|u\|_{H^1(U)})$$

(ii) $Lu=f \Rightarrow a^{nn}u_{nn}=-\sum_{\substack{i,j=1\\ i+j<2n}}^n a^{ij}u_{ij}+b^iu_i+cu-f$

一致椭圆 $\Rightarrow a^{nn}\ge\theta>0$

$$\Rightarrow |u_{nn}|\le C\Big(\sum_{\substack{i,j=1\\ i+j<2n}}^n |u_{ij}|+|Du|+|u|+|f|\Big)$$

则 $u\in H^2(V)$ 且 $\|u\|_{H^2(V)}\le C(\|f\|_{L^2(U)}+\|u\|_{L^2(U)})$

②一般情况 (这里为了体现$\partial U\in C^2$的特别, 加以详细论证)

取 $x^0\in\partial U \Rightarrow \exists\gamma\in C^2(\mathbb{R}^{n-1})$ s.t. $U\cap B(x^0,r)=\{x\in B(x^0,r)\mid x_n>\gamma(x_1\cdots x_{n-1})\}$

(不妨$x^0=(0,\cdots0)$) 则作变换 $y=\Phi(x)$ $\begin{cases} y_i=x_i, & 1\le i\le n-1\\ y_n=x_n-\gamma(x_1,\cdots x_{n-1}) & i=n\end{cases}$ 逆变换时 $x=\Psi(y)$

取s充分小 $U'=B^0(0,s)\cap\{y_n>0\}$ $V'=B^0(0,\frac{s}{2})\cap\{y_n>0\}$

$u'(y)\triangleq u(\Psi(y))\in H^1(U')$ 且 $u'=0$ on $\partial U'\cap\{y_n=0\}$

令 $L'u'=-\sum_{k,l=1}^n(a'^{kl}u'_{y_k})_{y_l}+\sum_{k=1}^n b'^ku'_{y_k}+c'u'$

其中 $a'^{kl}(y)=\sum_{r,s=1}^n a^{rs}(\Psi(y))\Phi^k_{x_r}(\Psi(y))\Phi^l_{x_s}(\Psi(y))$

$b'^k(y)=\sum_{r=1}^n b^r(\Psi(y))\Phi^k_{x_r}(\Psi(y))$

$c'(y)=c(\Psi(y))$ $f'(y)=f(\Psi(y))$

可以验证 u' 为 $L'u'=f'$ 的弱解

且由于 ∂U C^2 $\Rightarrow \Phi,\Psi$ C^2. 则 a'^{kl}. b'^{k} C^1 ★

则可用①中论证得 $\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)}+\|u\|_{L^2(U)})$ $V=\Phi(V')$

③最后有限覆盖即可. ✓

3. 特征值

Thm. $Lu=-(a^{ij}u_j)_i$ $a^{ij}\in C^\infty(\bar{U})$ (a^{ij}) 对称一致椭圆 U连通

则 (1) L 的特征值为实数

(2) 特征值(计重数)为 $\Sigma=\{\lambda_k\}_{k=1}^{+\infty}$ $0<\lambda_1\le\lambda_2\le\cdots$ $\lambda_k\to+\infty$

(3) $\exists$ $L^2(U)$ 的正交规范基 $\{w_k\}_{k=1}^{+\infty}$ s.t. $\begin{cases} -Lw_k=\lambda_k w_k & \text{in } U \\ w_k|_{\partial U}=0 \end{cases}$

(由 $a^{ij}\in C^\infty$ 与正则性理论有 $w_k\in C^\infty(U)$

若还 ∂U C^∞ 则 $w_k\in C^\infty(\bar{U})$)

Pf 可以说明 $Lu=0$ 只有零解 则0不为特征值 且可定义

$S=L^{-1}: L^2(U)\to L^2(U)$. 显然有界对称

由泛函分析知识 只须证 S 为紧算子

对 $f\in L^2(U)$, $u=L^{-1}f=Sf\in H_0^1(U)$

$\Rightarrow \int_U Lu \cdot u\, dx = \int_U fu\, dx$

$LHS = -\int_U (a^{ij}u_i)_j\, u\, dx = \int_U a^{ij}u_i u_j\, dx \geq \theta \int_U |Du|^2 dx$

$RHS \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)} \leq C\|f\|_{L^2} \|Du\|_{L^2}$

$\leq \frac{\theta}{2}\int_U |Du|^2 dx + C'\int_U |f|^2 dx$

$\Rightarrow \|Du\|_{L^2(U)} \leq \frac{2C}{\theta} \|f\|_{L^2(U)}$ 则 $\|u\|_{H_0^1(U)} \leq C'\|f\|_{L^2(U)}$

由紧性 Thm $H_0^1(U) \subset\subset L^2(U)$ 则 S为紧算子 ✓

Thm (Weyl's law) $\begin{cases} \Delta u + \lambda u = 0 & \text{in } U \\ u|_{\partial U} = 0 \end{cases}$ U光滑有界开集

则 $\lim\limits_{k\to\infty} \frac{\lambda_k^{\frac{n}{2}}}{k} = \frac{(2\pi)^n}{|U|\,\alpha_n}$. 且 λ_1 称为主特征值

Thm (刻画 λ_1) (1) $\lambda_1 = \min\{B[u,u] \,/\, u \in H_0^1(U)\ \ \|u\|_{L^2(U)} = 1\}$

(2) λ_1 对应的特征函数 $w_1 > 0$ in U. $\begin{cases} Lw_1 = \lambda w_1 & \text{in } U \\ w_1|_{\partial U} = 0 \end{cases}$

(3) $u \in H_0^1(U)$ 为 $\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u|_{\partial U} = 0 \end{cases}$ 的弱解. 则 u 为 w 的倍数.

pf. 只证(1)

$\forall k \neq l$. $B[w_k, w_k] = \lambda_k$ $B[w_k, w_l] = 0$ 由 Parseval 恒等式

$u = \sum_{k=1}^{\infty} d_k w_k$ $d_k = (u, w_k)_{L^2(U)}$ 且 $\sum d_k^2 = 1$

则在 $H_0^1(U)$ 上用 $B[\cdot,\cdot]$ 诱导范数. 有 $\{\frac{w_k}{\lambda_k^{\frac{1}{2}}}\}$ 为 $H_0^1(U)$ 的正交规范基

则 $\forall u \in H_0^1(U)$. $\|u\|_{L^2(U)} = 1$ $B[u,u] = \sum_{k=1}^{n} d_k^2 \lambda_k \leq \lambda_1$

同时取 $u = w_1$ 时 $B[u,u] = \lambda_1$ 故得证

Thm (Faber-Kahn) $\Omega \subset \mathbb{R}^n$ 有界区域 $|\Omega| = |B_R(0)|$ 则 $\lambda_1(B_R(0)) \leq \lambda_1(\Omega)$

Pf. 对 $\begin{cases} \Delta f = -\lambda_1 f & \text{in } U \\ f|_{\partial U} = 0 \end{cases}$

引入 $g: B_R(0) \to \mathbb{R}_+$ （称为f的重排对称）

s.t. $|\{f \geq c\}| = |\{g \geq c\}|$

$g|_{S_R(0)} = 0$ 且g为径向函数

又 $\lambda_1 \int_\Omega f^2 dx = \int_\Omega |\nabla f|^2 dx$

$\parallel$

$\lambda_1 \int_0^{+\infty} |\{f^2 > c\}| dx = \lambda_1 \int_{B_R(0)} g^2 dx$

$\Rightarrow \begin{cases} \Delta g = -\tilde{\lambda}_1 g & \text{in } B_R(0) \\ g|_{S_R(0)} = 0. \end{cases}$

$$\tilde{\lambda}_1 = \frac{\int_{B_R(0)} |\nabla g|^2 dx}{\int_{B_R(0)} g^2 dx}$$

只须证 $\tilde{\lambda}_1 \leq \lambda_1$ 则证 $\int_{B_R(0)} |\nabla g|^2 dx \leq \int_\Omega |\nabla f|^2 dx$

又 $|\{g=c\}|^2 = \left(\int_{\{g=c\}} 1 d\sigma\right)^2 = \int_{\{g=c\}} |\nabla g| d\sigma \cdot \int_{\{g=c\}} \frac{1}{|\nabla g|} d\sigma$

由等周不等式 $|\{f=c\}| \geq |\{g=c\}|$.

由余面积公式 $-\frac{d}{dc} |\{f \geq c\}| = \int_{\{f=c\}} \frac{1}{|\nabla f|} d\sigma$ g类似

又 $|\{f \geq c\}| = |\{g \geq c\}| \Rightarrow \int_{\{f=c\}} \frac{1}{|\nabla f|} d\sigma = \int_{\{g=c\}} \frac{1}{|\nabla g|} d\sigma$

$\Rightarrow \int_{\{f=c\}} |\nabla f| d\sigma \geq \int_{\{g=c\}} |\nabla g| d\sigma$

则 $\int_\Omega |\nabla f|^2 dx = \int_0^\infty \int_{\{f=c\}} |\nabla f| d\sigma dc \geq \int_0^\infty \int_{\{g=c\}} |\nabla g| d\sigma dc = \int_{B_R(0)} |\nabla g|^2$ ✓

4. 弱解的有界估计

(1) De Giorgi-Nash-Moser 理论

Def. $u\in H_0^1(U)$ 为 $\begin{cases} -(a^{ij}u_j)_i = f \text{ in } U \\ u|_{\partial U}=0 \end{cases}$ 的弱$\overset{(\text{下})}{\text{上}}$解.

若 $\forall \varphi\in H_0^1(U)$ $\varphi\geq 0$ 有 $\int_U a^{ij}u_j\varphi_i\,dx \underset{(\leq)}{\geq} f\varphi$

Lemma ① $\Phi\in C^{0,1}_{loc}(\mathbb{R})$ 凸 $f=0$. 则若 u 为弱$\overset{(\text{上})}{\text{下}}$解. $\Phi'(s)\overset{(\leq)}{\geq}0$ 则 $v=\Phi(u)$ 也为弱$\underset{(\text{上})}{\text{下}}$解

pf. 先设 $\Phi\in C^2_{loc}(\mathbb{R})$. 再逼近即可

② $\|v\|_{L^\infty(U)} = \lim\limits_{p\to\infty}\|v\|_{L^p(U)}$

Thm. $u\in W^{1,2}(U)$ 为弱下解 $a^{ij}\in L^\infty(U)$ $c\in L^q(U)$ $q>\frac{n}{2}$ $B_1\subset\subset U$

则若 $f\in L^q(B_1)$. 有 $u^+\in L^\infty_{loc}(B_1)$ 且 $\forall\theta\in(0,1)$ $p>0$

$$\sup_{B_\theta}u^+\leq C\{\frac{1}{(1-\theta)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)}+\|f\|_{L^q(B_1)}\} \quad C=C(n,\lambda,\Lambda,p,q)$$

pf. 先证 $\theta=\frac{1}{2}$, $p=2$ 情形. 使用 De Giorgi 和 Moser 的两种方法

I. De Giorgi 方法

$v=(u-k)^+$ $(k\geq 0)$ $\quad \zeta\in C_0^1(B_1)$ $\quad \varphi=v\zeta^2$

则取 φ 为 test function $\Rightarrow$ 积分区域为 $\{u>k\}$. $Dv=Du$ a.e

$$\int a^{ij}u_i\varphi_j = \int a^{ij}u_iv_j\xi^2 + 2a^{ij}u_iv\xi\xi_j$$

$$\geq \lambda\int|Dv|^2\xi^2 - 2\Lambda\int|Dv||D\xi|v\xi$$

$$\geq \frac{\lambda}{2}\int|Dv|^2\xi^2 - \frac{2\Lambda^2}{\lambda}\int|D\xi|^2v^2$$

则 $\int|Dv|^2\xi^2 \leq C\left(\int v^2|D\xi|^2 + |f|v\xi^2\right)$

$\Rightarrow \int|D(v\xi)|^2 \leq C'\left(\int v^2|D\xi|^2 + |f|v\xi^2\right)$

又 $\int|f|v\xi^2 \overset{\text{Hölder}}{\leq} \left(\int|f|^q\right)^{\frac{1}{q}}\left(\int|v\xi|^{2^*}\right)^{\frac{1}{2^*}}|\{v\xi\neq0\}|^{1-\frac{1}{q}-\frac{1}{2^*}}$

$$\overset{\text{sobolev}}{\leq} C\|f\|_{L^q}\left(\int|D(v\xi)|^2\right)^{\frac{1}{2}}|\{v\xi\neq0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}}$$

$$\leq \delta\int|D(v\xi)|^2 + C(n,\delta)\|f\|_{L^q}^2|\{v\xi\neq0\}|^{\underbrace{1+\frac{2}{n}-\frac{2}{q}}_{>1-\frac{1}{q}\ (\text{因为}\ q>\frac{n}{2})}}$$

则 $\int|D(v\xi)|^2 \leq C'\left(\int v^2|D\xi|^2 + (k^2+F^2)|\{v\xi\neq0\}|^{1-\frac{1}{q}}\right)$ $F=\|f\|_{L^q(B_1)}$

$$\int v^2\xi^2 \overset{\text{Hölder}}{\leq}\left(\int(v\xi)^{2^*}\right)^{\frac{2}{2^*}}|\{v\xi\neq0\}|^{1-\frac{2}{2^*}} \overset{\text{sobolev}}{\leq} C\int|D(v\xi)|^2|\{v\xi\neq0\}|^{\frac{2}{n}}$$

$\Rightarrow \int(v\xi)^2 \leq C\left(\int v^2|D\xi|^2|\{v\xi\neq0\}|^{\frac{2}{n}} + (k+F)^2|\{v\xi\neq0\}|^{\underbrace{1+\frac{2}{n}-\frac{1}{q}}_{<1+\frac{2}{n}}}\right)$

(当$|\{v\xi\neq0\}|$小时成立)

则固定 $0<r<R\leq1$ $\xi\in C_0^\infty(B_R)$ $\xi=1$ in B_r 且 $\begin{cases}0\leq\xi\leq1\\ |D\xi|\leq\frac{2}{R-r}\end{cases}$ in B_1

则令 $A(k,r)=\{x\in B_r \mid u\geq k\}$. $\exists k_0$ 当 $k\geq \overset{\overset{C\|u^+\|_{L^2(B_1)}}{\shortparallel}}{k_0}$ 时

$$\int_{A(k,r)}(u-k)^2 \leq C\left(\frac{1}{(R-r)^2}|A(k,R)|^{\frac{2}{n}}\int_{A(k,R)}(u-k)^2 + (k+F)^2|A(k,R)|^{1+\frac{2}{n}}\right)$$

又 $\forall h>k\geq k_0$. $A(k,r)\supset A(h,r)$ 且 $|A(h,r)| = |B_r\cap\{u-k>h-k\}|$

$$\leq \frac{1}{(h-k)^2}\int_{A(k,r)}(u-k)^2$$

同时显然: $\int_{A(h,r)}(u-h)^2 \leq \int_{A(k,r)}(u-k)^2$

则 $\forall h>k\geq k_0$. $\frac{1}{2}\leq r<R\leq 1$. 有

$$\int_{A(h,r)}(u-h)^2 \leq C\left(\frac{1}{(R-r)^2}\int_{A(h,R)}(u-h)^2+(h+F)^2|A(h,R)|\right)|A(h,R)|^{\frac{2}{n}}$$

$$\leq C\left(\frac{1}{(R-r)^2}+\frac{(h+F)^2}{(h-k)^2}\right)\frac{1}{(h-k)^{\frac{4}{n}}}\left(\int_{A(k,R)}(u-k)^2\right)^{1+\frac{2}{n}}$$

即 $$\|(u-h)^+\|_{L^2(B_r)} \leq C\left(\frac{1}{R-r}+\frac{h+F}{h-k}\right)\frac{1}{(h-k)^{\frac{2}{n}}}\|(u-k)^+\|_{L^2(B_R)}^{1+\frac{2}{n}} \quad (*)$$

令 $\varphi(k,r)=\|(u-k)^+\|_{L^2(B_r)}$ 定义 $k_\ell=k_0+k(1-\frac{1}{2^\ell})$

$r_\ell=\frac{1}{2}+\frac{1}{2^{\ell+1}}$

$$\text{则 } \varphi(k_\ell,r_\ell)\overset{(*)}{\leq} C\left(2^{\ell+1}+\frac{2^\ell(k_0+F+k)}{k}\right)\left(\frac{2^\ell}{k}\right)^{\frac{2}{n}}\varphi(k_{\ell-1},r_{\ell-1})^{1+\frac{2}{n}}$$

$$\leq C'\frac{k_0+F+k}{k^{1+\frac{2}{n}}}2^{(1+\frac{2}{n})\ell}\varphi(k_{\ell-1},r_{\ell-1})^{1+\frac{2}{n}}$$

$C_*(k_0+F+\varphi(k_0,r_0))$ C_* 大

$\parallel$

可取 k 利归纳证明 $\varphi(k_\ell,r_\ell)\leq\frac{\varphi(k_0,r_0)}{\gamma^\ell}$ 其中 $\gamma=2^{1+\frac{n}{2}}>1$

则 $\varphi(k_\ell,r_\ell)\to 0$. 即 $\varphi(k_0+k,\frac{1}{2})=0$

$$\Rightarrow \sup_{B_{\frac{1}{2}}}u^+ \leq C(C_*+1)(k_0+F+\underbrace{\varphi(k_0,r_0)}_{\leq\|u^+\|_{L^2(B_1)}}) \leq C'(\|u^+\|_{L^2(B_1)}+\|f\|_{L^q(B_1)})$$

✓

II. Moser迭代

$\bar{u}=u^++k$. $\bar{u}_m=\begin{cases}\bar{u}. & u<m\\ k+m. & u\geq m\end{cases}$ $\Rightarrow D\bar{u}_m=0$ on $\{u<0\}\cup\{u>m\}$

$\bar{u}_m\leq\bar{u}$.

令 $\varphi=\eta^2(\bar{u}_m^\beta\bar{u}-k^{\beta+1})\in H_0^1(B_1)$

$$D\varphi=\eta^2\bar{u}_m^\beta(\beta D\bar{u}_m+D\bar{u})+2\eta D\eta(\bar{u}_m^\beta\bar{u}-k^{\beta+1}).$$

$\varphi=0$ 且 $D\varphi=0$ in $\{u\leq 0\}$ 则在 $\{u>0\}$ 上积分

$$\int a^{ij} u_i \varphi_j = \int a^{ij} \bar{u}_i (\beta \bar{u}_{m,j} + \bar{u}_j) \eta^2 \bar{u}_m^{\beta} + 2a^{ij} \bar{u}_i \eta_j (\bar{u}_m^{\beta} \bar{u} - k^{\beta+1}) \eta$$

$$\geq \lambda\beta \int \eta^2 \bar{u}_m^{\beta} |D\bar{u}_m|^2 + \lambda \int \eta^2 \bar{u}_m^{\beta} |D\bar{u}|^2 - \Lambda \int |D\bar{u}| |D\eta| \bar{u}_m^{\beta} \bar{u} \eta$$

$$\geq \lambda\beta \int \eta^2 \bar{u}_m^{\beta} |D\bar{u}_m|^2 + \frac{\lambda}{2} \int \eta^2 \bar{u}_m^{\beta} |D\bar{u}|^2 - \frac{2\Lambda^2}{\lambda} \int |D\eta|^2 \bar{u}_m^{\beta} \bar{u}^2$$

$$\Rightarrow \beta \int \eta^2 \bar{u}_m^{\beta} |D\bar{u}_m|^2 + \int \eta^2 \bar{u}_m^{\beta} |Du|^2 \leq C \left(\int |D\eta|^2 \bar{u}_m^{\beta} u^2 + |f| \eta^2 \bar{u}_m^{\beta} \bar{u} \right)$$

$$\leq C \left(\int |D\eta|^2 \bar{u}_m^{\beta} u^2 + \int c_0 \eta^2 \bar{u}_m^{\beta} \bar{u}^2 \right) \qquad c_0 = \frac{|f|}{k}$$

取 $k = \|f\|_{L^q}$ 则 $\|c_0\|_{L^q} = 1$

取 $w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}$ 有 $|Dw|^2 \leq (1+\beta)(\beta \bar{u}_m^{\beta} |D\bar{u}_m|^2 + \bar{u}_m^{\beta} |D\bar{u}|^2)$

$$\Rightarrow \int |D(w\eta)|^2 \leq C(1+\beta) \left(\int w^2 |D\eta|^2 + c_0 w^2 \eta^2 \right)$$

又 $\int c_0 w^2 \eta^2 \leq \|c_0\|_{L^q} \left(\int (w\eta)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}} = \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}}$

$$\Rightarrow \|\eta w\|_{L^{\frac{2q}{q-1}}} \overset{插值}{\leq} \varepsilon \|\eta w\|_{L^{2^*}} + C(n,q) \varepsilon^{-\frac{n}{2q-n}} \|w\eta\|_{L^2}$$

$$\overset{\text{sobolev}}{\leq} \varepsilon \|D(\eta w)\|_{L^2} + C(n,q) \varepsilon^{-\frac{n}{2q-n}} \|w\eta\|_{L^2}$$

则 $\int |D(w\eta)|^2 \leq C \left((1+\beta) \int w^2 |D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}} \int w^2 \eta^2 \right)$

$$\overset{\text{sobolev 式}}{\left(\int |w\eta|^{2\chi} \right)^{\frac{1}{\chi}}} \leq C (1+\beta)^{\alpha} \int (|D\eta|^2 + \eta^2) w^2$$

$(\chi = \frac{n}{n-2})$

则 $\forall 0 < r < R \leq 1$ $\eta \in C_0^1(B_R)$ st $\eta = 1$ in B_r, $|D\eta| \leq \frac{2}{R-r}$

故 $\left(\int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(1+\beta)^{\alpha}}{(R-r)^2} \int_{B_R} w^2$

$$\|$$

$$\left(\int_{B_r} \bar{u}^{2\chi} \bar{u}_m^{\beta\chi} \right)^{\frac{1}{\chi}}$$

令 $\gamma = \beta + 2 \geq 2$ 则 $\left(\int_{B_r} \bar{u}_m^{\gamma\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(\gamma-1)^{\alpha}}{(R-r)^2} \int_{B_R} \bar{u}^{\gamma}$

$m \to \infty$ 有 $\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \leq \left(C \frac{(\gamma-1)^{\alpha}}{(R-r)^2} \right)^{\frac{1}{\gamma}} \|\bar{u}\|_{L^{\gamma}(B_R)}$

则令 $\gamma_i = 2\chi^i$. $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$

则 $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_i})} \le C(n,q,\lambda,\Lambda)^{\frac{i}{\chi^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}$

故迭代有 $\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)}$

$i \to \infty$ 有 $\sup\limits_{B_{\frac{1}{2}}} u^+ \le C(\|u^+\|_{L^2(B_1)} + \underset{\overset{\parallel}{\|f\|_{L^q(B_1)}}}{k})$ (用到Lemma 2) ✓

θ. P的其它情况可由上述归化得到 故证毕

(2) Stampacchia 迭代

Thm $\begin{cases} -(a^{ij}u_j)_i + a(x)u \le f_0 + \sum\limits_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \quad \text{in } U \\ u|_{\partial U} \le 0 \end{cases}$

u为弱下解. $\lambda I \le (a^{ij}) \le \Lambda I$ $a^{ij} \in L^\infty(U)$ $U \subseteq \mathbb{R}^{n\ge 2}$ 有界

$0 \le a(x) \le \Lambda$ $f_0 \in L^q(U)$ $(\frac{1}{q} = \frac{1}{p} + \frac{1}{n})$ $f_i \in L^p(U)$

则 $u^+ \le C(\|f_0\|_{L^p(U)} + \|f\|_{L^q(U)})|U|^{\frac{1}{n}-\frac{1}{p}}$ $|f| = \sum\limits_{i=1}^{n} f_i^2$

pf. 思路与De Giorgi类似. 略. 可参考庞讲义.

七. 解的正则性: Bootstrapping

1. Schauder 理论

首先修正一下之前定义的 C^k. $C^{k,\alpha}$ 范数.

① Ω有界. $d = \operatorname{diam}\Omega$ 定义 $\|u\|'_{C^k(\bar\Omega)} = \sum_{j=0}^{k} d^j \sup_{\substack{|\beta|=j \\ x\in\Omega}} |D^\beta u(x)|$

$$\|u\|'_{C^{k,\alpha}(\bar\Omega)} = \|u\|'_{C^k(\bar\Omega)} + d^{k+\alpha} \sup_{\substack{|\beta|=k \\ x\neq y}} \frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha}$$

② $\Omega\subset\mathbb{R}^n$. $d_x = d(x,\partial\Omega)$ $d_{x,y} = \min(d_x, d_y)$

定义 $\|u\|^*_{C^k(\bar\Omega)} = \sum_{j=0}^{k} \sup_{\substack{|\beta|=j \\ x\in\Omega}} d_x^j |D^\beta u(x)|$

$$\|u\|^*_{C^{k,\alpha}(\bar\Omega)} = \|u\|^*_{C^k(\bar\Omega)} + \sup_{\substack{|\beta|=k \\ x\neq y}} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|}.$$

若Ω有界 $d = \operatorname{diam}\Omega$

$$\Rightarrow \|u\|^*_{k,\alpha;\Omega} \le \max(1, d^{k+\alpha}) \|u\|_{k,\alpha;\Omega}$$

Thm 1 (内 Hölder估计) (1) $\Omega\subset\mathbb{R}^n$ $u\in C^2(\Omega)$ $f\in C^\alpha(\Omega)$

$\Delta u = f$. 则 $u\in C^{2,\alpha}(\Omega)$ 且 $\forall B_1 \triangleq B_R(x_0)$ $B_2 \triangleq B_{2R}(x_0) \subset\subset \Omega$

有 $\|u\|'_{2,\alpha;B_1} \le C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2})$

(2) $\Omega\subset\mathbb{R}^n$ 开 $u\in C^2(\Omega)$ $f\in C^\alpha(\Omega)$ $\Delta u = f$.

则 $\|u\|^*_{2,\alpha;\Omega} \le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega})$

其中 $|f|^{(k)}_{0,\alpha;\Omega} = \sup_{x\in\Omega} d_x^k |f(x)| + \sup_{x\neq y\in\Omega} d_{x,y}^{k+\alpha} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$

(pf. 参考 G-T Thm 4.6. 4.8).

则对 $Lu=a^{ij}u_{ij}+b^iu_i+cu=f$. a^{ij}一致椭圆 ($a^{ij}\xi_i\xi_j\ge\lambda|\xi|^2$)

(G-T Thm 6.6)

Thm 2 (内 Hölder 估计) $\Omega\subset\mathbb{R}^n$ 开 $u\in C^{2,\alpha}(\Omega)$ 有界解 of $Lu=f$

且若 $|a^{ij}|^{(0)}_{0,\alpha;\Omega}$. $|b^i|^{(1)}_{0,\alpha;\Omega}$. $|c|^{(2)}_{0,\alpha;\Omega}\le\Lambda$. $|f|^{(2)}_{0,\alpha;\Omega}<\infty$

则 $\|u\|^*_{2,\alpha;\Omega}\le C(|u|_{0;\Omega}+|f|^{(2)}_{0,\alpha;\Omega})$

上述估计结合利用连续性方法可以证明 Dirichlet 问题的 $C^{2,\alpha}$ 解的存在唯一性
(加上A-A引理)

(G-T Thm 6.10)

Cor L严格椭圆 系数 in $C^\alpha(\bar\Omega)$ $c\le0$ $B\subset\mathbb{R}^n$ 为球 $\varphi\in C^0(\partial B)$

$f\in C^\alpha(\bar B)$ 则 $\begin{cases}Lu=f & \text{in } B\\ u=\varphi & \text{on } \partial B\end{cases}$ 在 $C^{2,\alpha}(B)\cap C^0(\bar B)$ 中存在唯一解

Cor. (Schauder) ① Ω 开集 L的系数与 $f\in C^\alpha(\Omega)$. $Lu=f$ $u\in C^2(\Omega)$

则 $u\in C^{2,\alpha}(\Omega)$

② $u\in C^2(\Omega)$ 为 $Lu=f$ 的解 f和L的系数 $\in C^{k,\alpha}(\Omega)$. 则 $u\in C^{k+2,\alpha}(\Omega)$

③ 进一步地. 若 f和L的系数 $\in C^\infty(\Omega)$ 则 $u\in C^\infty(\Omega)$

2. 黏性解的估计

(1) De Giorgi 的 Holder 估计

Thm. (De Giorgi) $D_i(a^{ij}u_j)=0$ in $B_1(0)$

$0<\lambda I\le(a^{ij})\le\Lambda I$. 则 $\|u\|_{C^\alpha(B_{\frac12})}\le C(\lambda,\Lambda,n)\|u\|_{L^2(B_1)}$

方法1: 迭代思想 → De Giorgi oscillation lemma

方法2: Krylov-Safmov 方法 (ABP估计 + CZ分解)

⇒ Krylov-Safonov Harnack 不等式: $a^{ij}u_{ij}=0$. $u\ge0$ in $B_1(0)$

则 $\sup\limits_{B_{\frac12}(0)}u\le C\inf\limits_{B_{\frac12}}u$

(2) $C^{1,\alpha}$ 估计

Thm. $F(D^2u)=0$ in B_1. F-致椭圆. 则 $u\in C^{1,\alpha}(B_{\frac12})$ 且

$\|u\|_{C^{1,\alpha}(B_{\frac12})}\le C(\|u\|_{L^\infty(B_1)}+|F(0)|)$

pf. 差商方法 + (1)

(3) Evans-Krylov

Thm. F-致椭圆.凹 u 为 $F(D^2u)=0$ 的黏性解. in B_1

则 $u\in C^{2,\alpha}(B_{\frac12})$ 且 $\|u\|_{C^{2,\alpha}(B_{\frac12})}\le C(\|u\|_{L^\infty(B_1)}+|F(0)|$

pf. 先证 $\|u\|_{C^{1,1}(B_{\frac12})}\le C\|u\|_{L^\infty(B_1)}$ $\Rightarrow\left\{\begin{array}{l}\Delta u\in L^1\\ \Downarrow\\ u\in W^{2,2}\\ \Downarrow\\ u\in C^{1,1}\end{array}\right.$

再证 $\|u\|_{C^{2,\alpha}(B_{\frac12})}\le C\|u\|_{C^{1,1}(B_1)}$

则对于F一致椭圆 $F(D^2u)=f$

若$u\in C^{2,\alpha}$ 考虑差商 $u^h(x)=\dfrac{u(x+h\xi)-u(x)}{h}$

$$a_h^{pq}(x)=\int_0^1\frac{\partial F}{\partial u_{pq}}(tD^2u(x+h\xi)+(1-t)D^2u(x))dt$$

$$\Rightarrow a_h^{pq}(x)\,u^h_{pq}(x)=\frac{1}{h}\int_0^1\frac{d}{dt}F(tD^2u(x+h\xi)+(1-t)D^2u(x))dt=f^h(x)$$

$u\in C^{2,\alpha}\Rightarrow a_h^{pq}\in C^{0,\alpha}\xrightarrow{\text{Schauder}}u^h\in C^{2,\alpha}\Rightarrow u\in C^{3,\alpha}$ …、 光滑性

这种正规性提高的策略称为bootstrap

应用: Calabi-Yau定理

八. 线性发展方程

1. 二阶抛物方程

(1) 基本定义

$U\subset\mathbb{R}^n$ 有界开集 $U_T=U\times(0,T]$

$$\begin{cases} u_t+Lu=f & \text{in } U_T \\ u=0 & \text{on } \partial U\times[0,T] \\ u=g & \text{on } U\times\{t=0\} \end{cases} \quad (*)$$

$$Lu=-\big(\underset{\text{扩散项}}{a^{ij}(x,t)u_i}\big)_j+\underset{\text{漂移项}}{b^i(x,t)u_i}+\overset{\text{反应项}}{c(x,t)u}$$

称算子 L 一致抛物 若 $\exists\theta>0$ s.t. $\forall(x,t)\in U_T$ $\xi\in\mathbb{R}^n$ $a^{ij}(x,t)\xi_i\xi_j\geq\theta|\xi|^2$

设 $a^{ij}, b^i, c\in L^\infty(U_T)$ $f\in L^2(U_T)$ $g\in L^2(U)$. $a^{ij}=a_{ji}$, 定义双线性型

$$B[u,v;t]=\int_U a^{ij}(x,t)u_iv_j+b^i(x,t)u_iv+c(x,t)uv\,dx \qquad \begin{matrix} u,v\in H_0^1(U) \\ 0\leq t\leq T\ \text{a.e} \end{matrix}$$

令 $u:[0,T]\to H_0^1(U)$ $\quad f:[0,T]\to L^2(U)$

$[u(t)](x)=u(x,t)$ $\quad [f(t)](x)=f(x,t)$

$\forall v\in H_0^1(U)$ $\int_U u_t v\,dt+\int_U Lu\cdot v\,dx=\int_U fv\,dx$

$$\Rightarrow \left(\frac{d}{dt}u,v\right)+B[u,v;t]=(f,v) \quad \forall 0\leq t\leq T$$

Def 称 $u\in L^2(0,T;H_0^1(U))$ $u'\in L^2(0,T;H^{-1}(U))$ 为 (*) 的弱解

若 (1) $\langle u',v\rangle+B[u,v;t]=(f,v)$ $\forall v\in H_0^1(U)$ a.e. $0\leq t\leq T$

(2) $u(0)=g$

记号: X Banach空间 $L^p(0,T;X)=\{u:[0,T]\to X \mid \|u\|_{L^p(0,T;X)}=(\int_0^T\|u\|^p dt)^{\frac{1}{p}}<\infty\}$

$C([0,T];X)=\{u:[0,T]\to X \mid \|u\|_{C([0,T];X)}=\max_{0\leq t\leq T}\|u(t)\|<\infty\}$

用step function逼近的方式可定义 X-值的积分

(记作 $u'=v$)

Def ① $u\in L^2(0,T;X)$ 称 $v\in L^1(0,T;X)$ 为 u 的弱导数. 若

$$\int_0^T \phi'(t)u(t)dt = -\int_0^T \phi(t)v(t)dt \quad (\forall \phi\in C_0^\infty(0,T;X))$$

② $W^{1,p}(0,T;X)=\{u\in L^p(0,T;X) \,|\, u'$ 存在且 $u'\in L^p(0,T;X)\}$

范数定义与之前类似. $H^1(0,T;X)=W^{1,2}(0,T;X)$

Thm (1) $u\in W^{1,p}(0,T;X)$ $1\le p<\infty$.

① $u\in C([0,T];X)$ (挖掉零测集意义下)

② $u(t)=u(s)+\int_s^t u'(\tau)d\tau \quad \forall 0\le s<t\le T$

③ $\max_{0\le t\le T}\|u(t)\|_{L^2(U)} \le C(\|u\|_{L^2(0,T;H_0^1(U))}+\|u'\|_{L^2(0,T;H^{-1}(U))})$

(2) $u\in L^2(0,T;H_0^1(U))$ $u'\in L^2(0,T;H^{-1}(U))$

① $u\in C([0,T];L^2(U))$ (挖掉零测集意义)

② $\max_{0\le t\le T}\|u(t)\|_{L^2(U)} \le C(\|u\|_{L^2(0,T;H_0^1(U))}+\|u'\|_{L^2(0,T;H^{-1}(U))})$

(见 Evans 5.9.2)

(2) 弱解的构造

Thm. $\forall m\ge 1$ $\exists !\, u_m=\sum_{k=1}^m d_m^k(t)w_k$ s.t.

$$d_m^k(0)=(g,w_k)$$

$$(u_m',w_k)+B[u_m,w_k;t]=(f,w_k) \quad \left(\begin{matrix}\forall 0\le t\le T\\ 1\le k\le m\end{matrix}\right)$$

其中 $\{w_k\}$ 为 $L^2(U)$ 的正交规范基 $H_0^1(U)$ 的正交基

pf $u_m = \sum_{k=1}^m d_m^k(t) w_k$. 则 $(u_m'(t), w_k) = (d_m^k(t))'$

$$B[u_m, w_k, t] = \int_U a^{ij} u_{m,i} w_{k,j} + b^i u_{m,i} w_k + c u_m w_k \, dx$$

$$= \sum_{l=1}^m d_m^l(t) B[w_l, w_k; t] \overset{\circ}{=} \sum_{l=1}^m d_m^l(t) e^{kl}(t)$$

$f^k(t) \overset{\circ}{=} (f, w_k)(t)$ 故只需解 ODE $\begin{cases} (d_m^k(t))' + \sum_{l=1}^m d_m^l(t) e^{kl}(t) = f^k(t) \\ d_m^k(0) = (g, w_k) \end{cases}$ ✓

Thm $\{u_m\}$如上. 则存在 $C = C(U, T, L)$ s.t.

$$\max_{0 \le t \le T} \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2(0,T;H_0^1(U))} + \|u_m'\|_{L^2(0,T;H^{-1}(U))} \le C(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)}).$$

pf $(u_m', u_m) + B[u_m, u_m, t] = (f, u_m)$

回忆 $\beta \|u_m\|^2_{H_0^1(U)} \le B[u_m, u_m, t] + \nu \|u_m\|^2_{L^2(U)}$ $\forall m$, a.e $0 \le t \le T$

又 $|(f, u_m)| \le \frac{1}{2} \|f\|^2_{L^2(U)} + 2\beta \|u_m\|^2_{L^2(U)}$

$(u_m', u_m) = \frac{1}{2} \frac{d}{dt} (\|u_m\|^2_{L^2(U)})$

故 $\frac{d}{dt}(\|u_m\|^2_{L^2(U)}) + 2\beta \|u_m\|^2_{H_0^1(U)} \le C_1 \underbrace{\|u_m\|^2_{L^2(U)}}_{E(t)} + C_2 \|f\|^2_{L^2(U)}$ $(*)$

Gronwall $\Rightarrow$ $E(t) \le e^{C_1 t}(E(0) + C_2 \int_0^t \|f\|_{L^2(U)} ds)$

$E(0) = \|u_m(0)\|^2_{L^2(U)} \le \sum_{k=1}^m |(g, w_k)|^2 \le \|g\|^2_{L^2(U)}$ $(**)$

故 $\|u_m\|_{L^2(U)} \le C(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)})$

对$(*)$积分 $\|u_m\|^2_{L^2(0,T;H_0^1(U))} = \int_0^T \|u_m\|^2_{H_0^1(U)} dt$

$\le C_1 \int_0^T \|u_m\|^2_{L^2(U)} dt + C_2 \int_0^T \|f\|^2_{L^2(U)} dt$ $(***)$

$\overset{(**)}{\le} C(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(0,T;L^2(U))})$

$\forall v \in H_0^1(U)$ 且 $\|v\|_{H_0^1(U)} \le 1$　记 $v = v^1 + v^2$　$v^1 \in \operatorname{span}\{w_k\}_{k=1}^m$

则 $\|v_1\|^2_{H_0^1(U)} = \|v\|^2_{H_0^1(U)} + \|v^2\|^2_{H_0^1(U)} - 2(v^1+v^2, v^2)_{H_0^1(U)}$

$= \|v\|^2_{H_0^1(U)} - \|v^2\|^2_{H_0^1(U)} \le \|v\|^2_{H_0^1(U)} \le 1$

又 $(u_m', v^1) + B[u_m, v^1; t] = (f, v^1)$

$(u_m', v^1) = (f, v^1) - B[u_m, v^1; t]$

$\Rightarrow |(u_m', v^1)| \le \|f(t)\|_{L^2(U)} \|v^1\|_{L^2(U)} + \alpha \|u_m(t)\|_{H_0^1(U)} \|v^1\|_{H_0^1(U)}$

$\le C(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)})$

$\Rightarrow \|u_m'\|_{H^{-1}(U)} \le C \cdots$

故 $\|u_m'\|_{L^2(0,T;H^{-1}(U))} = \int_0^T \|u_m'\|^2_{L^2(0,T,H^{-1}(U))} dt \overset{(**)}{\le} C(\|f\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)})$

Thm. 弱解存在唯一.

Pf. ① 由上述估计 $\exists$ 子列 u_{m_l} s.t. $u_{m_l} \rightharpoonup u$ in $L^2(0,T;H_0^1(U))$

$u'_{m_l} \rightharpoonup u'$ in $L^2(0,T;H^{-1}(U))$

由前面 Thm(2)① 有 $u \in C([0,T], L^2(U))$

且 $\forall N \in \mathbb{N}$ 取 $v = \sum_{k=1}^N d^k(t) w_k \in C^1([0,T], L^2(U))$

$\forall m_l \ge N$ 有 $\int_0^T \langle u'_{m_l}, v\rangle + B[u_{m_l}, v, t]\, dt = \int_0^T (f, v) dt$

令 $m_l \to \infty$ 并由 $\{\sum d^k(t) w_k\} \subset L^2([0,T], L^2(U))$ 利用稠密

$\forall v \in L^2(0,T;H_0^1(U))$. $\int_0^T \langle u', v\rangle + B[u, v, t]\, dt = \int_0^T (f, v) dt$.

② $\forall v \in C^1([0,T]; H_0^1(U))$ 且 $v(T) = 0$ 有

$-\int_0^T \langle v', u\rangle + B[u, v, t] dt = \int_0^T (f, v) dt + (u(0), v(0))$

$-\int_0^T \langle v', u_m\rangle + B[u_m, v, t] dt = \int_0^T (f, v) dt + (u_m(0), v(0))$

令 $m_l \to \infty$ $\Rightarrow -\int_0^T \langle v, u' \rangle + B[u, v, t] dt = \int_0^T (f, v) dt + (g, v(0))$

$(u_{m_l}(0) \xrightarrow{L^2} g)$

则 $(u(0), v(0)) = (g, v(0))$ $(\forall v)$ $\Rightarrow u(0) = g$.

③唯一性 只须证若 $\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \Gamma_T \end{cases}$. 则 $u=0$.

上述取 $v=u \Rightarrow \langle u', u \rangle + B[u, u, t] = 0$

即 $\frac{1}{2} \frac{d}{dt} (\|u\|^2_{L^2(U)}) + B[u, u, t] = 0$

又由能量估计 $B[u, u, t] \geq \beta \|u\|^2_{H_0^1(U)} - \gamma \|u\|^2_{L^2(U)} \geq -\gamma \|u\|^2_{L^2(U)}$

$\Rightarrow$由Gronwall. $\|u\|^2_{L^2(U)} \leq e^{2\gamma T} \|u(0)\|^2_{L^2(U)} = 0$. $\Rightarrow u = 0$ ✓

(3) 弱解的正则性

Thm. ① $g \in H_0^1(U)$ $f \in L^2(0, T; L^2(U))$. 若 $u \in L^2(0, T; H_0^1(U))$ $u' \in L^2(0, T; H^{-1}(U))$

为弱解 则 $u \in L^2(0, T; H^2(U)) \cap L^\infty(0, T; H_0^1(U))$ $u' \in L^2(0, T; L^2(U))$

且 $\operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_{H_0^1(U)} + \|u\|_{L^2(0, T; H^2(U))} + \|u'\|_{L^2(0, T; L^2(U))} \leq C(\|f\|_{L^2(0, T; L^2(U))} + \|g\|_{H_0^1(U)})$

② 进一步 若 $g \in H^2(U)$ $f' \in L^2(0, T; L^2(U))$ 则 $u \in L^\infty(0, T; H^2(U))$.

$u' \in L^\infty(0, T; L^2(U)) \cap L^2(0, T; H_0^1(U))$ $u'' \in L^2(0, T; H^{-1}(U))$

且 $\operatorname*{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_{H^2(U)} + \|u'(t)\|_{L^2(U)}) + \|u'\|_{L^2(0, T; H_0^1(U))} + \|u''\|_{L^2(0, T; H^{-1}(U))}$
$\leq C(\|f\|_{H^1(0, T; L^2(U))} + \|g\|_{H^2(U)})$.

(4) 抛物方程的其它性质

(i) Harnack 不等式

Thm. L-致抛物 $u\in C_1^2(U_T)$ $u_t+Lu=0$ in U_T. $u\geq 0$

$V\subset\subset U$ 连通. 则 $\forall\, 0<t_1<t_2\leq T$ $\exists C$ s.t. $\sup_V u(\cdot,t_1)\leq C\inf_V u(\cdot,t_2)$

(ii) 弱极值原理

Thm① $u\in C_1^2(U_T)\cap C(\overline{U}_T)$. $c=0$. 则

若 $\underset{(\geq)}{u_t+Lu\leq 0}$ in U_T. 则 $\underset{(\min)}{\max_{\overline{U}_T}} u=\underset{(\min)}{\max_{\Gamma_T}} u$

② $u\in C_1^2(U_T)\cap C(\overline{U}_T)$. $c\geq 0$. 则

若 $\underset{(\geq 0)}{u_t+Lu\leq 0}$ in U_T. 则 $\max_{\overline{U}_T} u\leq\max_{\Gamma_T} u^+$

$(\min_{\overline{U}_T} u\geq -\max_{\Gamma_T} u^-)$.

(iii) 强极值原理

Thm. U连通. $u\in C_1^2(U_T)\cap C(\overline{U}_T)$

① $c=0$. 若 $\underset{(\geq)}{u_t+Lu\leq 0}$ in U_T u在$\overline{U}_T$内部(x_0,t_0)取最$\underset{(小)}{大}$. 则u为常数

② $c\geq 0$. 若 $\underset{(\geq)}{u_t+Lu\leq 0}$ in U_T u在$\overline{U}_T$内部(x_0,t_0)取非$\underset{(正)}{负}$极$\underset{(小)}{大}$ $\Rightarrow$ u常数

2. 二阶双曲方程

U_T 定义同前　L 同前

(称一致双曲若 ⋯)

$$\begin{cases} u_{tt}+Lu=f & \text{in } U_T \\ u=0 & \text{on } \partial U\times[0,T] \\ u=g,\ u_t=h & \text{on } U\times\{t=0\} \end{cases} \quad (*)$$

Def. 称 $u\in L^2(0,T;H_0^1(U))$. $u'\in L^2(0,T;L^2(U))$　$u''\in L^2(0,T;H^{-1}(U))$

为(*)的弱解 若 (1) $\langle u'',v\rangle+B[u,v;t]=(f,v)$ $\forall v\in H_0^1(U)$. a.e. $0\le t\le T$

(2) $u(0)=g$. $u'(0)=h$

Thm 弱解存在唯一

(Pf 与抛物方程类似 略 可参考 Evans 7.2)

Thm(正则性) (1) $g\in H_0^1(U)$ $h\in L^2(U)$ $f\in L^2(0,T;L^2(U))$

若 $u\in L^2(0,T;H_0^1(U))$ $u'\in L^2(0,T;L^2(U))$　$u''\in L^2(0,T;H^{-1}(U))$ 为弱解

则 $u\in L^\infty(0,T;H_0^1(U))$. $u'\in L^\infty(0,T;L^2(U))$ 且

$$\operatorname*{esssup}_{0\le t\le T}\ \|u(t)\|_{H_0^1(U)}+\|u'(t)\|_{L^2(U)}\le C(\|f\|_{L^2(0,T;L^2(U))}+\|g\|_{H_0^1(U)}+\|h\|_{L^2(U)})$$

(2) 进一步. 若 $g\in H^2(U)$. $h\in H_0^1(U)$　$f'\in L^2(0,T;L^2(U))$. 则

$u\in L^\infty(0,T;H^2(U))$. $u'\in L^\infty(0,T;H_0^1(U))$ $u''\in L^\infty(0,T;L^2(U))$ $u'''\in L^2(0,T;H^{-1}(U))$

且
$$\operatorname*{esssup}_{0\le t\le T}\left(\|u(t)\|_{H^2(U)}+\|u'(t)\|_{H_0^1(U)}+\|u''\|_{L^2(U)}\right)+\|u'''\|_{L^2(0,T;H^{-1}(U))}$$
$$\le C(\|f\|_{H^1(0,T;L^2(U))}+\|g\|_{H^2(U)}+\|h\|_{H^1(U)})$$

Thm(有限传播速度) $u_{tt}+Lu=0$ on $\mathbb{R}^n\times(0,\infty)$　Q 满足 $\begin{cases} a^{ij}Q_iQ_j=1 \text{ on } \mathbb{R}^n\setminus\{x_0\} \\ Q(x_0)=0 \end{cases}$

$K=\{(x,t)\mid Q(x)<t_0-t\}$　$K_0=\{x\mid Q(x)<t_0\}$. 若 $u=u_t=0$ on K_0 则 $u=0$ in K