Ricci flow

Ref. «Lectures on the Ricci flow» - P. Topping

«Ricci flow and the Sphere theorem» - S Brendle

«Hamilton's Ricci Flow» - Chow. etc

I. Introduction and Basic Computation

1. Def and basic examples

Ricci flow: $\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}$

Eg. ① Einstein mfld: $Ric(g_0) = \lambda g_0 \overset{g(0)=g_0}{\Rightarrow} g(t) = (1-2\lambda t) g_0$

(Ric is rescaling-invariant)

(i) $(S^n, g_0) \Rightarrow g(t) = (1-2(n-1)t) g_0$

standard collapse at $T = \frac{1}{2(n-1)}$

(ii) $(B^n, g_0) \Rightarrow g(t) = (1+2(n-1)t) g_0$

② Ricci Soliton: $X(t)$: v.f $\xrightarrow{\text{generate}}$ diffeo ψ_t

Prop $\hat{g}(t) = \sigma(t) \psi_t^*(g(t)) \Rightarrow \frac{\partial \hat{g}}{\partial t} = \dot{\sigma}(t) \psi_t^*(g) + \sigma(t) \psi_t^*\left(\frac{\partial g}{\partial t}\right)$
$+ \sigma(t) \psi_t^*(\mathcal{L}_X g)$

then if $-2Ric(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0$

set $g(t) = g_0$ $\quad \sigma(t) = 1-2\lambda t$ $\quad X(t) = \frac{Y}{\sigma(t)}$

$\Rightarrow \frac{\partial \hat{g}}{\partial t} = \psi_t^*(\mathcal{L}_Y g_0 - 2\lambda g_0) = \psi_t^*(-2Ric(g_0))$

$= -2Ric(\psi_t^* g_0) = -2Ric(\hat{g})$

$\lambda \begin{cases} =0 & \text{steady} \\ <0 & \text{expanding} \\ >0 & \text{shrinking} \end{cases}$ Ricci Soliton

$Y=\nabla f \rightsquigarrow$ gradient Ricci soliton

$\mathcal{L}_Y g_0 = 2\mathrm{Hess}_{g_0}(f) \Rightarrow \mathrm{Hess}_{g_0}(f) + \mathrm{Ric}(g_0) = \lambda g_0.$

(i) Cigar Soliton: $(\mathbb{R}^2, g_0 = \rho^2(dx^2+dy^2))$ $\rho^2 = \frac{1}{1+|x|^2}$ $\Rightarrow \mathrm{Ric}(g_0) = \frac{2}{1+|x|^2} g_0$

Set $Y = -2(x\partial_x + y\partial_y)$ $\lambda = 0$ ✓

$X(t) = Y \Rightarrow \frac{\partial \psi_t(p)}{\partial t} = X(\psi_t(p)) \Rightarrow \dot{x} = -2x \Rightarrow x = e^{-2t} x_0$

$\dot{y} = -2y \qquad y = e^{-2t} y_0$

$\Rightarrow \hat{g}(t) = \psi_t^*(g_0) = \psi_t^*\left(\frac{dx^2+dy^2}{1+|x|^2}\right) = \frac{dx^2+dy^2}{e^{4t}+|x|^2}$

(ii) Rosenau Soliton: On S^2, $t<0$ set $g_{ij}(t) = \frac{\delta \sinh(-t)\,\delta_{ij}}{1+2\cosh(-t)|x|^2+|x|^4}$

(iii) Bryant Soliton: $K>0$. asymptotically paraboloid

(iv) Gaussian Soliton

③ Rescale: $g(t)$: Ricci flow on $[0,T]$

$\hat{g}(x,t) = \lambda g(x, \frac{t}{\lambda})$ $(t\in[0,\lambda T])$ $\Rightarrow \frac{\partial \hat{g}}{\partial t}(x,t) = \frac{\partial g}{\partial t}(x, \frac{t}{\lambda})$

$= -2\mathrm{Ric}(\hat{g}(t))$

$\Rightarrow \hat{g}(t)$: Ricci flow on $[0, \lambda T)$

with curvature scaled: $R(\hat{g}(x,t)) = \frac{1}{\lambda} R(g(x, \frac{t}{\lambda}))$.

2 On geometry

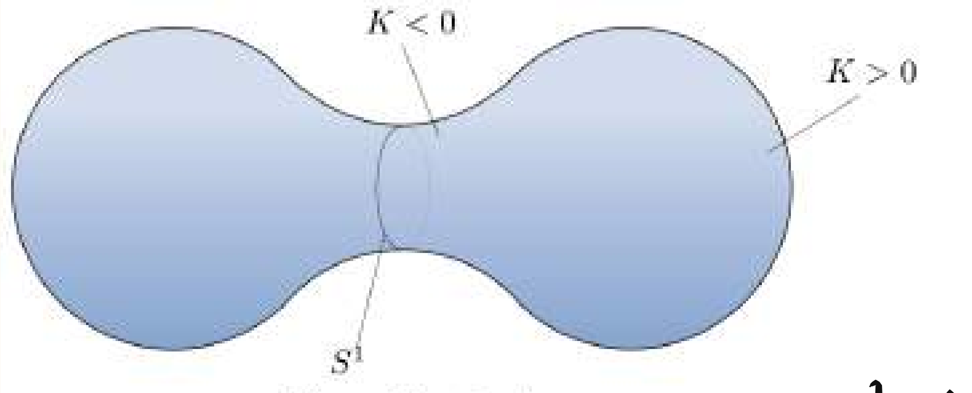


Figure 1.1: 2-sphere

① 2-dim: $Ric(g) = Kg$.
$\Rightarrow K<0$: expand
$K>0$: shrink

constant K

fact: Ricci flow on surface tends to make it "round"

② 3-dim

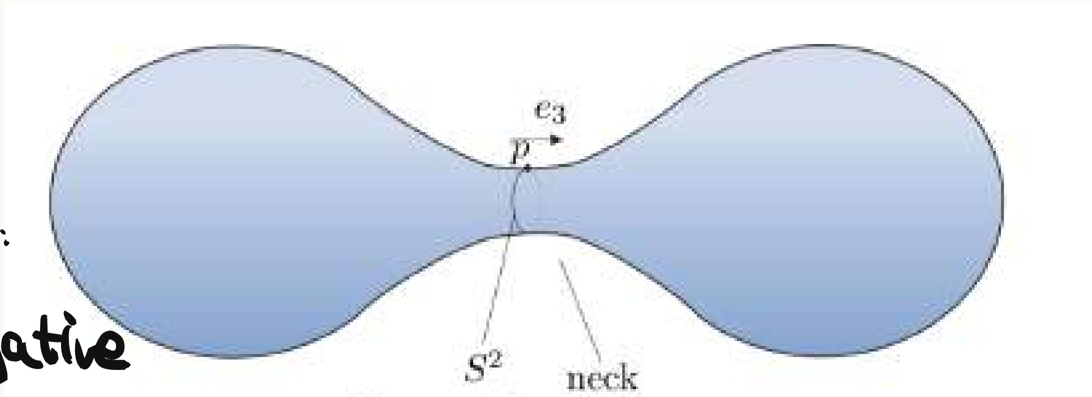


Figure 1.2: 3-sphere

$K(e_1, e_3)$ $K(e_2, e_3)$:
slightly negative
$K(e_1, e_2)$: positive
$\Rightarrow$ "stretch the neck"

singularity $\Rightarrow$ blow up

(closed, orientable, ...)

Geometry of dim-3: not explained here

Thurston conj. M irreducible, then the case must be

(i) $M = S^3/\Gamma$ $\Gamma \subset Isom(S^3)$

(ii) $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ $\begin{cases} \text{contains "incompressible" torus} \\ \text{Seifert fibered space} \end{cases}$

(iii) $M = \mathbb{H}^3/\Gamma$ $\Gamma \subset Isom(\mathbb{H}^3)$

"prime" mfld: can't be expressed as nontrivial connected sum

Results: ① Prime 3-mfld is either irreducible or $S^1 \times S^2$

② (Kneser): Any mfld can be decomposed into primes

$\exists$ discrete isom group with cpt quotient

A "geometry" is a simply-connected homogeneous unimodular Riem mfld

classified:
- $S^3, \mathbb{H}^3, \mathbb{R}^3$ - constant curvature
- $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ - product
- Nil, Sol, $\widetilde{SL}_2(\mathbb{R})$ - twisted product

M "geometric mfld": $\mathring{M} = X / \Gamma$ has finite volume
(maybe with bdry)

X: geometry; Γ: free iso group

Thurston's geometrization conj: Any prime 3-mfld arises by finitely gluing geometric pieces along bdry tori.

strategy using Ricci flow

start with any metric and flow $\longrightarrow$ singularity: neck pinches ...

$\longrightarrow$ chop the neck
restart the flow

Perelman $\longrightarrow$ finite surgeries needed if correctly operated

3 Variation of Tensor

First we clarify the notations

$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ $\Rightarrow \nabla^2_{X,Y} A = (\nabla^2 A)(X, Y, \cdots)$

$\mathrm{Hess}(f) = \nabla df$ $\qquad \Delta A = \mathrm{tr}_{1,2} \nabla^2 A$

$R(X,Y) = \nabla^2_{Y,X} - \nabla^2_{X,Y}$ $\qquad Rm(X,Y,W,Z) = \langle R(X,Y)W, Z\rangle$

$Ric(X,Y) = \mathrm{tr}\, Rm(X,\cdot,Y,\cdot)$ $\qquad S = \mathrm{tr}(Ric)$: scalar curvature

$\delta : \Gamma(\otimes^k T^*M) \to \Gamma(\otimes^{k-1} T^*M)$
$T \mapsto -\mathrm{tr}_{1,2} \nabla T$

$G : \Gamma(\mathrm{Sym}^2 T^*M) \to \Gamma(\mathrm{Sym}^2 T^*M)$
$T \mapsto T - \frac{1}{2}(\mathrm{tr}\, T) g$

$\Rightarrow \delta G(T) = \delta T + \frac{1}{2} d(\mathrm{tr}\, T)$

$A * B$: linear combination of tensor fields from $A \otimes B$ by "$\flat$" or "$\sharp$"

$R^2 = Rm * Rm$. $\qquad R(\cdot,\cdot)A = A * Rm$ (Ricci Indentity)

Fact: $\delta G(Ric) = \delta Ric + \frac{1}{2} dS = 0$ $\leftarrow$ contracted Bianchi: $2g^{ij}\nabla_i R_{jk} = \nabla_k S$
$\Leftrightarrow \mathrm{div}(Rc - \frac{1}{2} S g) = 0.$

$\{g_t\} \subseteq \Gamma(\mathrm{Sym}^2 T^*M)$ $\qquad h = \frac{\partial g_t}{\partial t}$

$\mathrm{II}(X,Y) \triangleq \frac{\partial}{\partial t} \nabla_X Y$ $\Rightarrow$ for $V = V(t)$. $\quad \mathrm{II}(X,V) + \nabla_X \frac{\partial V}{\partial t} = \frac{\partial}{\partial t} \nabla_X V$

$\hookrightarrow$ is a tensor! since difference of ∇ is a tensor

write $\mathrm{II} = V_{ij}\, dx^i \otimes dx^j$

Firstly we calculate the evolution of curvature tensor

Prop ① $\langle \mathrm{II}(X,Y), Z\rangle = \frac{1}{2}[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)]$

$\Rightarrow \frac{\partial}{\partial t}\nabla Y = Y * \nabla h$. more generally. $\frac{\partial}{\partial t}\nabla A = A * \nabla h$

(pf. $\langle \mathrm{II}(X,Y), Z\rangle = \frac{\partial}{\partial t}\langle \nabla_X Y, Z\rangle - h(\nabla_X Y, Z)$

$= \frac{\partial}{\partial t}[X\langle Y,Z\rangle - \langle Y, \nabla_X Z\rangle] - h(\nabla_X Y, Z)$

since II is a tensor. we always take X, Y to be like ∂_i. so $[X,Y]=0$, $\nabla X = 0$. (at a time)

$= X h(Y,Z) - h\langle Y, \nabla_X Z\rangle - g(Y, \frac{\partial}{\partial t}\nabla_X Z) - h(\nabla_X Y, Z)$

$= (\nabla_X h)(Y,Z) - \langle \mathrm{II}(Z,X), Y\rangle$

$\Rightarrow \langle \mathrm{II}(X,Y), Z\rangle = (\nabla_X h)(Y,Z) - [(\nabla_Z h)(X,Y) - \langle \mathrm{II}(Y,Z), X\rangle]$

$= (\nabla_X h)(Y,Z) - ((\nabla_Z h)(X,Y) + (\nabla_Y h)(Z,X) - \langle \mathrm{II}(X,Y), Z\rangle)$

② $\frac{\partial}{\partial t} R(X,Y)W = (\nabla_Y \mathrm{II})(X,W) - (\nabla_X \mathrm{II})(Y,W)$

(pf. straight calc. omitted)

③ $\frac{\partial}{\partial t}\mathrm{Rm}(X,Y,W,Z) = \frac{1}{2}[h(R(X,Y)W,Z) - h(R(X,Y)Z,W)]$

$+ \frac{1}{2}[\nabla^2_{Y,W} h(X,Z) - \nabla^2_{X,W} h(Y,Z) + \nabla^2_{X,Z} h(Y,W) - \nabla^2_{Y,Z}(X,W)]$

(pf let $\nabla X = \nabla Y = \nabla Z = \nabla W = 0$

$\frac{\partial}{\partial t}\langle R(X,Y)W, Z\rangle \overset{②}{=} h(R(X,Y), W, Z) + \langle (\nabla_Y \mathrm{II})(X,W), Z\rangle - \langle (\nabla_X \mathrm{II})(Y,W), Z\rangle$

and $\langle (\nabla_Y \mathrm{II})(X,W), Z\rangle \overset{\nabla Z = 0}{=} Y\langle \mathrm{II}(X,W), Z\rangle$

$\overset{①}{=} \frac{1}{2} Y((\nabla_W h)(X,Z) + (\nabla_X h)(W,Z) - (\nabla_Z h)(X,W))$

$\overset{\nabla X = \nabla Z = \nabla W = 0}{=} \frac{1}{2}[\nabla^2_{Y,W} h(X,Z) + \nabla^2_{Y,X} h(W,Z) - \nabla^2_{Y,Z} h(X,W)]$)

then it's the evolution of Ricci and scalar curvatures

Prop ④ $\frac{\partial}{\partial t}\mathrm{Ric} = -\frac{1}{2}\Delta_L h - \frac{1}{2}\mathbb{1}_{(\delta G(h))^{\#}}\, g$

where $(\Delta_L h)(X,W) = (\Delta h)(X,W) - h(X,\mathrm{Ric}(W)) - h(W,\mathrm{Ric}(X)) + 2\,\mathrm{tr}\,h(R(X,\cdot)W,\cdot)$

⑤ $\frac{\partial}{\partial t} E(X,Y) = (\Delta_L E)(X,Y) + 2\sum_{p,q} Rm(X,e_p,Y,e_q)\underset{\text{traceless Ricci}}{E}(e_p,e_q) + \frac{2}{n} S E(X,Y) - \frac{2}{n}|E|^2\langle X,Y\rangle$

⑥ $\frac{\partial}{\partial t} S = -\langle \mathrm{Ric}, h\rangle + \delta^2 h - \Delta(\mathrm{tr}\, h)$

$\overset{\text{under}}{\underset{\text{RF}}{=}} \Delta S + 2|\mathrm{Ric}|^2 \quad (*) \rightsquigarrow \frac{\partial S}{\partial t} \geq \Delta S + \frac{2}{n} S^2$

⑦ $\frac{\partial}{\partial t} d\mathrm{Vol} = \frac{1}{2}\mathrm{tr}\, h\, d\mathrm{Vol}$

Cor ① $\min_{p\in M} S_{g(t)}(p) \uparrow$

② If $\inf_M S_{g(0)} = \alpha > 0 \Rightarrow T \leq \frac{n}{2\alpha}$

$\inf_M S_{g(t)} \geq \frac{n\alpha}{n-2\alpha t}$

In coordinates: ① $\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij})$

$\overset{\text{under}}{\underset{\text{RF}}{\rightsquigarrow}} \frac{\partial}{\partial t}\Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})$

Cor. under R-F: $\frac{\partial}{\partial t}(\Delta_{g(t)}) = \frac{\partial}{\partial t}(g^{ij}\nabla_i\nabla_j)$

$= -\frac{\partial}{\partial t} g^{ij}\, \nabla_i\nabla_j - \underbrace{g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)}\nabla_k$

$= 2R_{ij}\nabla_i\nabla_j$

$\underbrace{g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)} = -g^{kl}(2g^{ij}\nabla_i R_{jl} - \nabla_l S) \overset{\text{CBI}}{=} 0$

④ $\frac{\partial}{\partial t} R_{ij} = \nabla_p\left(\frac{\partial}{\partial t}\Gamma_{ij}^p\right) - \nabla_i\left(\frac{\partial}{\partial t}\Gamma_{pj}^p\right)$

$= -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i(\mathrm{div}\, v)_j - \nabla_j(\mathrm{div}\, v)_i\right)$

$\frac{\partial}{\partial t}(-2R_{ij}) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i \quad (X = \frac{1}{2}\nabla V - \mathrm{div}\, v,\ V = \mathrm{tr}\, V)$

under R-F: $\frac{\partial}{\partial t} R_{ij} = \Delta_L R_{ij} = \Delta R_{ij} + 2R_{kijl}R_{kl} - 2R_{ik}R_{jk}$

⑧ under R-F: $\nabla_i\nabla_j\left(\frac{\partial}{\partial t} - \Delta\right) f = \left(\frac{\partial}{\partial t} - \Delta_L\right)\nabla_i\nabla_j f$

$\left(\Rightarrow \frac{\partial}{\partial t}\Delta f = \Delta\frac{\partial f}{\partial t} + 2\langle \mathrm{Ric}, \mathrm{Hess}(f)\rangle\right)$

4. Short-time Existence and Uniqueness

Ricci flow is weakly parabolic!

$\downarrow$ DeTurck's trick

Ricci DeTurck flow is parabolic!

$\downarrow$

Thm (Short time existence) M closed Riem mfld, g_0 C^∞ metric

then exists a unique solution $\tilde{g}(t)$ of RF with $\tilde{g}(0)=g_0$

on some interval $t\in[0,\delta)$

Def $(M,g)\xrightarrow{f}(N,h)$. $\Delta_{g,h} f = \mathrm{tr}_g(\underbrace{\nabla^{g,h}}_{\text{induced connection on } T^*M\otimes f^*T^*N} df)$

$\downarrow$

is invariant under diffeomorphism: $\Delta_{\varphi^* g, h}(f\circ\varphi)(p) = \Delta_{g,h} f(\varphi(p))$

fix a background metric h on M

Ricci-DeTurck flow: $\frac{\partial}{\partial t}\tilde{g}(t) = -2\mathrm{Ric}_{\tilde{g}(t)} - \mathcal{L}_{\xi(t)}\tilde{g}(t)$

$(\xi(t) = \Delta_{\tilde{g}(t),h}\mathrm{Id})$

in coordinate $\xi(t) = \sum_{i,k,l}\tilde{g}^{ik}\left((\Gamma^h)_{ik}^l - (\Gamma^{\tilde{g}})_{ik}^l\right)\partial_l$

$$= -\frac{1}{2}\sum_{i,j,k,l}\tilde{g}^{ik}\tilde{g}^{jl}(\partial_i\tilde{g}_{jk} + \partial_k\tilde{g}_{ij} - \partial_j\tilde{g}_{ik})\partial_l + LT$$

$$\mathrm{Ric}_{\tilde{g}} = -\frac{1}{2}\sum_{i,j,k,l}\tilde{g}^{ik}(\partial_i\partial_k\tilde{g}_{jl} - \partial_i\partial_l\tilde{g}_{jk} - \partial_j\partial_k\tilde{g}_{il} + \partial_j\partial_l\tilde{g}_{ik})dx^j\otimes dx^l + LT$$

$$\mathcal{L}_\xi \check{g}(\partial_j, \partial_l) = -\check{g}([\xi, \partial_j], \partial_l) - \check{g}(\partial_j, [\xi, \partial_l])$$

$$= \check{g}_{kl}\partial_j \xi^k + \check{g}_{jk}\partial_l \xi^k$$

($\partial_i g^{jk}$ are LT)

$$= -\frac{1}{2}\check{g}_{kl}\partial_j \Big(\sum_{\alpha,\beta,\gamma} \check{g}^{\alpha\gamma}\tilde{g}^{\beta k}(\partial_\alpha \hat{g}_{\beta\gamma} + \partial_\gamma \hat{g}_{\alpha\beta} - \partial_\beta \hat{g}_{\alpha\gamma})\Big)$$

$$-\frac{1}{2}\check{g}_{jk}\partial_l (\cdots)$$

$$= -\frac{1}{2}\partial_j\Big(\sum_{\alpha,\beta} \check{g}^{\alpha\beta}(\partial_\alpha \hat{g}_{\beta l} + \partial_\beta \hat{g}_{\alpha l} - \partial_l \hat{g}_{\alpha\beta})\Big)$$

$$-\frac{1}{2}\partial_l\Big(\sum_{\alpha,\beta} \check{g}^{\alpha\beta}(\partial_\alpha \hat{g}_{j\beta} + \partial_\beta \hat{g}_{\alpha j} - \partial_j \hat{g}_{\alpha\beta})\Big) + LT$$

$$= -\sum_{i,k} \check{g}^{ik}(\partial_i\partial_l \check{g}_{jk} + \partial_j\partial_k \hat{g}_{il} - \partial_j\partial_l \hat{g}_{ik})\, dx^j \otimes dx^l + LT$$

$$\Rightarrow -2Ric_{\check{g}} - \mathcal{L}_\xi \check{g} = \sum_{i,j,k,l} \check{g}^{ik}\partial_i\partial_k \check{g}_{jl}\, dx^j \otimes dx^l + LT$$

$\Rightarrow$ Ricci-DeTurck flow strictly parabolic!

$\downarrow$

R-D flow short time existence and uniqueness

(i.e. ξ_t generates φ_t)

$\uparrow$

now for $\frac{\partial}{\partial t}\varphi_t = \xi_t$ $\quad g(t) = \varphi_t^*(\check{g}(t))$ $\qquad$ $g(t)$ RF

$\uparrow$

$$\frac{\partial}{\partial t}g(t) = \varphi_t^*\Big(\frac{\partial}{\partial t}\check{g}(t) + \mathcal{L}_{\xi_t}\check{g}(t)\Big) \Rightarrow \frac{\partial}{\partial t}g(t) + 2Ric_{g(t)} = 0$$

Conversely, given RF $g(t)$. let $\frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),h}\varphi(t)$

$\check{g}(t) = (\varphi_t^*)^{-1}(g(t))$

$$\Rightarrow \frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),h}\varphi(t) = \Delta_{\varphi_t^*(\check{g}(t)),h}\, Id = \xi_t$$

$$\varphi_t^*\Big(\frac{\partial}{\partial t}\check{g}(t) + \mathcal{L}_{\xi_t}\check{g}(t) + 2Ric_{\check{g}(t)}\Big) = \frac{\partial}{\partial t}g(t) + 2Ric_{g(t)} = 0.$$

$\Rightarrow \check{g}(t)$ R-D flow

So RF $\longleftrightarrow$ R-D flow $\Rightarrow$ short time existence and uniqueness of RF

5. Convergence of Manifolds and Flows

Def (Cheeger-Gromov convergence) Pointed complete Riem mflds $(M_i, g_i, p_i) \xrightarrow{C^\infty} (M, g, p)$ if $\exists$

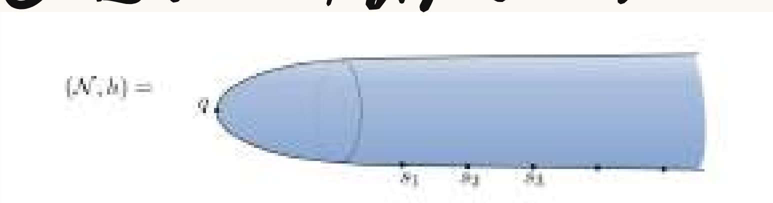
(1) $\{\Omega_i\}$ cpt and exhausting M with $p \in \mathring{\Omega}_i$ $(\forall i)$

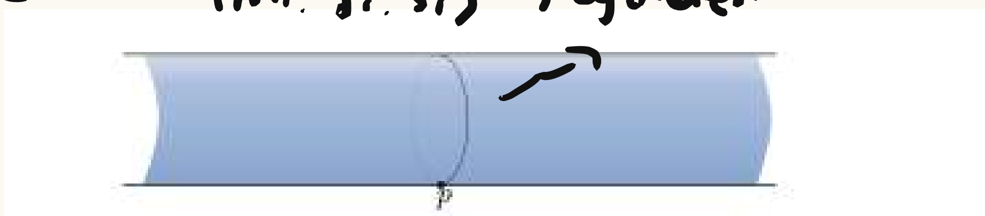
(2) $\{\phi_i : \Omega_i \to M_i\}$ where $\phi_i : \Omega_i \to \phi_i(\Omega_i)$ is a diffeo

and $\phi_i(p) = p_i$

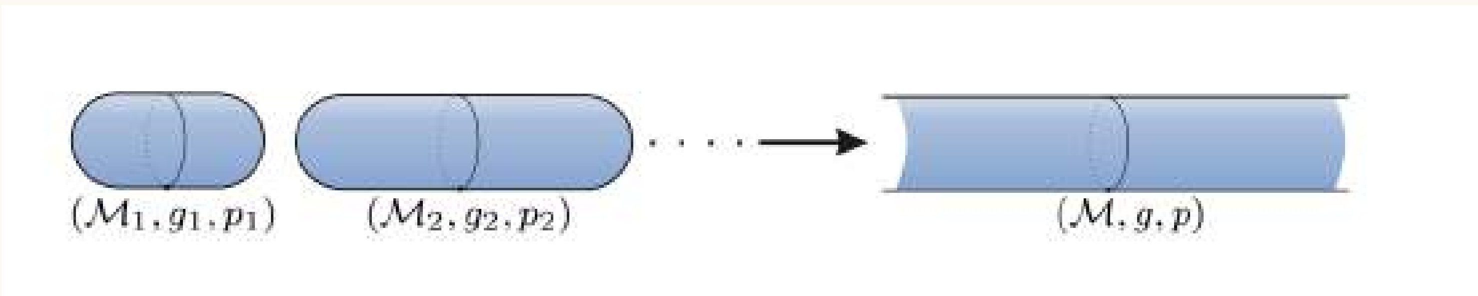
s.t. $\phi_i^* g_i \xrightarrow{C^\infty} g$ (i.e $|\nabla^k(\phi_i^* g_i - g)|$ $(k=0,1,\cdots) \longrightarrow 0$ locally uniformly).

Ex ① Let $(M_i, g_i) = (N, h)$ then
$(M_i, g_i, q) \to (N, h, q)$
$(M_i, g_i, s_i) \to$ cylinder





② It's possible that M_i cpt but M noncpt



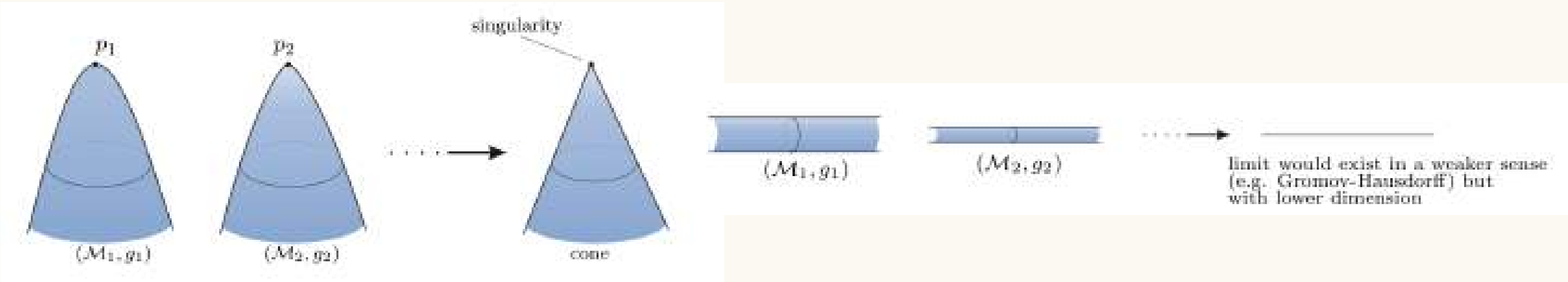
Rmk The convergence implies (i) $\forall s > 0$, $k \geq 0$, $\sup_{i \in \mathbb{N}} \sup_{B_{g_i}(p_i, s)} |\nabla^k Rm(g_i)| < \infty$

(ii) $\inf_i \text{inj}(M_i, g_i, p_i) > 0$

and the inverse also holds

Thm. (M_i, g_i, p_i) pointed complete satisfying (i)(ii), then $\exists (M, g, p)$ complete and subsequence s.t $(M_{i_k}, g_{i_k}, p_{i_k}) \xrightarrow{C^\infty} (M, g, p)$

Rmk The two conditions are necessary



Now it's for flows.

Def $(M_i, g_i(t))$ $(t\in(a,b)$ where $-\infty\le a<b\le\infty)$ complete mfld $(M, g(t))$ complete. $(M_i, g_i(t), p_i) \xrightarrow{C^\infty} (M, g(t), p)$ $(i\to\infty)$ if $\exists$

(i) $\{\Omega_i\}$ cpt. exhausting M. $p\in\mathring{\Omega}_i$ $(\forall i)$

(ii) $\{\phi_i : \Omega_i \to M\}$ $\phi_i : \Omega_i \to \phi_i(\Omega_i)$ diffeo. $\phi_i(p)=p_i$

s.t. $\phi_i^* g_i(t) \to g(t)$ that $\phi_i^* g_i(t) - g(t)$ and its derivatives of every order (w.r.t. time and space) converge to 0 locally uniformly on $M\times(a,b)$

Thm (Hamilton) $(M_i^n, g_i(t), p_i)$ $g_i(t)$ RF on M_i. $t\in(a,b)$

if (i) $\sup\limits_i \sup\limits_{x\in M_i, t\in(a,b)} |Rm(g_i(t))|(x)<\infty$

(ii) $\inf\limits_i \mathrm{inj}(M_i, g_i(0), p_i)>0$

then $\exists (M, g(t), p)$ $g(t)$ RF and a sequence s.t.

$(M_i, g_i(t), p_i) \xrightarrow{C^\infty} (M, g(t), p)$

II. Maximum Principles

1 For Parabolic Equation

(1) For function

Prop. M^n closed mfld. $u: M^n \times [0,T) \to \mathbb{R}$ s.t. $\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u$

if $u \geq c$ at $t=0$, then $u \geq c$ for all $t \geq 0$.

(pf. $u_\varepsilon = u + \varepsilon(1+t): M \times [0,T) \to \mathbb{R}$ $\Rightarrow u_\varepsilon > c$ at $t=0$

if $u_\varepsilon < c$ somewhere, then choose the smallest $t_1 > 0$

s.t. $u_\varepsilon(x_1, t_1) = c$ $\Rightarrow$ at (x_1, t_1)

$$0 \geq \frac{\partial u_\varepsilon}{\partial t} \geq \Delta_{g(t)} u_\varepsilon + \varepsilon > 0 \quad \times$$

$\Rightarrow u_\varepsilon > c$ ($\forall x \in M, t > 0$). $\varepsilon \to 0$ $\Rightarrow u \geq c$ ($\forall t \geq 0$) #)

Rmk. By the proof we actually see that the minimum of u is non-decreasing.

Prop': M^n closed mfld. $u: M^n \times [0,T) \to \mathbb{R}$ s.t. $\frac{\partial}{\partial t} u \leq \Delta_{g(t)} u + \overset{\text{Lipschitz}}{F(u)} + \langle \overset{\text{v.f.}}{X(t)}, \nabla u \rangle$

if $u \leq c$ at $t=0$, then $u(x,t) \leq U(t)$ ($\forall t \geq 0$)

where $\begin{cases} \frac{dU}{dt} = F(U) \\ U(0) = c \end{cases}$

(for pf check Thm 3.1.1 on Topping's book)

For $L = -a^{ij}\partial_i\partial_j + b^i\partial_i + c$ $\quad a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$

$\Omega_T = \Omega \times (0,T]$ $\quad \Gamma_T = \bar{\Omega}_T \setminus \Omega_T$ $\quad (\Omega \subset \mathbb{R}^n$ open$)$

Thm. If $c(x) \geq 0$, $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$

① $u_t + Lu \leq 0$ in Ω_T then $\max\limits_{\bar{\Omega}_T} u \leq \max\limits_{\Gamma_T}(u^+)$ $\left(\max\limits_{\Gamma_T} u \text{ for } c=0\right)$

② $u_t + Lu \geq 0$ in Ω_T then $\min\limits_{\bar{\Omega}_T} u \geq -\max\limits_{\Gamma_T}(u^-)$ $\left(\min\limits_{\Gamma_T} u \text{ for } c=0\right)$

③ $u_t + Lu = 0$ in Ω_T then $\max\limits_{\Omega_T}|u| = \max\limits_{\Gamma_T}|u|$

(2) For tensor

Thm. $(\underset{\text{closed}}{M^n}, g(t))$. $\alpha(t)$ symmetric 2-tensor with $\frac{\partial}{\partial t}\alpha = \Delta_{g(t)}\alpha + \beta$

where $\beta(x,t) = \beta(\alpha(x,t), g(x,t))$ is a symmetric (2,0) tensor

which is locally Lipschitz and satisfies:

if $A \geq 0$ symmetric 2-tensor at (x,t) with $A_{ij}V^j = 0$

then $\beta_{ij}(A,g)V^iV^j \geq 0$ $\quad (*)$

Then if $\alpha(0) \geq 0$ then $\alpha(t) \geq 0$ $(\forall t \geq 0)$

(pf. Choose (x_1,t_1) s.t. $\exists V$ s.t. $(\alpha_{ij}V^j)(x_1,t_1) = 0$ for the first time

Let V be time-independent. then $\frac{\partial}{\partial t}(\alpha_{ij}V^iV^j) \overset{(*)}{\geq} (\Delta\alpha_{ij})V_iV_j \underset{\text{choose } \nabla V = 0}{=} \Delta(\alpha_{ij}V_iV_j)$

≥ 0

So if α has eigenvalue 0 for the first time, it tends to increase in the direction of eigenvector. "add ε" to complete the proof)

We want to use this to show nonnegative Ricci is preserved.

So it suffices: if $R_{ij}W^iW^j \geq 0$ $(\forall W)$ and $R_{ij}V^j = 0$ for some V.

then $(R_{ikjl}R_{kl} - R_{ik}R_{jk})V^iV^j \geq 0$.

This is true only when $n=3$ since then " $\mathrm{Ric} \xrightarrow{\text{recover}} R$ "

Fact: $n=3 \Rightarrow \frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + \underbrace{3SR_{ij} - 6R_{ip}R_{jp} + (2|Rc|^2 - S^2)g_{ij}}_{\text{check the condition}}$

2. For Parabolic System (ODE-PDE thm)

For tensor (or see M as a section of $\Lambda^2M^n \otimes_S \Lambda^2M^n$)

Let $M: \Lambda^2M^n \to \Lambda^2M^n$ $\rightsquigarrow M$ on

Lie algebra structure on Λ^2M^n: $[U,V]_{ij} = g^{kl}(U_{ik}V_{lj} - U_{lj}V_{ik})$
is
$so(n)$)

Choose a basis $\{\varphi^\alpha\}$ of Λ^2M^n. $[\varphi^\alpha, \varphi^\beta] = C^{\alpha\beta}_\gamma \varphi^\gamma$

then define $M^\#: \Lambda^2M^n \to \Lambda^2M^n$ by $(M^\#)_{\alpha\beta} = C_\alpha^{\gamma\delta} C_\beta^{\varepsilon\zeta} M_{\gamma\varepsilon} M_{\delta\zeta}$

(ODE-PDE theorem)

Thm (Maximum Principle for system) E a vector bundle

$K \subset E$ is convex and invariant under parallel translation.

f is a section of E if the solution of $\frac{df}{dt} = \phi(f)$ remains in X. then so is the solution of $\frac{\partial f}{\partial t} = \Delta f + \phi(f)$

3. For non-compact mfld

Maximal principle fails generally for non-cpt mfld $\nearrow$ non-uniqueness of sol. to heat Eq

$\rightarrow$ but holds for solutions with certain growth conditions

Def. $u \in H^1_{loc}(M^n \times (0,T])$ is a weak subsolution of the heat equation $(\frac{\partial}{\partial t} - \Delta) u(x,t) = 0$, if $\forall \phi \in C_c^\infty(M^n \times (0,T))$ $\phi \geq 0$

$$\int_0^T \int_{M^n} (u \frac{\partial \phi}{\partial t} - \nabla u \cdot \nabla \phi) \, d\mu(x) \, dt \geq 0$$

Thm (Karp-Li) u as above $u_+ = \max\{0, u\}$. Fix $o \in M^n$ (for some α)

if $u(\cdot, 0) \leq 0$ with $\int_0^T \int_{M^n} e^{-\alpha d^2(x,o)} u_+^2(x,t) \, d\mu(x) dt < \infty$

then $u \leq 0$ on $M \times [0,T]$

pf. $h(x,t) = -\dfrac{d^2(x,o)}{4(2T-t)}$: Lipschitz on $M^n \times [0, 2T)$

$$|\nabla h|^2 = h_x^2 \underset{|\nabla d|=1}{=} \frac{(2d(x,o))^2}{(4(2T-t))^2} = \frac{1}{4} \frac{d^2(x,o)}{(4(2T-t))^2} = -\frac{\partial h}{\partial t} \qquad (*)$$

Take $\varphi_s \in C_0^\infty(B(o, s+1))$ $\begin{cases} 0 \leq \varphi_s \leq 1 \\ \varphi_s = 1 \text{ on } B(o,s) \\ |\nabla \varphi_s|^2 \leq 2 \end{cases}$

Also u_+ is a weak subsolution $\Rightarrow (\frac{\partial}{\partial t} - \Delta) u_+ \overset{\text{weakly}}{\leq} 0$

$$0 \geq \int_0^T \int_{M^n} \varphi_s^2 e^h u_+ (\frac{\partial}{\partial t} - \Delta) u_+ \, d\mu dt$$

$$= \frac{1}{2} \int_0^T \int_{M^n} \varphi_s^2 e^h \frac{\partial}{\partial t} (u_+^2) \, d\mu dt + \int_0^T \int_{M^n} \nabla u_+ \cdot \nabla (\varphi_s^2 e^h u_+) d\mu dt$$

$\nabla(\varphi_s^2 e^h u_+)\cdot\nabla u_+ = e^h(\varphi_s^2 |\nabla u_+|^2 + 2\langle \nabla \varphi_s, \nabla u_+ \rangle \varphi_s u_+ + \varphi_s^2 u_+ \langle \nabla h, \nabla u_+ \rangle)$

$$\doteq \frac{1}{2}\int_{M^n} \varphi_s^2 e^h u_t^2 \Big|_0^T d\mu - \frac{1}{2}\int_0^T\int_{M^n} \varphi_s^2 e^h u_t^2 \frac{\partial h}{\partial t} d\mu dt$$

$$+ \int_0^T\int_{M^n} e^h(-2|\nabla\varphi_s|^2 u_t^2 - \frac{1}{2}\varphi_s^2 u_t^2 |\nabla h|^2) d\mu dt$$

$$\Rightarrow \int_{M^n} \varphi_s^2 e^h u_t^2 \Big|_0^T d\mu \overset{(*)}{\leq} 4\int_0^T \int_{M^n} e^h u_t^2 |\nabla\varphi_s|^2 dt d\mu$$

$$\underset{\text{choose } T\leq \frac{1}{8\alpha}}{\leq} 8\int_0^T\int_{B(0,s+1)\setminus B(0,s)} e^{-\alpha d^2(x,0)} u_t^2 d\mu dt$$

$$\to 0 \ (s\to\infty) \qquad \Rightarrow u_t \overset{a.e}{=} 0 \text{ on } M^n\times[0,T]$$

then itearte to $[T,2T]$ ... we obtain the result.)

Cor If $Rc(x) \geq -d^2(x,0)$ and $u(x,t)$ is a bounded subsolution $u(x,0)\leq 0$, then $u(x,t)\leq 0$ In particular, bounded solutions are unique

(pf Use volume comparison and result above)

Thm (general ver). If curvatures of $(M^n, g(t))$ uniformly bounded. u weak subsolution with $u(\cdot,0)\leq 0$ and if $\exists \alpha>0$ s.t. $\int_0^T\int_{M^n} e^{-\alpha d^2_{g(t)}(x,0)} u_t^2(x,t) d\mu_{g(t)} \times dt < \infty$. then $u(x,t)\leq 0$

Lemma. If curvatures and their first derivatives are uniformly bounded, then $\forall a, A>0$ $\exists \phi>0, b>0$ s.t. $(\frac{\partial}{\partial t}-\Delta)\phi \geq A\phi$ with $e^{ad(x,0)} \leq \phi(x,t) \leq e^{bd(x,0)}$

Cor. ① $(M^n, g(t))$ is a complete solution of Ricci flow with bounded curvature. If $\varphi \geq 0$ with $\begin{cases} \frac{\partial}{\partial t}\varphi \leq \Delta\varphi + C\varphi \\ \varphi(0) = 0 \end{cases}$

and $\varphi(x,t) \leq e^{A(d(x,p)+1)}$ for some $A<\infty$. then $\varphi(t) \equiv 0$ $(t>0)$

② $(M^n, g(t))$ - - - - α_0 (p,q)-tensor

$|\alpha_0(x)|_{g(0)} \leq e^{A(d(x,p)+1)}$ for some $A<\infty$

Let $E^{p,q} = (\otimes^p T^*M) \otimes (\otimes^q TM)$ with $F_t : E^{p,q} \to E^{p,q}$ fiberwise linear

with $\|F_t\| = \sup\limits_{\beta(x) \in E^{p,q}} \dfrac{|F(\beta(x))|_{g(t)}}{|\beta(x)|_{g(t)}} < \infty$.

then $\exists B<\infty$ and $\alpha(t)$ s.t. $\begin{cases} \frac{\partial}{\partial t}\alpha = \Delta_{g(t)}\alpha + F_t(\alpha) \\ \alpha(0) = \alpha_0 \quad |\alpha|_{g(t)} \leq e^{B(d(x,p)+1)} \end{cases}$

This solution is unique among all solutions with $|\alpha|_{g(t)} \leq e^{C(d(x,p)+1)}$ for $C<\infty$

4. Strong Maximum Principle

Y is a sequentially cpt space.

$g : (a,b) \times Y \to \mathbb{R}$ and $\frac{\partial g}{\partial t}$ are continuous.

Let $h : (a,b) \to \mathbb{R}$ $\quad h(t) = \sup\limits_{y \in Y} g(t,y)$

$$\frac{d^+ h}{dt}(t) \triangleq \limsup_{s \to 0^+} \frac{h(t+s) - h(t)}{s}$$

$Y_t \triangleq \{ z \in Y \mid h(t) = g(t,z) \}$

Lemma. $\forall t \in (a,b)$ $\frac{d^+h}{dt}(t) = \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y)$.

(pf $s_i \to 0_+$. take $y_i \in Y$ s.t $h(t+s_i) = g(t+s_i, y_i)$

$\Rightarrow \exists y_{i_j} \to y_\infty \in Y$. then $g(t+s_{i_j}, y_{i_j}) \to g(t, y_\infty)$

If $y_\infty \notin Y_t$, then $\exists y' \in Y_t$ s.t. $g(t,y') - g(t, y_\infty) = 3\varepsilon > 0$

$\Rightarrow \exists N_0$ s.t. $\forall j \geq N_0$. $|g(t+s_{i_j}, y_{i_j}) - g(t, y_\infty)| < \varepsilon$

$\exists \delta > 0$ s.t. $\forall |s| \leq \delta$ $|g(t,y') - g(t+s, y')| \leq \varepsilon$

then choose $j \geq N_0$ with $|s_{i_j}| \leq \delta$

$g(t,y') \leq g(t+s_{i_j}, y_{i_j}) + \varepsilon \leq g(t, \infty) + 2\varepsilon = g(t, y') - \varepsilon$ X.

$\Rightarrow y_\infty \in Y_t$. $\overline{\lim_{j\to\infty}} \frac{h(t+s_{i_j}) - h(t)}{s_{i_j}} = \overline{\lim_{j\to\infty}} \frac{g(t+s_{i_j}, y_{i_j}) - g(t, y_\infty)}{s_{i_j}}$

$\leq \overline{\lim_{j\to\infty}} \frac{g(t+s_{i_j}, y_{i_j}) - g(t, y_{i_j})}{s_{i_j}}$

$= \frac{\partial g}{\partial t}(t, y_\infty) \leq \sup_{y \in Y_t} \frac{\partial g}{\partial t}(t,y)$

$\Rightarrow$ by mean value theorem. $\overline{\lim_{i\to\infty}} \frac{h(t+s_i) - h(t)}{s_i} \leq \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y)$.

Then take $y' \in Y_t$ s.t. $\frac{\partial g}{\partial t}(t,y') = \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y)$

$\Rightarrow \overline{\lim_{i\to\infty}} \frac{h(t+s_i) - h(t)}{s_i} \geq \overline{\lim_{i\to\infty}} \frac{g(t+s_i, y') - g(t, y')}{s_i} = \frac{\partial g}{\partial t}(t, y')$ ✓)

This lemma is important in the proof of maximum principle for system.

(1) Strong Maximum Principle for Function

Thm $u: M^n \times [0,T) \to \mathbb{R}$ $\quad g(t)$: C^1-class of metric

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + X(t)\nabla u + c(t) u$$

If $u \geq 0$ on $M^n \times [0,T)$ with $u(x_0,t_0) > 0$ ($\exists x_0 \in M$, $t_0 > 0$)

then $u(x,t)$ ($\forall x \in M$ $t \in (t_0 - \varepsilon, T)$ for some $\varepsilon > 0$)

Cor. $(M^n, g(t))$ $t \in [0,T)$ RF. $S(g(t)) \geq 0$ ($\forall t \geq 0$)

$S(x_0,t_0) > 0$ ($\exists x_0 \in M$, $t_0 > 0$). then $S(x,t) > 0$ ($\forall x \in M$ $t \in (t_0 - \varepsilon, T)$)

(This comes from under RF. $\frac{\partial}{\partial t} S = \Delta S + 2|Ric|^2 \geq \Delta S$)

Rmk If M^n closed. one can take $\varepsilon = t_0$

(2) Strong Maximum Principle for System

Thm $(M^n, g(t))$ $t \in [0,T)$ RF with $Rm(g(t)) \geq 0$

then $\exists \delta > 0$ s.t. $\forall t \in (0,\delta)$ $Im(Rm(g(t))) \subset \Lambda^2 T^* M$ is a constant smooth subbundle invariant under parallel translation

Moreover ① $Im(Rm_x(g(t)))$ is a Lie subalgebra of $\Lambda^2 T^*_x M$ $\overset{so(n)}{\cong}$ ($\forall x$, $t < \delta$)

② If M closed. then one may let $\delta = T$

③ If M open $\exists 0 = t_0 < t_1 < \cdots < t_k = T$ s.t. in $(t_{i-1}, t_i]$

$Im(Rm_x(g(t)))$ is a Lie subalgebra $\subset \Lambda^2 T^*_x M$ independent of time with $Im(Rm[g(t_1)]) \subset Im(Rm[g(t_2)]) \cdots$

III. Curvature Operator and Hamilton's ODE

1 Evolution of Rm

Now we derive the evolution of Rm under RF

Recall we've shown that

$$\frac{\partial}{\partial t} Rm(X,Y,W,Z) = \frac{1}{2}[h(R(X,Y)W,Z) - h(R(X,Y)Z,W)]$$
$$+\frac{1}{2}[\nabla^2_{Y,W}h(X,Z) - \nabla^2_{X,W}h(Y,Z) + \nabla^2_{X,Z}h(Y,W) - \nabla^2_{Y,Z}h(X,W)]$$

Define $B(X,Y,W,Z) = \langle Rm(X,\cdot,Y,\cdot), Rm(W,\cdot,Z,\cdot)\rangle$

$(B_{ijkl} = g^{pr}g^{qs}R_{ipjq}R_{krls})$

Prop. $(\Delta Rm)(X,Y,W,Z) = -\nabla^2_{Y,W}Ric(X,Z) + \nabla^2_{X,W}Ric(Y,Z)$
$-\nabla^2_{X,Z}Ric(Y,W) + \nabla^2_{Y,Z}Ric(X,W) - 2(B(X,Y,W,Z) -$
$B(X,Y,Z,W) + B(X,W,Y,Z) - B(X,Z,Y,W))$

(pf. Normal coordinate. $\nabla_i R_{jkla} + \nabla_j R_{kila} + \nabla_k R_{ijla} = 0$

$$\Rightarrow \Delta R_{jkla} = \nabla_i\nabla_k R_{ijla} - \nabla_i\nabla_j R_{kila}$$

Ricci identity $\nabla_i\nabla_j R_{kila} - \nabla_j\nabla_i R_{kila} = R_{jc}R_{kcla} + B_{jkla}$
$- B_{jkal} + B_{jlka} - B_{jakl}$

Again Bianchi II $\Rightarrow$ $\nabla_b R_{laki} + \nabla_l R_{abki} + \nabla_a R_{blki} = 0$

$$\Rightarrow g^{bi}\nabla_b R_{kila} + \nabla_l R_{ak} - \nabla_a R_{lk} = 0$$

$$\Rightarrow \nabla_j\nabla_i R_{kila} = \nabla_j\nabla_a R_{lk} - \nabla_j\nabla_l R_{ak}$$

So $\nabla_i\nabla_j R_{kila} = \nabla_j\nabla_a R_{lk} - \nabla_j\nabla_l R_{ak} + R_{jc}R_{kcla} + B_{jkla} - B_{jkal}$
$+ B_{jlka} - B_{jakl}$)

In an alternative language under RF $(h = -2Ric)$

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + Rm * Rm + Rc * Rm$$

Now treat Rm as : $\Lambda^2 M^n \to \Lambda^2 M^n$

$$Rm(\alpha)_{ij} = R_{ijkl}\alpha_{kl}$$

"Uhlenbeck trick" gives $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}}$ $(*)$

Ex. $n=3$. $so(3) \cong \mathbb{R}^3$ $[u,v] = u \times v$

if $Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$ $Rm^2 + Rm^{\#} = \begin{pmatrix} \lambda^2 + \mu\nu & & \\ & \mu^2 + \lambda\nu & \\ & & \nu^2 + \lambda\mu \end{pmatrix}$

$n=4$ use the maximum principle for system. Hamilton showed

Thm. (Hamilton. 1986) (M^4, g_0) closed with positive curvature operator, then $\exists$ a smooth solution of normalized RF $g(t)$ with $g(0) = g_0$ $t \in [0, \infty)$.

As $t \to \infty$ $g(t)$ converges to a constant positive metric exponentially in every C^k-norm. In particular $M^4 \overset{\text{diffeo}}{\cong} S^4$ or $\mathbb{RP}^4$

Here normalized RF is $\frac{\partial}{\partial t} \hat{g}_{ij} = -2\hat{Ric}_{ij} + \frac{2}{n}\hat{r}\hat{g}_{ij}$

where $\hat{r} = Vol(\hat{g})^{-1} \int_M \hat{S}\, d\mu$ (then $\frac{d}{dt} Vol(\hat{g}(t)) = 0$

since $\frac{\partial}{\partial t} d\mu = -S d\mu$)

Rmk. Bohm. Wilking (2008) showed that mfld with positive curvature operator must be space form

2. High-order Estimate

Lemma. If $|Rm(g(0))| \le K$, then $|Rm(g(t))| \le 2K \ (\forall 0 \le t \le \frac{1}{16K})$

(pf $\frac{\partial}{\partial t}|Rm|^2 = \Delta|Rm|^2 - 2|\nabla Rm|^2 + 8(B_{ijkl}+B_{ikjl})R_{ijkl}$

$\le \Delta |Rm|^2 - 2|\nabla Rm|^2 + 16|Rm|^3$

let ρ be that $\begin{cases} \frac{d}{dt}\rho^2 = 16\rho^3 \\ \rho(0) = K \end{cases}$

and use the weak maximum principle (IV 1. prop')

$|Rm(g(t))| \le \rho(t) = \frac{K}{1-8Kt}$)

Prop. (Long-time existence) $(M^n, g(t))$ RF on maximal interval (closed)

$[0,T)$ where $T<\infty$. then $\sup_{M^n\times[0,T)} |Rm| = \infty$ (*)

in fact $\lim_{t\to T} \max_{M^n} |Rm(\cdot,t)| = \infty$. (**)

Rmk. (*) → (**) from the lemma above.

Thm (Shi's estimate)

① (local) $\forall \alpha, K, r, m, n$ $\exists C = C(\alpha, K, r, m, n)$ s.t. $\forall (M^n, g(t))$ RF

$t\in[0,T_0]$, $0<T_0\le\frac{\alpha}{K}$ on a nbhd U of p with $B_{g(0)}(p,r)\subset\subset U$

if $|Rm(x,t)|\le K$ $(\forall x\in U, t\in[0,T_0])$ then $|\nabla^m Rm(y,t)| \le \frac{C}{t^{\frac{m}{2}}}$

$(\forall y\in B_{g(0)}(p,\frac{r}{2}),\ t\in(0,T_0])$

② (global) (closed) $(M^n, g(t))$ RF $t\in[0,T)$. then $\forall \alpha, m$

$\exists C = C(m,n,\alpha)$ s.t. if $|Rm(x,t)|_{g(t)} \le K$ $(\forall x\in M,\ t\in[0,\frac{\alpha}{K}]\cap[0,T))$

then $|\nabla^m Rm(x,t)|_{g(t)} \le \frac{CK}{t^{\frac{m}{2}}}$ $(\forall x\in M,\ t\in[0,\frac{\alpha}{K}]\cap(0,T))$

(pf. We only show ② here Fix m

Lemma. $(M, g(t))$ RF then $\frac{\partial}{\partial t}\nabla^m Rm = \Delta(\nabla^m Rm) + \sum_{l=0}^{m} \nabla^l Rm * \nabla^{m-l} Rm$

then inductively assume the estimate holds for $\forall l \leq m-1$

then by lemma

$$\frac{\partial}{\partial t}(|\nabla^{m-1} Rm|^2) \leq \Delta(|\nabla^{m-1} Rm|^2) - 2|\nabla^m Rm|^2 + C_1 \sum_{l=0}^{m-1} |\nabla^l Rm||\nabla^{m-l-1} Rm||\nabla^{m-1} Rm|$$

$$\leq \Delta(|\nabla^{m-1} Rm|^2) - 2|\nabla^m Rm|^2 + C_2 \frac{k^2}{t^m}$$

$$\frac{\partial}{\partial t}(|\nabla^m Rm|^2) \leq \Delta(|\nabla^m Rm|^2) - 2|\nabla^m Rm|^2 + C_3 \sum_{l=0}^{m} |\nabla^l Rm||\nabla^{m-l} Rm||\nabla^m Rm|$$

$$\leq \Delta(|\nabla^m Rm|^2) + C_4 k|\nabla^m Rm|^2 + C_4 k t^{-\frac{m}{2}-1}|\nabla^m Rm|$$

Let $F: M \times [0,T] \to \mathbb{R}$ by $F = t^{m+1}|\nabla^m Rm|^2 + \frac{1}{2}(C_4 + m + 2)t^m |\nabla^{m-1} Rm|^2$

then $\frac{\partial}{\partial t} F \leq \Delta F - t^m |\nabla^m Rm|^2 + C_4 k t^{\frac{m}{2}} |\nabla^m Rm|$

$$+ \frac{1}{2}(C_4 + m + 2) m t^{m-1} |\nabla^{m-1} Rm|^2 + \frac{C_2}{2}(C_4 + m + 2) k^2$$

$$\leq \Delta F + C_6 k^2$$

then by the maximum principle one can obtain the result).

3. Extension of Ricci flow

Now we focus on the longtime existence.

Lemma (Uniform Equivalence) $g(t)$ $[0,T]$ $T\le\infty$ smooth metrics

if $\exists C<\infty$ s.t. $\int_0^T \sup_{x\in M^n} \left|\frac{\partial g}{\partial t}(x,t)\right|_{g(t)} dt \le C$

then $\forall x_0\in M^n$, $t_0\in[0,T)$. $\quad e^{-C} g(x_0,0) \le g(x_0,t_0) \le e^{C} g(x_0,0)$ $(*)$

Moreover, $g(t) \xrightarrow{\text{uniform}} g(T)$ with $e^{-C} g(0) \le g(T) \le e^{C} g(0)$

(pf. $\forall V\in T_xM$ $0\le t_1\le t_2<T$

$$\left|\log\frac{g_{(x,t_2)}(V,V)}{g_{(x,t_1)}(V,V)}\right| = \left|\int_{t_1}^{t_2} \frac{\frac{\partial}{\partial t} g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)} dt\right| \le \int_{t_1}^{T} \left|\frac{\partial g}{\partial t}(x,t)\right|_{g(t)} dt$$
$$\triangleq C(t_1) \le C$$

$\lim_{t\to T} C(t)=0$. $\Rightarrow (*)$ holds

Also by $\langle V,W\rangle=\frac{1}{4}(\|V+W\|^2-\|V-W\|^2)$ $\quad \lim_{t\to\infty} g(t)(V,W) \triangleq g(T)(V,W)$ exists ✓)

Cor. For RF with $\sup_{M^n\times[0,T)} |Ric|\le K$, then

$$e^{-2KT} g(x,0) \le g(x,t) \le e^{2KT} g(x,0) \quad (\forall x\in M^n, t\in[0,T))$$

Now we prove the prop$(*)$ in the last section

i.e. the maximal time is finite $\Rightarrow$ unbounded curvature

If not. then $\sup_{t\in[0,T)} \sup_M |Rm(g(t))| < \infty$

then by high-order estimate $\sup_{t\to T} \sup_M |\nabla^m Rm(g(t))| < \infty$

$$\Rightarrow \sup_{t\to T} \sup_M |\nabla^m \frac{\partial g}{\partial t}|_{g(t)} \le K < \infty$$

Prop. (closed $M^n, g(t)$) RF with $\sup_{M^n\times[0,T)} |Rm| \le K < \infty$. then $\forall$ metric $\bar{g}$ on

$\exists\, C_m < \infty,\ C_m' < \infty$ s.t. $|\bar{\nabla}^m g(x,t)|_{\bar{g}} < C_m$

$C_m = C_m(m, K, T, g(0), \bar{g})$ $\qquad |\bar{\nabla}^m Rc(x,t)|_{\bar{g}} < C_m'$

(shown by using local high-order estimate).

Now $g(t) \xrightarrow{C^0} g(T)$ and by uniform equivalence $g(T)$ is actually a metric. here $g_{ij}(x,T) = g_{ij}(x,\tau) - 2\int_\tau^T R_{ij}(x,t)\,dt$

Then $\forall |\alpha| = m$ $\quad \frac{\partial^m}{\partial x^\alpha} g_{ij}(x,T) = \frac{\partial^m}{\partial x^\alpha} g_{ij}(x,\tau) - 2\int_\tau^T \frac{\partial^m}{\partial x^\alpha} R_{ij}(x,t)\,dt$

by the above prop. the two terms in RHS are bounded

so $g(T)$ is a smooth metric with

$$\left|\frac{\partial^m}{\partial x^\alpha} g_{ij}(x,T) - \frac{\partial^m}{\partial x^\alpha} g_{ij}(x,\tau)\right| \le C|\tau - T| \to 0$$

So actually $g(x,\tau) \xrightarrow{C^\infty} g(x,T)$. And one can extend the flow beyond T be setting $g(T)$ as initial data.

Rmk. Sesum showed that if maximal time $T<\infty$. then

$$\sup_{M^n\times[0,T)} |Ric| = \infty.$$

4. Maximum Principle for the curvature operator

Recall the ODE-PDE thm and the evolution of Rm

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}$$

it suffices to consider the ODE $\frac{d}{dt} M = M^2 + M^{\#}$ (*)

Thm $E = \Lambda^2 M^n \otimes_s \Lambda^2 M^n$ $(\overset{\text{closed}}{\overset{\uparrow}{M^n}}, g(t))$ $t \in [0,T)$ RF

$K \subset E$ invariant under parallel translation with $K_x = K \cap E_x$ closed convex

If (*) satisfies: $\forall M(0) \in K$, we have $M(t) \in K$ $(\forall t \in [0,T))$

then if $Rm(0) \in K$ we have $Rm(t) \in K$ $(\forall t \in [0,T))$

Cor. If $(\overset{\text{closed}}{\overset{\uparrow}{M^n}}, g(t))$ RF $t \in [0,T)$ $Rm(g(0)) \geq 0$ then $Rm(g(t)) \geq 0$ $(\forall t \in [0,T))$

(because $Rm \geq 0 \Rightarrow Rm^2 \geq 0$, $Rm^{\#} \geq 0$)

Now we choose suitable K where $n=3$

If $M(0)$ diagonal then so is $M(t)$

write $\lambda_1 \leq \lambda_2 \leq \lambda_3$ as eigenvalues of M, then

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2 \lambda_3 \\ \frac{d\lambda_2}{dt} = \lambda_2^2 + \lambda_1 \lambda_3 \\ \frac{d\lambda_3}{dt} = \lambda_3^2 + \lambda_1 \lambda_2 \end{cases}$$

① $K = \{M : \lambda_1 + \lambda_2 + \lambda_3 \geq C_0\}$

then $\frac{d}{dt}(\lambda_1 + \lambda_2 + \lambda_3) = \frac{1}{2}[(\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_3)^2 + (\lambda_2+\lambda_3)^2]$

$\geq \frac{2}{3}(\lambda_1+\lambda_2+\lambda_3)^2 \geq 0 \Rightarrow K$ is preserved

So if $S(0) \geq C_0$. then $S(t) \geq C_0 \quad (\forall t \geq 0)$

(Note that this was shown before using weak maximum principle)

② $K = \{M : \lambda_1 \geq 0\}$ (check convex!)

$\frac{d}{dt}\lambda_1 = \lambda_1^2 + \lambda_2\lambda_3 \geq 0$ if $\lambda_1 \geq 0$

$\Rightarrow$ if $\lambda_1(M(0)) \geq 0$ then $\lambda_1(M(t)) \geq 0 \; (\forall t \geq 0)$

So nonnegative sectional curvature is preserved

③ $K = \{M : \lambda_1 + \lambda_2 \geq 0\}$ $\quad \frac{d}{dt}(\lambda_1+\lambda_2) = \lambda_1^2 + \lambda_2^2 + (\lambda_1+\lambda_2)\lambda_3 \geq 0$ if $\lambda_1+\lambda_2 \geq 0$

So nonnegative Ricci is preserved

④ $C \geq \frac{1}{2}$ $\quad K = \{M : \lambda_3 \leq C(\lambda_1+\lambda_2)\}$

$\frac{d}{dt}(\lambda_3 - C\lambda_1 - C\lambda_2) = \lambda_3(\lambda_3 - C\lambda_1 - C\lambda_2) - C(\lambda_1^2 + \lambda_2^2 - \frac{1}{C}\lambda_1\lambda_2) \leq 0$

If $Rc(g(0)) > 0$, then by compactness $Rm(g(0)) \subset K$ for some $C \geq \frac{1}{2}$

$\Rightarrow \forall t \geq 0 \quad Rc \geq \frac{1}{C}\lambda_3 g \geq \frac{1}{3}\frac{S}{C} g = \varepsilon S g$

So Ricci Pinching is Preserved

⑤ $C_0 > 0$. $C_1 \geq \frac{1}{2}$ $\; C_2 < \infty$. $\delta > 0$ $\quad K = \left\{ M : \begin{array}{l} \lambda_3 - \lambda_1 - C_2(\lambda_1+\lambda_2+\lambda_3)^{1-\delta} \leq 0 \\ \lambda_3 \leq C_1(\lambda_1+\lambda_2) \\ \lambda_1+\lambda_2+\lambda_3 \geq C_0 \end{array} \right\}$

(check K is preserved)

since $\lambda_3 - \lambda_1 \geq |Rc - \frac{1}{3}Sg| \Rightarrow \exists C < \infty$ $\quad \underbrace{|Rc - \frac{1}{3}Sg| \leq C S^{1-\delta}}_{\text{Ricci Pinching Improves}}$

5. Hamilton-Ivey Estimate

Prop (Maximum Principle for Time-dependent set)

$(M^n, g(t))$ closed, $t\in[0,T)$ RF, $K(t)\subset E$ invariant under parallel translation with $K(t)_x = K(t)\cap E_x$ closed and convex

Also $\{(v,t)\in E\times[0,T) \mid v\in K(t)\}\subset E$ closed

If $\frac{dM}{dt} = M^2+M^\#$ satisfies if $M(t_0)\in K(t_0)$ then $M(t)\in K(t)$ $(t\in[t_0,T))$

Then if $Rm(0)\in K(0) \Rightarrow Rm(t)\in K(t)$ $(\forall t\in[0,T))$

Thm (3-dim Hamilton-Ivey) $(M^3, g(t))$ closed, $t\in[0,T)$ RF

If $\lambda_1(Rm)(x,0)\geq -1$, then at any (x,t) with $\lambda_1(Rm)(x,t)<0$

we have
$$S\geq |\lambda_1(Rm)|\left(\log|\lambda_1(Rm)| + \log(1+t) - 3\right)$$
$$\geq |\lambda_1(Rm)|\left(\log|\lambda_1(Rm)| - 3\right)$$

(Pf. Let $K(t)=\left\{M : \begin{array}{l}\lambda_1+\lambda_2+\lambda_3 \geq -\frac{3}{1+t} \\ \text{if } \lambda_1\leq -\frac{1}{1+t} \Rightarrow \lambda_1+\lambda_2+\lambda_3 \geq -\lambda_1\left(\log(-\lambda_1)+\log(1+t)-3\right)\end{array}\right\}$

satisfies the condition

$$\frac{d}{dt}(\lambda_1+\lambda_2+\lambda_3) \geq \frac{1}{3}(\lambda_1+\lambda_2+\lambda_3)^2 \Rightarrow \lambda_1+\lambda_2+\lambda_3 \geq -\frac{3}{1+t}$$

And check that $\frac{d}{dt}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{-\lambda_1} - \log(-\lambda_1) - \log(1+t)\right)\geq 0$

if $\lambda_1(M)\leq -\frac{1}{1+t}$)

So if $\lambda_1(Rm) \le e^{-C+3} \Rightarrow |\lambda_1(Rm)| \le C^{-1}S \le \frac{3}{C}\lambda_3(Rm)$

Cor. ① $\exists\, \phi: (0,\infty) \to (0,\infty)$ decreasing with $\phi(\infty)=0$ s.t

if $(M^3, g(t))$ $t\in[0,T)$ RF on closed 3-mfld with $\lambda_1(Rm)(x,0) \ge -1$ $(\forall x)$

then $\lambda_1(Rm) \ge -\phi(S)S - e^2$ on $M^3\times[0,T)$

② If drop the initial condition then if $t \ge \varepsilon > 0$.

(i) $S \ge |\lambda_1(Rm)|(\log|\lambda_1(Rm)| + \log\varepsilon - 3)$ for $\lambda_1(Rm)<0$ $(*)$

$\Rightarrow$ (ii) $\lambda_1(Rm) \ge -\phi(S)S - \varepsilon^{-1}e^2$ on $M^3\times[\varepsilon, T)$

Cor ① If we have a complete ancient solution on a closed 3-mfld with bounded curvature, then $K \ge 0$

② Complete ancient solutions on surfaces with bounded curvature, then $K \ge 0$.

pf. ① Otherwise $\exists (x,t)$ where $\lambda_1(Rm)<0$

in $(*)$ take $\varepsilon\to\infty$ there is a contradiction

② Consider the product of the surface and a line and use ①.

Rmk. This holds for all dimensions.

6 Nonnegative Curvature Operator

Now we utilize the strong maximum principle for Rm introduced in IV. 4.(2)

Thm. $(\underset{\uparrow}{M}^3, g(t))$ closed $t\in[0,T)$. $K\geq 0$ RF then the universal solution $(\tilde{M}^3, \tilde{g}(t))$ is either (1) standard $\mathbb{R}^3$ (2) $K>0 \Rightarrow \tilde{M}^3 \overset{\text{diffeo}}{\simeq} S^3$

(3) $(S^2, h(t))\times\mathbb{R}$. $h(t)$ a solution to RF with positive curvature

(pf. Lie subalgebra of $so(3)$: $0, so(3), so(2)$. $t\in(0,\delta)$

① $\mathrm{Im}[Rm(g(x,t))] \simeq so(3)$: $Rm>0 \overset{n=3}{\Rightarrow} K>0$ $(\forall t>0) \Rightarrow \tilde{M}^3\simeq S^3$

② - - - $=0$. Bieberbach thm $\Rightarrow \tilde{M}^3 \simeq \mathbb{R}^3$

③ - - $= so(2)$: $\exists\alpha\in \mathrm{Im}[Rm(g(x,t))]$ parallel 2-form

de Rham Splitting thm: $\tilde{M}^3 = \mathbb{R}\times S^2$)

Rmk The thm holds for $Ric\geq 0$

Now for $n=4$

Thm. $(\underset{\uparrow}{M}^4, g(t))$ closed $t\in[0,T)$ $Rm\geq 0$ RF. then $(\tilde{M}^4, \tilde{g}(t))$ is

(1) $\mathbb{R}^4$ (2) $(S^3, h(t))\times\mathbb{R}$ $h(t)$ RF with $K>0$

(3) $(S^2, h_1(t))\times(S^2, h_2(t))$ (4) $(S^2, h_1(t))\times\mathbb{R}^2$ (5) S^4

(6) $\mathbb{CP}^2$ with solution whose Rm is positive on the subspace $\Lambda^{1,1}_{\mathbb{R}}\subset\Lambda^2$.

Conjecture (Hamilton) (M^n, g_0) closed $n \geq 5$ $Rm > 0$ then the solution of NRF exists for all time and converges to a constant curvature metric

Thm. (M^n, g) closed $Rm \geq 0$ then $(\tilde{M}, \tilde{g})$ is isometric to product of

(1) Euclidean space (2) closed symmetric space

(3) closed Riem. mfld with $Rm > 0$ (4) closed Kähler mfld with $Rm > 0$ on $\Lambda^{1,1}_{\mathbb{R}}$

Thm. (Berger) (M^n, g) simply connected, irreducible. then (M^n, g) is a symmetric space of rank ≥ 2, or its holonomy group is from

(1) $SO(n)$ (2) $U(\frac{n}{2})$ (3) $SU(\frac{n}{2})$ (4) $Sp(\frac{n}{4})Sp(1)$ (5) $Sp(\frac{n}{4})$

(6) G_2 $(n=7)$ (7) $Spin(7)$ $(n=8)$ (8) $Spin(9)$ $(n=16)$

Thm (Tachibana, 1974) (1) (M^n, g) Einstein $Rm > 0 \Rightarrow$ constant $K > 0$

(2) - - - $Rm \geq 0 \Rightarrow$ locally symmetric

(Pf of Thm. Take RF on $M^n \times [0, \varepsilon)$ $g(0) = g$ then for $t > 0$

$Im(Rm)$ invariant under parallel translation and constant

is $h \subseteq so(n)$ Lie algebra of holonomy group H

$H \simeq SO(n)$, $U(\frac{n}{2})$, $Sp(\frac{n}{4})Sp(1)$, $Spin(9)$, others

$SO(n) \downarrow$ (3)

$U(\frac{n}{2}) \downarrow$ (4)

$Sp(\frac{n}{4})Sp(1) \downarrow$ Einstein $\downarrow$ Tachibana locally symmetric $\downarrow$ simply connected symmetric

$Spin(9) \downarrow$ Alekseevskii symmetric

others $\downarrow$ flat.)

7. Pinching Towards Constant Curvature

Prop(∇S estimate) I. $(M^{n\geq 3}, g(t))$ RF on a closed mfld

with $S>0$ If $\left|Rm - \frac{2S}{n(n-1)} Id_{\Lambda^2}\right| \leq KS^{1-\varepsilon}$ for some $\begin{matrix} K<\infty \\ \varepsilon>0 \end{matrix}$

then $\forall \eta, \theta > 0$. $\exists C = C(g_0, \eta, \theta) < \infty$ s.t. at any (x,t) with

① $S(\bar{x}, \bar{t}) \geq C$ ② $S(\bar{x}, \bar{t}) \geq \eta \max_{M^3 \times [0, \bar{t}]} S$

we have $|\nabla S|(\bar{x}, \bar{t}) \leq \theta S^{\frac{3}{2}}(\bar{x}, \bar{t})$

II. $(M^3, g(0))$ closed 3-mfld with $Rc>0$ then $\forall \varepsilon > 0$

$\exists C = C(\varepsilon, g(0))$ with $|\nabla S|^2(x,t) \leq \varepsilon S(x,t)^3 + C(\varepsilon)$

III. $(M^3, g(0))$ as above. then $\exists \beta_0 > 0$. $\delta > 0$ s.t. $\forall \beta \in [0, \beta_0]$

$$\frac{|\nabla S|^2}{S^3} \leq \beta S^{-\delta} + CS^{-3}. \quad C = C(\beta, g(0)) < \infty$$

So since $|Rm| \to \infty$. $S_{max} \to \infty$

$\Rightarrow S_{min} \to \infty$ (positive Ricci means $T<\infty$!)

Cor. ① $(M^3, g(0))$ as above $[0,T)$ maximal interval of solution

to Ricci flow. then $\lim_{t \to T} \frac{S_{max}(t)}{S_{min}(t)} = 1$

actually. $\exists C < \infty$ $\gamma > 0$. so that

$$\frac{S_{min}(t)}{S_{max}(t)} \geq 1 - CS_{max}(t)^{-\gamma} \quad \text{for } \forall t \in [0,T)$$

② $\forall \varepsilon \in (0,1)$ $\exists \tau < T$ s.t. $\forall t \in [\tau, T)$. $K(t) > 0$

and $\min_{x \in M^3} \lambda_1(Rm)(x,t) \geq (1-\varepsilon) \max_{x \in M^3} \lambda_3(Rm)(x,t)$

pf. ① By Prop III $\exists C<\infty$ $\delta>0$ s.t. $|\nabla S(x,t)| \le C S_{max}(t)^{\frac{3}{2}-\delta}$

$\forall t\in[0,T)$ $\exists x_t$ s.t $S_{max}(t)=S(x_t,t)$ then $\forall x\in B(x_t, \frac{\eta}{\sqrt{S_{max}(t)}})$

$$S_{max}(t)-S(x,t) \le \frac{\eta}{\sqrt{S_{max}(t)}} \max_{x\in M^3} |\nabla S(x,t)| \le \frac{C}{\eta} S_{max}(t)^{1-\delta}$$

$$\Rightarrow S(x,t) \ge S_{max}(t)\left(1-\frac{C}{\eta} S_{max}^{-\delta}(t)\right) \qquad (*)$$

However, since $\lim_{t\to T} S_{max}(t)=\infty$ $\exists \tau<T$ s.t $\forall t\in[\tau,T)$

$$S(x,t)\ge S_{max}(t)(1-\eta) \qquad \forall x\in B(x_t, \frac{\eta}{\sqrt{S_{max}(t)}})$$

By pinching. $Rc\ge \varepsilon S g$ $\Rightarrow Rc(x,t)\ge \varepsilon S_{max}(t)(1-\eta) g$

then by Bonnet-Myers for η small $M^3 = B(x_t, \frac{\eta}{\sqrt{S_{max}(t)}})$

so $(*)$ holds for $\forall x\in M^3$ $\Rightarrow$ $\frac{S_{min}(t)}{S_{max}(t)} \ge 1-\frac{C}{\eta} S_{max}^{-\delta}(t)$.

② By Ricci-Pinching improves (V.4.⑤) $\exists C<\infty$. $\delta>0$ s.t

$$\frac{\lambda_1}{\lambda_3}(x,t) \ge 1-C\frac{S^{1-\delta}}{\lambda_3}(x,t) \ge 1-3C S_{min}(t)^{-\delta} \quad (\forall x\in M^3\ t\in[0,T))$$

then $\forall \varepsilon>0$ take $\tau<T$ s.t. $1-3CS_{min}(t)^{-\delta}\ge 1-\varepsilon$ $(\forall t\in[\tau,T))$

so $\min_{x\in M^3} \lambda_1(x,t) \ge (1-\varepsilon)\max_{x\in M^3} \lambda_3(x,t)$ ✓

8. Hamilton's 1982 Thm

Now we can prove the famous result.

Thm (Hamilton, 1982) (M^3, g_0) closed with $Rc>0$

then $\exists!$ solution $g(t)$ of NRF with $g(0)=g_0$ and for $t\to\infty$

$g(t) \xrightarrow{C^m} g_\infty$ exponetially ($\forall m$). where g_∞ is a C^∞ constant

positive metric. Hence M is diffeomorphic to S^3.

Firstly for $g(t)$ solution to RF let $c(t) = e^{\frac{2}{n}\int_0^t r(\tau)d\tau}$

$\hat{t}(t) = \int_0^t c(\tau)d\tau$

$\tilde{g}(\hat{t}) = c(t)g(t)$ is a solution to NRF

so NRF comes from RF by rescaling space and time.

Lemma $(M^3, g(0))$ as above. $\tilde{g}(\tilde{t})$ NRF on $[0, \underset{\text{maximal}}{T})$. then

① $\lim\limits_{t\to T} \dfrac{\tilde{S}_{max}(\tilde{t})}{\tilde{S}_{min}(\tilde{t})} = 1$ ② $\tilde{Rc} \geq \delta \tilde{S}\tilde{g}$ ③ $\tilde{S}_{max}(\tilde{t}) \leq C$

④ $\tilde{T} = \infty$ ⑤ $\lim\limits_{\tilde{t}\to\infty}\left(\max\limits_{x\in M^3} \dfrac{|\tilde{Rc} - \frac{1}{3}\tilde{S}\tilde{g}|^2}{\tilde{S}^2}(x,\tilde{t})\right) = 0$

⑥ $\tilde{S}_{min}(\tilde{t}) \geq \frac{1}{C}$ ⑦ $|\tilde{Rc} - \frac{1}{3}\tilde{S}\tilde{g}| \leq Ce^{-\delta\tilde{t}}$

⑧ $\tilde{S}_{max}(\tilde{t}) - \tilde{S}_{min}(\tilde{t}) \leq Ce^{-\delta\tilde{t}}$ ⑨ $|\tilde{Rc} - \frac{1}{3}\tilde{r}\tilde{g}| \leq Ce^{-\delta\tilde{t}}$

⑩ $\forall k.$ $|\tilde{\nabla}^k \tilde{Rc}| \leq Ce^{-\delta\tilde{t}}$

pf. ①② : comes from RF because rescaling-invariant

③ $\tilde{R}_{ij} \geq 0 \Rightarrow \text{Const} = \text{Vol}(\tilde{g}(\tilde{t})) \leq C\,\text{diam}(\tilde{g}(\tilde{t}))^3$

Since by ①② $\tilde{Rc} \geq \varepsilon \tilde{S}_{max} \tilde{g}$ and Bonnet-Myers $\text{diam}(\tilde{g}(\tilde{t})) \leq \frac{C}{\sqrt{|\tilde{Rc}(\tilde{t})|}}$

so $\tilde{S}_{max}(\tilde{t}) \leq C$

④ Let $g(t)$ be RF and $\begin{cases} f(0) = S_{max}(0) \\ f'(t) = 2 S_{max}(t) f(t) \end{cases}$

$\Rightarrow \frac{\partial}{\partial t}(S-f) \leq \Delta(S-f) + 2 S_{max}(S-f)$

So by maximum principle $S \leq f$ on $[0, T_0) \Rightarrow f(t) \to \infty \; (t \to T_0)$

$\Rightarrow \int_0^t S_{max}(\tau)\,d\tau = \frac{1}{2}\log\frac{f(t)}{f(0)} \to \infty \overset{S_{max} \sim S_{min}}{\underset{(t \geq T)}{\Longrightarrow}} \int_0^{T_0} r\,dt = \infty$

then $\int_0^{\tilde{T}} \tilde{S}_{max}(\tilde{t})\,d\tilde{t} = \int_0^{T_0} S_{max}(t)\,dt = \infty \overset{③}{\Longrightarrow} \tilde{T} = \infty$

⑤ By Ricci pinching improves and rescaling-invariant

⑥ Replacing by universal cover. by Klingenberg's estimate

$$\text{inj}(\tilde{g}(\tilde{t})) \geq C \tilde{K}_{max}(\tilde{t})^{-\frac{1}{2}}$$

also $\tilde{K}(\tilde{g}(\tilde{t})) \leq C \tilde{S}_{max}(\tilde{t}) \Rightarrow \underbrace{\text{Vol}(\tilde{g}(\tilde{t}))}_{\text{const}} \geq \varepsilon \tilde{S}_{max}(t)^{-\frac{3}{2}}$

$\Rightarrow \tilde{S}_{max}(t) \geq \frac{1}{C} \overset{①}{\Rightarrow} \tilde{S}_{min}(t) \geq \frac{1}{C}$

⑦ Let $\tilde{f} = \dfrac{|\tilde{Rc} - \frac{1}{3}\tilde{S}\tilde{g}|^2}{\tilde{S}^2}$

Hamilton showed that $\frac{\partial \tilde{f}}{\partial \tilde{t}} \leq \Delta \tilde{f} + 2\langle \tilde{\nabla} \log \tilde{S} + \tilde{\nabla} \tilde{f} \rangle - \delta \tilde{S}_{min} \tilde{f}$

so by ⑥ and maximum principle $|\tilde{Rc} - \frac{1}{3}\tilde{S}\tilde{g}| \leq C e^{-\delta \tilde{t}}$

(Lemma 17.2 of Hamilton's JDG paper)

8 $\tilde{\Psi} = \frac{|\tilde{\nabla}\tilde{S}|^2}{\tilde{S}} + 168(|\tilde{Rc}|^2 - \frac{1}{3}\tilde{S}^2)$

(Lemma 17.4 in the paper) $\Rightarrow$ $\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta}\right)(e^{\delta t}\tilde{\Psi} - C\tilde{t}) \le 0$

$\Rightarrow \tilde{\Psi} \le Ce^{-\delta \tilde{t}}(1+\tilde{t})$ so $|\tilde{\nabla}\tilde{S}| \le Ce^{-\delta \tilde{t}}$

9 $\Leftarrow$ 7 + 8

10. See Thm 17.9 of the paper.

So 4 9 10 $\Rightarrow$ $\tilde{g}(\tilde{t}) \xrightarrow{C^k} \tilde{g}_\infty$ ($\forall k$. $t \to \infty$)

and 5 gives that $\tilde{g}_\infty$ has positive constant curvature. ✓

IV. Differentiable Sphere Thm

The goal of this chapter is the following thm

Thm (Brendle–Schoen) (M,g) complete and simply connected and $\forall p\in M$, $0<\sup_{\Pi_p} K(\Pi_p)<4\inf_{\Pi_p} K(\Pi_p)$ then $M \overset{\text{diffeo}}{\simeq} S^n$

We follow the proof in Brendle's book. The key is to construct Hamilton ODE-invariant sets, where "cones" are some of them. Most details involving tensor algebra will be omitted.

1 Pinching Set and Hamilton's convergence criterion

$\mathscr{C}_B(V)=\{\text{algebraic tensors on } V\}$

$F\subset \mathscr{C}_B(\mathbb{R}^n)$ is called a Pinching set if

① F closed, convex, parallel translation-invariant

② Invariant under Hamilton ODE $\frac{d}{dt}R=R^2+R^\#$

③ $\forall \delta\in(0,1)$ $\{R\in F \mid R \text{ is not weakly } \delta\text{-pinched}\}$ is bounded

Now (M^n, g_0) (closed, $n\geq 3$) with $S>0$ $g(t)$ $(t\in[0,T))$ RF

and assume $\exists F\subset \mathscr{C}_B(\mathbb{R}^n)$ pinching set s.t. $Rm(p,0)\in F(p,0)$ $(\forall p\in M)$

then

Lem ① $\forall\delta\in(0,1)$ $\exists C>0$ s.t. $K_{min}(p,t)\geq \delta K_{max}(p,t)-C$

② $T<\infty$ $\limsup_{t\to T} K_{max}(t)=\infty$ (since $S(g(0))>0$)

③ $t_k \to T$ and $K_{max}(t_k) \geq \frac{1}{2} \sup_{t \in [0, t_k]} K_{max}(t)$. $K_{max}(t_k) = K_{max}(p_k, t_k)$

Then let $\Omega_k = \{x \in M \mid d_{g(t)}(p_k, x) \leq 2\pi K_{max}(t_k)^{-\frac{1}{2}}\}$

we have $\liminf_{k \to \infty} \dfrac{\inf_{x \in \Omega_k} K_{min}(x, t_k)}{K_{max}(t_k)} \geq 1$

pf ③ Fix $\varepsilon > 0$. then $\exists C_1$ s.t. $\sup_M |E(t)| \leq \varepsilon K_{max}(t) + C_1 \ \forall t$

$\Rightarrow \sup_M |E(t)| \leq 2\varepsilon K_{max}(t_k) + C_1$ $(\forall t \leq t_k)$

Also since $\frac{\partial}{\partial t} E = \Delta E + R * E \Rightarrow \exists C_2$ s.t.

$\sup_M |\nabla E(t_k)|^2 \leq C_2 K_{max}(t_k) (2\varepsilon K_{max}(t_k) + C_1)^2$

by $\sum_{k=1}^{n} (\nabla_{e_k} E)(X, e_k) = \frac{n-2}{2n} X(S)$

$\Rightarrow \sup_M |dS|^2 \leq C_3 K_{max}(t_k) (2\varepsilon K_{max}(t_k) + C_1)^2$

def of Ω_k $\Rightarrow \inf_{x \in \Omega_k} S(t_k, x) \geq S(t_k, p_k) - 2\pi \sqrt{C_3} (2\varepsilon K_{max}(t_k) + C_1)$

$\Rightarrow \inf_{x \in \Omega_k} K_{max}(t_k, x) \geq K_{min}(t_k, p_k) - \frac{2\pi}{n(n-1)} \sqrt{C_3} (2\varepsilon K_{max}(t_k) + C_1)$ (i)

since $K_{min}(t_k, p_k) \geq (1-\varepsilon) K_{max}(t_k) - C_4$ (ii)

(i),(ii) + $\lim_{k \to \infty} K_{max}(t_k) \Rightarrow$ LHS $\geq (1-\varepsilon)^2 - \frac{4\pi}{n(n-1)} \sqrt{C_3} (1-\varepsilon)\varepsilon$. $\varepsilon \to 0$ ✓

Cor. $t_k \to T$ and $K_{max}(t_k) \geq \frac{1}{2} \sup_{t \in [0, t_k]} K_{max}(t)$ then $\liminf_{k \to \infty} \dfrac{K_{min}(t_k)}{K_{max}(t_k)} \geq 1$

pf. Otherwise set $\limsup_{k \to \infty} \dfrac{K_{min}(t_k)}{K_{max}(t_k)} < 1$ and Ω_k as above

then $\Omega_k \neq M$ (k large) and choose x_k with $d_{g(t_k)}(p_k, x_k) = \dfrac{2\pi}{\sqrt{K_{max}(t_k)}}$ (connect with γ_k)

$\Rightarrow \inf_{s \in [0,1]} K_{min}(\gamma_k(s), t_k) \leq \dfrac{\pi^2}{(L_{g(t_k)}(\gamma_k))^2} = \frac{1}{4} K_{max}(t_k)$

$\vee$

$\inf_{x \in \Omega_k} K_{min}(t_k, x)$ contradiction!

Prop. Conditions as above $\Rightarrow \dfrac{K_{min}(t)}{K_{max}(t)} \to 1 \quad (t \to T) \qquad (\circ)$

pf. Otherwise $\liminf\limits_{k\to\infty} \dfrac{K_{min}(T_k)}{K_{max}(T_k)} < 1 \qquad T_k \to T$

let $K_{max}(t_k) = \sup\limits_{t\in[0,T_k]} K_{max}(t) \Rightarrow t_k \to T \overset{Cor}{\Rightarrow} \liminf\limits_{k\to\infty} \dfrac{K_{min}(t_k)}{K_{max}(t_k)} \geq 1$

then by minimum of S is $\uparrow$ one can directly show $\liminf\limits_{k\to\infty} \dfrac{K_{min}(T_k)}{K_{max}(T_k)} \geq 1$

contradiction!

prop. $(T-t) \overset{(\inf)}{\sup\limits_M} S(t) \to \frac{n}{2} \quad (t \to T) \qquad (*)$

pf Fix $\varepsilon>0$ by above Prop $\exists \eta>0$ s.t. $|E|^2 \leq \frac{\varepsilon}{n} S^2$ on $M\times[T-\eta, T)$

also $\frac{\partial}{\partial t} S = \Delta S + 2|Ric|^2 \leq \Delta S + \frac{2(1+\varepsilon)}{n} S^2$

$\Rightarrow$ by maximum principle $\dfrac{n}{2\sup\limits_M S(\tau)} + (1+\varepsilon)(T-t) \geq \dfrac{n}{2\sup\limits_M S(t)}$

$(\forall t\in[T-\eta, T) \ \tau\in[t,T))$

then $\tau \to T \Rightarrow (1+\varepsilon)(T-t) \geq \dfrac{n}{2\sup\limits_M S(t)}$

$\Rightarrow \liminf\limits_{t\to T} (T-t) \sup\limits_M S(t) \geq \frac{n}{2}$

$\overset{Prop}{\Rightarrow} \liminf\limits_{t\to T} (T-t) \inf\limits_M S(t) \geq \frac{n}{2}$

the "$\leq$" was shown using $\frac{\partial}{\partial t} S \geq \Delta S + \frac{2}{n} S^2$ ✓

Lemma ① Fix $\alpha\in(0, \frac{1}{n-1})$ $\exists C$ s.t. $\sup\limits_M |E(t)|^2 \leq C(T-t)^{2\alpha-2} \quad \forall t\in[0,T)$

② Fix $m\geq 1$, - - - - - s.t. $\sup\limits_M |\nabla^m E(t)|^2 \leq C(T-t)^{2\alpha-m-2} \quad \forall t$

③ - - - - - - - $\sup\limits_M |\nabla^m Ric(t)|^2 \leq C(T-t)^{2\alpha-m-2} \quad \forall t$

Prop. Fix $\alpha \in (0, \frac{1}{n-1}) \Rightarrow \exists C$ s.t. $\sup_M |Rc - \frac{1}{2(T-t)} g|^2 \leq C(T-t)^{2\alpha-2}$ $(\forall t)$

Pf choose C s.t $|E|^2 \leq C_1(T-t)^{2\alpha-2}$ $|\Delta S| \leq C_2(T-t)^{\alpha-2}$

$\Rightarrow |\frac{\partial}{\partial t} S - \frac{2}{n} S^2| \leq |\Delta S| + 2|E|^2 \leq C_3(T-t)^{\alpha-2}$

and by $(*)$. $|\frac{\partial}{\partial t}(\frac{1}{S}) + \frac{2}{n}| \leq C_4(T-t)^{\alpha}$

so $|\frac{1}{S} - \frac{2}{n}(T-t)| \leq \frac{C_4}{\alpha+1}(T-t)^{\alpha+1}$

again by $(*)$ $|S - \frac{n}{2(T-t)}| \leq C_5(T-t)^{\alpha-1}$

then since $|E|^2 \leq C_1(T-t)^{2\alpha-2}$ $\Rightarrow |Rc - \frac{g}{2(T-t)}| \leq C(T-t)^{\alpha-1}$ ✓

Thm (Hamilton) $(M^{n\geq 3}, g_0)$ closed $S>0$. $F \subset \ell_B(\mathbb{R}^n)$ pinching set with $Rm(0) \in F$. the let $g(t)$ $t \in [0,T)$ be the maximal solution to RF then $\frac{1}{2(n-1)(T-t)} g(t) \xrightarrow{C^\infty} g_\infty$ with constant positive curvature

Pf. Let $\tilde{g}(t) = \frac{1}{2(n-1)(T-t)} g(t)$ $\Rightarrow \frac{\partial}{\partial t} \hat{g}(t) = \omega(t) = -\frac{1}{(n-1)(T-t)}(Ric - \frac{1}{2(T-t)} g)$

fix $\alpha \in (0, \frac{1}{n-1}) \overset{\text{prop}}{\Rightarrow} \sup_{t\in[0,T)} [(T-t)^{1-\alpha} \sup_M |\omega(t)|_{\hat{g}(t)}] < \infty$

and by lemma ③ $\sup_{t\in[0,T)} [(T-t)^{1-\alpha} \sup_M |\nabla^m \omega(t)|_{\hat{g}(t)}] < \infty$

then $\tilde{g}(t) \xrightarrow{C^\infty} g_\infty$ and by $(*)$. $S(g_\infty) = n(n-1)$

by (Δ) $K(g_\infty) \equiv 1$

2. Isotropic Curvature and Cones

Def M is said to have nonnegative isotropic curvature if $Rm(e_1, e_3, e_1, e_3) + Rm(e_1, e_4, e_1, e_4) + Rm(e_2, e_3, e_2, e_3)$
$$+ Rm(e_2, e_4, e_2, e_4) - 2Rm(e_1, e_2, e_3, e_4) \geq 0$$
$(\forall \{e_1, e_2, e_3, e_4\}$ orthonormal 4-frame$)$

Prop (Micallef, Wang) $n \geq 4$ then nonnegative isotropic curvature implies nonnegative scalar curvature.

Thm. $C = \{R \in \mathcal{C}_B(\mathbb{R}^n) : R$ has nonnegative isotropic curvature$\}$ is closed, convex, invariant under parallel translation, invariant under Hamilton's ODE

(very lengthy and boring calculation, omitted)

Cor M cpt mfld. $g(t)$ $[0, T)$ RF. If $(M, g(0))$ has nonnegative isotropic curvature, then so does $(M, g(t))$ $\forall t$.

Now we define $\hat{R} \in \mathcal{C}_B(\mathbb{R}^n \times \mathbb{R})$ by
$$\hat{R}(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4) = R(v_1, v_2, v_3, v_4)$$
where $\hat{v}_j = (v_j, y_j) \in \mathbb{R}^n \times \mathbb{R}$

Prop If $\check{R}$ has nonnegative isotropic curvature, then R has nonnegative Ricci tensor

Thm $\check{C} = \{\check{R} \in \mathcal{C}_B(\mathbb{R}^n) \mid \check{R}$ has nonnegative isotropic curvature$\}$ is closed, convex, invariant under parallel translation and invariant under Hamilton ODE

Then define $\hat{R} \in \mathcal{C}_B(\mathbb{R}^n \times \mathbb{R}^2)$ by

$$\hat{R}(\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4) = R(v_1, v_2, v_3, v_4) \quad \text{where } \hat{v}_j = (v_j, y_j) \in \mathbb{R}^n \times \mathbb{R}^2$$

Prop ① $\hat{R}$ has nonnegative isotropic curvature $\Rightarrow$ R has nonnegative sectional curvature

② If R has nonnegative curvature operator, then $\hat{R}$ has nonnegative isotropic curvature

Thm $\hat{C} = \{R \in \mathcal{C}_B(\mathbb{R}^n) \mid \hat{R}$ has nonnegative isotropic curvature$\}$ is closed, convex, invariant under parallel translation and invariant under Hamilton ODE

3 A family of Invariant Cones

Fix $a, b \geq 0$ define $l_{a,b}: \mathscr{C}_B(\mathbb{R}^n) \to \mathscr{C}_B(\mathbb{R}^n)$ by

$$l_{a,b}(R) = R + b\,\mathrm{Ric} \owedge \mathrm{Id} + \frac{1}{n}(a-b) S\, \mathrm{Id} \owedge \mathrm{Id}$$

where $\owedge$ is the Kulkarni-Nomizu product

Now we want a cone C satisfing the following conditions

(i) closed. convex and invariant under parallel translation

(ii) invariant under Hamilton ODE (#)

(iii) $\forall R \in C$ has nonnegative sectional curvature

(iv) If $R \in \mathscr{C}_B(\mathbb{R}^n)$ has nonnegative curvature operator. then $R \in C$

Clearly $\hat{C}$ above satisfies (#)

Prop. (Bohm. Wilking) ① If C satisfies (#). fix $b \in (0, \frac{1}{2}]$ and $2a = \frac{2b + (n-2)b^2}{1 + (n-2)b^2}$ $\quad \delta = 1 - \frac{1}{1 + (n-2)b^2}$

then $\{ l_{a,b}(R) \mid R \in C.\ \mathrm{Rc} \geq \frac{\delta}{n} S\, \mathrm{Id} \}$ is transversally invariant under Hamilton's ODE

② If C satisfies (*) fix $a \in (a, \infty)$ let $b = \frac{1}{2}$ $\quad \delta = 1 - \frac{4}{n - 2 + 8a}$

then $\{ l_{a,b}(R) \mid R \in C \quad \mathrm{Rc} \geq \frac{\delta}{n} S\, \mathrm{Id} \}$ is transversally invariant under Hamilton's ODE

4. Proof of the Sphere Theorem

$\forall s>0$ define $\hat{C}(s)=\{\ell_{a(s),b(s)}(R) \mid R\in\hat{C} \quad Ric \geq \frac{\delta(s)}{n} S\, Id\}$

where $2a(s)=\begin{cases} \frac{2s+(n-2)s^2}{1+(n-2)s^2} & 0<s\leq\frac{1}{2} \\ 2s & s>\frac{1}{2}\end{cases}$ $\quad 2b(s)=\begin{cases} 2s & 0<s\leq\frac{1}{2} \\ 1 & s>\frac{1}{2}\end{cases}$

$\delta(s)=\begin{cases} 1-\frac{1}{1+(n-2)s^2} & 0<s\leq\frac{1}{2} \\ 1-\frac{4}{n-2+8s} & s>\frac{1}{2}\end{cases}$

Prop (1) If $R\in\hat{C}(s)$ for some $s>0$ then $R^2+R^\#$ lies in the interior of tangent cone of $\hat{C}(s)$ at R

(2) If $R\in\hat{C}(s)$ for some $s>\frac{1}{2}$, then R is weakly $\frac{2s-1}{2s+n-1}$ pinched.

pf. (2) By def. $\forall R'\in\hat{C}(s)$ is in the form of $\ell_{s,\frac{1}{2}}(R)$ for $R\in\hat{C}$

$0\leq R(e_1,e_2,e_1,e_2)\leq\frac{1}{2}(Ric(e_1,e_1)+Ric(e_2,e_2))$

then $\ell_{s,\frac{1}{2}}(R)(e_1,e_2,e_1,e_2)\geq\frac{2s-1}{n}S$

and $\ell_{s,\frac{1}{2}}(R)(e_1,e_2,e_1,e_2)\leq Ric(e_1,e_1)+Ric(e_2,e_2)+\frac{2s-1}{n}S$

$\leq\frac{2s+n-1}{n}S$ ✓

Lemma. Fix $[\alpha,\beta]\subset(0,\infty)$ $\exists\varepsilon=\varepsilon(\alpha,\beta,n)>0$ s.t.

if $F\subset \mathcal{C}_B(\mathbb{R}^n)$ closed and ODE-invariant, with $F\subset\{R \mid R+hI\in\hat{C}(s)\}$

for some $s\in[\alpha,\beta]$ $h>0$, then $\hat{F}=\{R\in F / R+2hI\in\hat{C}(s+\varepsilon)\}$

is also ODE-invariant. And we have $\{R\in F / S(R)\leq h\}\subset\hat{F}$.

Prop $K \subset \mathcal{C}_B(\mathbb{R}^n)$ cpt. let F be the smallest closed. convex translation and ODE-invariant set containing K. If $F \subset \{R \in \mathcal{C}_B(\mathbb{R}^n) \mid R + h_0 I \in \hat{\mathcal{C}}(s_0)\}$ then F is a pinching set.

pf. $\mathcal{S} = \{s > 0 \mid F \subset \{R \in \mathcal{C}_B(\mathbb{R}^n).\ R + hI \in \hat{\mathcal{C}}(s)\},$ for some $h > 0\}$

then $s_0 \in \mathcal{S}$ Let $\sigma = \sup \mathcal{S}$ and choose $s_j \in \mathcal{S}$ $s_j \to \sigma$

then $\exists h_j$ s.t $F \subset \{R \in \mathcal{C}_B(\mathbb{R}^n) \mid R + h_j I \in \hat{\mathcal{C}}(s_j)\}$

WLOG let $h_j \geq \sup\{S(R) \mid R \in K\}$

If $\sigma < \infty$. then $\{s_j\} \subset [\alpha, \beta] \subset (0, \infty) \Rightarrow$ by lemma $\exists \varepsilon > 0$

$\hat{F}_j = \{R \in F \mid R + 2h_j I \in \hat{\mathcal{C}}(s_j + \varepsilon)\}$ is ODE-invariant with

$\{R \in F \mid S(R) \leq h_j\} \subset \hat{F}_j.$ $\overset{\text{def of } F}{\Rightarrow} F \subset \hat{F}_j\ (\forall j) \Rightarrow s_j + \varepsilon \in \mathcal{S}$ contradiction!

So $\sigma = \infty$ then by Prop(2) F is a pinching set ✓

Cor. $K \subset \mathrm{Int}(\hat{\mathcal{C}})$ cpt. then $\exists F \subset \mathcal{C}_B(\mathbb{R}^n)$ pinching set containing K

pf. Let F be as in the prop above since $K \subset \mathrm{Int}(\hat{\mathcal{C}})$

$\exists s_0 > 0$ s.t $K \subset \hat{\mathcal{C}}(s_0) \Rightarrow F \subset \hat{\mathcal{C}}(s_0)$.

then the above prop gives F is a pinching set ✓

Thm. $(M^{n\geq 4}, g_0)$ closed with $Rm(g_0) \subset Int(\hat{C})$ and $g(t)$ $(t\in[0,T))$ maximal solution to RF with $g(0)=g_0$. Then $\frac{1}{2(n-1)(T-t)} g(t) \xrightarrow{C^\infty} g_\infty$ with constant curvature 1.

(Direct result from above Cor and Hamilton's Thm in IV 1)

Then to prove the Sphere theorem, it suffices to show the following

Prop. $(M^{n\geq 4}, g)$ is strictly $\frac{1}{4}$-pinched, then $Rm(g) \subset Int(\hat{C})$

pf. $0 < K_{max}(p) < 4K_{min}(p)$ $(\forall p \in M)$

$\Rightarrow Rm(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(K_{max}(p) - K_{min}(p)) < 2K_{min}(p)$

then $\forall \{e_1, e_2, e_3, e_4\}$ orthonormal

$Rm(e_1, e_3, e_1, e_3) + \lambda^2 Rm(e_1, e_4, e_1, e_4) + \mu^2 Rm(e_2, e_3, e_2, e_3)$

$+ \lambda^2\mu^2 Rm(e_2, e_4, e_2, e_4) - 2\lambda\mu Rm(e_1, e_2, e_3, e_4)$

$> (1+\lambda^2+\mu^2+\lambda^2\mu^2 - 4\lambda\mu) K_{min}(p) = [(1-\lambda\mu)^2 + (\lambda-\mu)^2] K_{min}(p) > 0$

then $z = e_1 + i\mu e_2$ $w = e_3 + i\lambda e_4$ $Rm(z, w, \bar{z}, \bar{w}) > 0$

So $\forall \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ orthonormal in $\mathbb{R}^n \times \mathbb{R}^2$ $\hat{e}_j = (v_j, y_j)$

define $\zeta = v_1 + iv_2$ $\eta = v_3 + iv_4$

$\Rightarrow 0 < Rm(\zeta, \eta, \bar{\zeta}, \bar{\eta}) = Rm(v_1, v_3, v_1, v_3) + Rm(v_1, v_4, v_1, v_4) + Rm(v_2, v_3, v_2, v_3)$

$+ Rm(v_2, v_4, v_2, v_4) - 2Rm(v_1, v_2, v_3, v_4)$

so $Rm \in Int(\hat{C})$ ✓

V. Perelman's Entropy $\mathcal{W}$

1 Definitions, basic properties

Def ① $\mathcal{W}(g, f, \tau) = \int_M [\tau(S + |\nabla f|^2) + f - n] u \, dV$

where g is a metric, $f \in C^\infty(M)$, $\tau > 0$ is a scale parameter

$$u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$$

② (g, f, τ) is called compatible if $\int_M u \, dV = \int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dV = 1$

Log-Sobolev Ineq (proved by Gross) Let $d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$ be the Gaussian measure. then $\int v^2 \ln|v| d\mu \le \int |\nabla v|^2 d\mu + (\int v^2 d\mu) \ln(\int v^2 d\mu)^{\frac{1}{2}}$

for any v and $|\nabla v| \in L^2(d\mu)$

Prop. ① $\mathcal{W}(g, f, \tau) = \mathcal{W}(\tau^{-1} g, f, 1)$

② g: flat metric on $\mathbb{R}^n$, (g, f, τ) compatible, then

$\mathcal{W}(g, f, \tau) \ge 0$ "$=$" $\Leftrightarrow f(x) = \frac{|x|^2}{4\tau}$

pf. ① Trivial. ② Let $\tau = \frac{1}{2}$, then $\int_{\mathbb{R}^n} \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}} dx = 1$

$v = e^{\frac{|x|^2}{4} - \frac{f}{2}} \Rightarrow v^2 d\mu = (2\pi)^{-\frac{n}{2}} e^{-f} dx$ so $\int_{\mathbb{R}^n} v^2 d\mu = 1$

$$\overset{\text{log-Sobolev}}{\Rightarrow} \int v^2 \ln|v| d\mu \le \int |\nabla v|^2 d\mu = \int \left(\frac{|x|^2}{4} - \frac{x \cdot \nabla f}{2} + \frac{|\nabla f|^2}{4}\right) \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}} dx$$

$$\int v^2 \ln|v| d\mu = \int \left(\frac{|x|^2}{4} - \frac{f}{2}\right) \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}} dx$$

$$\overset{\nabla x = n}{=} \int \left(\frac{|x|^2}{4} - \frac{n}{2} + \frac{|\nabla f|^2}{4}\right) \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}} dx$$

$\Rightarrow \mathcal{W}(g, f, \frac{1}{2}) \ge 0$. ✓

Lemma $\forall (M,g)$ closed. $\tau>0$ $\mu(g,\tau)=\inf\{W(g,f,\tau) \mid f \text{ compatible}\}$

then $\mu(g,\tau)$ can be attained by a smooth compatible f and

bounded below on any finite interval $\tau\in(0,\tau_0]$.

hence $\nu(g,\tau_0)=\inf\limits_{\tau\in(0,\tau_0]}\mu(g,\tau)>-\infty$.

pf. Let $\phi=e^{-\frac{f}{2}} \Rightarrow \underset{\mathcal{W}(g,\phi,\tau)}{\underset{\shortparallel}{W(g,f,\tau)}}=(4\pi\tau)^{-\frac{n}{2}}\int [\tau(4|\nabla\phi|^2+S\phi^2)-2\phi^2\ln\phi-n\phi^2]dV$ and $\int\phi^2 dV=(4\pi\tau)^{-\frac{n}{2}}$

$\inf\limits_{f} W(g,f,\tau)=\inf\limits_{\phi: M\to(0,\infty)} \mathcal{W}(g,\phi,\tau) \Rightarrow$ only need to discuss ϕ (WLOG $\tau=\frac{1}{4\pi}$)

It suffices to show $\mathcal{W}(\phi)$ bounded below (then by taking subsequence one can show the infimum is attained)

So we only need $\mathcal{W}(\phi)=\int \frac{1}{4\pi}(4|\nabla\phi|^2+S\phi^2)-2\phi^2\ln\phi-n\phi^2 dV$

$$\geq \frac{1}{2\pi}\int|\nabla\phi|^2 dV-C(g,n) \quad (\forall \int\phi^2 dV=1)$$

(i) $\int\frac{1}{4\pi}S\phi^2-n\phi^2\geq -C(g,n)$ since S is bounded

(ii) $\int\phi^2\ln\phi\, dV=\frac{n-2}{4}\int\phi^2\ln\phi^{\frac{4}{n-2}}dV \overset{\text{Jensen}}{\leq} \frac{n-2}{4}\ln[\int\phi^{2+\frac{4}{n-2}}dV]$

$$=\frac{n}{4}\ln[\int\phi^{\frac{2n}{n-2}}dV]^{\frac{n-2}{n}} \overset{\text{Sobolev}}{\leq} \frac{n}{4}\ln[C(g,n)(\int\phi^2 dV+\int|\nabla\phi|^2 dV)]$$

$\Rightarrow \int\phi^2\ln\phi\, dV \leq C(g,n)+\frac{n}{4}\ln[1+\int|\nabla\phi|^2 dV]$

$$\leq C(g,n)+\frac{1}{2\pi}\int|\nabla\phi|^2 dV \quad \checkmark$$

2 Monotonicity of $\mathcal{W}$

Prop. M closed, let g, f, τ be $\begin{cases} \frac{\partial g}{\partial t} = -2Ric \\ \frac{d\tau}{dt} = -1 \\ \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - S + \frac{n}{2\tau} \end{cases}$

then $\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int |Rc + Hess(f) - \frac{g}{2\tau}|^2 u\, dV \geq 0$

Rmk. ① Let $\square^* = -\frac{\partial}{\partial t} - \Delta + S$. then recall $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$

$\square^* u = -2\pi n (4\pi\tau)^{-\frac{n}{2}-1} e^{-f} + \frac{\partial f}{\partial t} e^{-f} (4\pi\tau)^{-\frac{n}{2}} - (4\pi\tau)^{-\frac{n}{2}} (|\nabla f|^2 - \Delta f) e^{-f}$
$+ S (4\pi\tau)^{-\frac{n}{2}} e^{-f}$

$= e^{-f} (4\pi\tau)^{-\frac{n}{2}} \left(-\frac{n}{2\tau} - \Delta f + |\nabla f|^2 - S + \frac{n}{2\tau} - |\nabla f|^2 + \Delta f - S \right) = 0$

$\Rightarrow \frac{d}{dt} \int u\, dV = \int \frac{\partial u}{\partial t} - S u\, dV = -\int \square^* u\, dV = 0$

so the compatibility condition is preserved.

② If $\frac{d\mathcal{W}}{dt} = 0 \Rightarrow Rc + Hess(f) = \frac{g}{2\tau}$: shrinking gradient soliton

Prop. g, f, τ as above. let $v = [\tau(2\Delta f - |\nabla f|^2 + S) + f - n] u$

then $\square^* v = -2\tau |Rc + Hess f - \frac{g}{2\tau}|^2 u$

(then $\mathcal{W} = \int_M v\, dV$ $\quad \frac{d\mathcal{W}}{dt} = -\int_M \square^* v\, dV$ ✓)

Pf. $\square^* v = \square^* (\frac{v}{u} u) = -\frac{\partial}{\partial t}(\frac{v}{u}) u - \frac{v}{u} \frac{\partial}{\partial t} u - \Delta(\frac{v}{u}) u - \frac{v}{u} \Delta u - 2\langle \nabla(\frac{v}{u}), \nabla u \rangle$
$+ S \frac{v}{u} u$

$= \underbrace{\frac{v}{u} \square^* u}_{0} - u (\frac{\partial}{\partial t} + \Delta) \frac{v}{u} - 2\langle \nabla(\frac{v}{u}), \nabla u \rangle$

$\Rightarrow u^{-1} \square^* v = -(\frac{\partial}{\partial t} + \Delta) \frac{v}{u} - \frac{2}{u} \langle \nabla(\frac{v}{u}), \nabla u \rangle$

$$-\left(\frac{\partial}{\partial t}+\Delta\right)\frac{v}{u} = -\left(\frac{\partial}{\partial t}+\Delta\right)\left[\tau(2\Delta f-|\nabla f|^2+S)+f-n\right]$$

$$= 2\Delta f-|\nabla f|^2+S-\tau\left(\frac{\partial}{\partial t}+\Delta\right)(2\Delta f-|\nabla f|^2+S)\underbrace{-\left(\frac{\partial}{\partial t}+\Delta\right)f}_{\| \; S-|\nabla f|^2-\frac{n}{2\tau}}$$

$$= 2\Delta f-|\nabla f|^2+2S-\frac{n}{2\tau}-\tau\left(\frac{\partial}{\partial t}+\Delta\right)(2\Delta f-|\nabla f|^2+S)$$

$$\left(\frac{\partial}{\partial t}+\Delta\right)\Delta f = \Delta\frac{\partial f}{\partial t}+2\langle Ric, Hess(f)\rangle+\Delta^2 f$$

$$= 2\langle Ric, Hess(f)\rangle+\Delta\left(|\nabla f|^2-S+\frac{n}{2\tau}\right)$$

$$\left(\frac{\partial}{\partial t}+\Delta\right)S = 2\Delta S+2|Ric|^2$$

$$\left(\frac{\partial}{\partial t}+\Delta\right)|\nabla f|^2 = -2Ric(\nabla f,\nabla f)+2\langle\nabla f, \nabla\frac{\partial f}{\partial t}\rangle+\Delta|\nabla f|^2$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\Delta\right)(2\Delta f-|\nabla f|^2+S) = 4\langle Ric, Hess(f)\rangle+\Delta|\nabla f|^2-2Ric(\nabla f,\nabla f)$$
$$-2\langle\nabla f, \nabla(-\Delta f+|\nabla f|^2-S)\rangle+2|Ric|^2$$

also $-\frac{2}{u}\langle\nabla\frac{v}{u},\nabla u\rangle = 2\tau\langle\nabla(2\Delta f-|\nabla f|^2+S),\nabla f\rangle+2|\nabla f|^2$

$$\Rightarrow u^{-1}\Box^* v = -\left(\frac{\partial}{\partial t}+\Delta\right)\frac{v}{u}-\frac{2}{u}\langle\nabla\frac{v}{u},\nabla u\rangle$$

$$= 2\Delta f-|\nabla f|^2+2S-\frac{n}{2\tau}-4\tau\langle Ric, Hess(f)\rangle-\tau\Delta|\nabla f|^2$$
$$+2\tau Ric(\nabla f,\nabla f)+2\tau\langle\nabla(-\Delta f+|\nabla f|^2-S),\nabla f\rangle$$
$$-2\tau|Ric|^2+2\tau\langle\nabla(2\Delta f-|\nabla f|^2+S),\nabla f\rangle+2|\nabla f|^2$$

$$= -\frac{n}{2\tau}+2\Delta f+2S+\tau\Big(-4\langle Ric, Hess(f)\rangle-2|Ric|^2$$
$$+\big[\underbrace{-\Delta|\nabla f|^2+2Ric(\nabla f,\nabla f)+2\langle\nabla f,\nabla\Delta f\rangle}_{\| \text{ Bochner} \; -2|Hess(f)|^2}\big]\Big)$$

$$= -\frac{n}{2\tau}+2\langle Ric+Hess(f), g\rangle-2\tau|Ric+Hess(f)|^2$$

$$= -2\tau\left|Ric+Hess(f)-\frac{g}{2\tau}\right|^2$$

3. No Local Collapsing

Let $K(p,r) = \frac{V(p,r)}{r^n} = \frac{Vol(B(p,r))}{r^n}$ volume ratio. we show the following theorem giving lower bound on volume ratio (no-collapsing)

Thm. $(M^n, g(t))$ closed $t \in [0,T]$ RF. then w.r.t. $g(T)$.

if $p \in M$. $r > 0$ small enough so $|S| \le \frac{1}{r^2}$ on $B(p,r)$

then $K(p,r) > \xi$ for some $\xi = \xi(n, g(0), r, T) > 0$

(Rmk. In Perelman's paper the condition was strengthed to $|Rm| \le \frac{1}{r^2}$)

Lemma $\forall$ closed mfld (M,g). $p \in M$. $r > 0$. $\lambda > 0$

$$\mu(g, \lambda r^2) \le 36\lambda \frac{V(p,r) - V(p,\frac{r}{2})}{V(p,\frac{r}{2})} + \frac{\lambda r^2}{V(p,\frac{r}{2})} \int_{B(p,r)} |S| dV + \ln\left[\frac{V(p,r)}{(4\pi\lambda r^2)^{\frac{n}{2}}}\right] - n$$

pf. Let $\phi = e^{-\frac{f}{2}}$. $\tau = \lambda r^2$. then again

$$\mathcal{W}(g, \phi, \lambda r^2) = (4\pi\lambda r^2)^{-\frac{n}{2}} \int [\lambda r^2(4|\nabla\phi|^2 + S\phi^2) - 2\phi^2 \ln\phi - n\phi^2] dV$$

compatibility $\Leftrightarrow (4\pi\lambda r^2)^{-\frac{n}{2}} \int \phi^2 dV = 1$ (*)

Choose $\psi : [0,\infty) \to [0,1]$ with $\psi \equiv 1$ on $[0,\frac{1}{2}]$, $|\psi'| \le 3$

$C_c^\infty([0,1])$

and $\phi(x) = e^{-\frac{c}{2}} \psi\left(\frac{d(x,p)}{r}\right)$ (c is chosen to satisfy (*))

$$\Rightarrow V(p,\frac{r}{2}) \le e^c \int \phi^2 dV \le V(p,r)$$

$$\Rightarrow (4\pi\lambda r^2)^{-\frac{n}{2}} V(p,\frac{r}{2}) \le e^c \le (4\pi\lambda r^2)^{-\frac{n}{2}} V(p,r)$$

Now $|\nabla\phi| \le e^{-\frac{c}{2}}\frac{1}{r}\sup|\psi'| \le \frac{3}{r}e^{-\frac{c}{2}}$ and $\nabla\phi$ supported in $B(p,r)\setminus B(p,\frac{r}{2})$

so ① $(4\pi\lambda r^2)^{-\frac{n}{2}}\int \lambda r^2 4|\nabla\phi|^2 dV \le 4\lambda r^2 (4\pi\lambda r^2)^{-\frac{n}{2}} \sup|\nabla\phi|^2 (V(p,r)-V(p,\frac{r}{2}))$

$$\le 36\lambda(4\pi r^2)^{-\frac{n}{2}} e^{-c}(V(p,r)-V(p,\frac{r}{2})) \le 36\lambda\,\frac{V(p,r)-V(p,\frac{r}{2})}{V(p,\frac{r}{2})}$$

② $\phi^2 \le e^{-c} \le \frac{(4\pi\lambda r^2)^{\frac{n}{2}}}{V(p,\frac{r}{2})}$ supported in $B(p,r)$

$$\Rightarrow (4\pi\lambda r^2)^{-\frac{n}{2}}\int \lambda r^2 S\phi^2 dV \le \frac{\lambda r^2}{V(p,\frac{r}{2})}\int_{B(p,r)}|S|dV$$

③ $(4\pi\lambda r^2)^{-\frac{n}{2}}\int -2\phi^2\ln\phi\, dV = \int G(\sigma)d\mu$

where $G(y) = -y\ln y$ $\sigma=\phi^2$ $d\mu = (4\pi\lambda r^2)^{-\frac{n}{2}}dV$ $\Rightarrow \int\sigma d\mu = 1$

$\overset{\text{Jensen}}{\Rightarrow} \int G(\sigma)d\mu \le \int d\mu\, G(\frac{1}{\int d\mu}) = \ln(\int d\mu)$

$$\Rightarrow (4\pi\lambda r^2)^{-\frac{n}{2}}\int -2\phi^2\ln\phi\, dV \le \ln\left[\frac{V(p,r)}{(4\pi\lambda r^2)^{\frac{n}{2}}}\right]$$

④ $(4\pi\lambda r^2)^{-\frac{n}{2}}\int -n\phi^2 dV = -n$ $\Rightarrow$ lemma. ✓

By lemma $\mu(g(T),\frac{r^2}{36}) \le \frac{V(p,r)}{V(p,\frac{r}{2})} + \frac{r^2}{V(p,\frac{r}{2})}\int_{B(p,r)}|S|dV + \ln K(p,r) - n - \frac{n}{2}\ln\frac{\pi}{9}$

choose f_T compatible s.t. $\mu(g(T),\frac{r^2}{36}) = w(g(T),f_T,\frac{r^2}{36})$

Set $J = T+\frac{r^2}{36}-t$ and $f(t)$ s.t. $f(T)=f_T$

$$\Rightarrow \mu(g(0),\frac{r^2}{36}+T) \le w(g(0),f(0),\frac{1}{36}r^2+T)$$

$$\overset{\text{monotonicity}}{\le} w(g(T),f(T),\frac{1}{36}r^2) = \mu(g(T),\frac{r^2}{36})$$

then for r_0, T_0 upper bounds of r, T. we have

$$\nu(g(0),\frac{r_0^2}{36}+T_0) + \frac{n}{2}\ln\frac{\pi}{9} + n \le \frac{V(p,r)}{V(p,\frac{r}{2})} + \frac{r^2}{V(p,\frac{r}{2})}\int_{B(p,r)}|S|dV + |\ln|K(p,r)||$$

Back to the Thm, now $\exists\, \gamma = \gamma(n, g(0), T_0, r_0)$ s.t. $\forall s \in (0, r_0]$

$$\gamma \le \frac{V(p,s)}{V(p,\frac{s}{2})} + \frac{s^2}{V(p,\frac{s}{2})}\int_{B(p,s)} |S|\,dV + \ln|K(p,s)| \quad \text{w.r.t. } g(T)$$

since $|S| \le \frac{1}{s^2}$ on $B(p,s)$ $\quad (s \in (0, r])$

$$\Rightarrow \gamma \le \frac{2V(p,s)}{V(p,\frac{s}{2})} + \ln|K(p,s)|$$

ω_n: $Vol(B)$ in $\mathbb{R}^n$ then $K(p,s) \to \omega_n$ as $s \to 0$

Let $\xi = \min\{\frac{\omega_n}{2}, e^{\gamma - 2^{n+1}}\}$, $s \in (0, r]$

if $K(p,s) \le \xi$, then $K(p,s) \le e^{\gamma - 2^{n+1}}$

$$\Rightarrow \gamma \le \frac{2V(p,s)}{V(p,\frac{s}{2})} + \gamma - 2^{n+1} \quad \Rightarrow 2^n \le \frac{K(p,s)}{K(p,\frac{s}{2})}\, 2^n$$

$$\Rightarrow K(p, \tfrac{s}{2}) \le \xi$$

so $K(p, 2^{-m} r) \le \xi \le \frac{\omega_n}{2}$ but $K(p,s) \to \omega_n$ $(s \to 0)$ contradiction!

The no-collapsing also gives us a bound on injectivity radius through the following lemma

Lem. $\exists\, \bar{r} > 0$, $\eta > 0$ s.t. if (M^n, g) closed Riem mfld with $|Rm| \le 1$

then $\exists\, p \in M$ s.t $\frac{V(p,r)}{r^n} \le \frac{\eta}{r}\, inj(M)$ $(\forall r \in (0, \bar{r}])$

(see Topping §.4 for proof)

VI. Solitons and special Solutions

We examine the special solutions more carefully.

1. Gradient Soliton

$(M^n, g_0, f_0, \varepsilon)$ $\quad Rc(g_0) + \nabla^{g_0}\nabla^{g_0} f_0 + \frac{\varepsilon}{2} g_0 = 0$

Call it complete if (M^n, g_0) and $\nabla^{g_0} f_0$ complete

Thm. $(M^n, g_0, f_0, \varepsilon)$ complete gradient soliton and let

$T(t) = 1 + \varepsilon t > 0$ $\quad \varphi(t)$ generated by $\frac{1}{T(t)} \nabla^{g_0} f_0$

$g(t) = T(t)\varphi^*(t) g_0$ $\quad f(t) = \varphi(t)^* f_0$

then $\quad Rc(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f(t) + \frac{\varepsilon}{2T} g(t) = 0$

$\frac{\partial f}{\partial t}(t) = |\nabla^{g(t)} f(t)|^2_{g(t)}$

$(M^n, g(t))$ is a solution to RF.

Rmk. $\begin{cases} \varepsilon = 0 \Rightarrow \text{eternal. steady} \\ \varepsilon > 0 \Rightarrow \text{immortal. expanding} \\ \varepsilon < 0 \Rightarrow \text{ancient. shrinking} \end{cases}$

Ex ① let $T(t) = 1 + \varepsilon t$ $\qquad \varphi(t) = (T(t))^{-\frac{1}{2}} Id_{\mathbb{R}^n}$

Then $g(t) = T(t)\varphi^*(t) g(0) = g(0) = g_{can}$ on $\mathbb{R}^n$

then $(\mathbb{R}^n, g_{can})$ can be regarded as $\begin{cases} \text{steady} \\ \text{expanding} \\ \text{shrinking} \end{cases}$ gradient Soliton

here $f(x,t) = -\frac{\varepsilon |x|^2}{4T(t)}$

Gaussian Soliton

② $(S^{n-1}\times\mathbb{R}, g(t))$ $t\in(-\infty,0)$ $n\geq 3$ $\quad g(t)=2(n-2)|t|\, g_{S^{n-1}} + dr^2$

$$Rc(g(t)) = (n-2)\, g_{S^{n-1}} = \frac{1}{2|t|} g(t) - \frac{1}{2|t|} dr^2$$

then let $f(\theta,r,t)=\frac{r^2}{4|t|}$ $\Rightarrow Rc(g(t))+\nabla\nabla f(t)+\frac{1}{2t}g(t)=0$

shrinking gradient soliton (cylinder)

③ On $(\mathbb{R}^2, g_\Sigma)$. $g_\Sigma = \frac{dx^2+dy^2}{1+x^2+y^2}$ $\quad T(t)\equiv 1$ $(\varepsilon=0)$

$\varphi_t(x,y)=(e^{-2t}x, e^{-2t}y) \rightarrow g_\Sigma(t)=\varphi_t^* g_\Sigma(0) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}$

steady cigar soliton

the scalar curvature $S_\Sigma = \frac{4}{1+x^2+y^2}$

on $\mathbb{R}^2 - 0 = \mathbb{R}\times S^1$ (x,θ) $\quad g_{\mathbb{R}^2-0} = \frac{1}{1+e^{-2x}}(dx^2+d\theta^2)$

↳ cigar metric on cylinder (incomplete)

Prop. i) $(\mathbb{R}^2, g(t))$ is a steady gradient soliton conformal to g_{can},

then it's either cigar soliton or flat metric

ii) $(M^2, g(t))$ is a steady gradient soliton with $K>0$.

then it's cigar soliton

(Rosenau Soliton)

$\to$ flat cylinder

④ $\mathbb{R}\times S^1(2)$ (x,θ) $\quad h=dx^2+d\theta^2$ $\qquad g(t)=\frac{\sinh(-t)}{\cosh x+\cosh t}\, h$ $\quad (t<0)$

} compactification

$\mathbb{R}^2$

$t\to 0 \longrightarrow$ sphere with radius $\sqrt{2}$

dilating about pole $\rightarrow$ cigar

equator $\rightarrow$ flat cylinder

2. An Expanding Soliton

On $\mathbb{R}^2$, define $g(t) = t\left(F(r)^2 dr^2 + r^2 d\theta^2\right)$

$r \in (0, \infty)$
$\theta \in \mathbb{R}/(2\pi F(0))$
$F : [0, \infty) \to [0, \infty)$

$$K[g(t)] = \frac{1}{t}\frac{F'(r)}{rF(r)^3}$$

Let $X(t) = \frac{r}{tF(r)}\frac{\partial}{\partial r}$ and F satisfies $rF(r)^3\left(1 - \frac{F(r)}{2}\right) = F'(r)$ $(*)$

then one can check $\frac{\partial}{\partial t} g(t) = -S_{g(t)} g(t) + \mathcal{L}_{X(t)} g(t)$

now for diffeo $\psi(t)$ s.t. $\frac{d}{dt}\Big|_{t=t_0} \left(\psi(t) \circ \psi^{-1}(t_0)\right) = -\psi(t_0)^* X(t_0)$

the metrics $\psi(t)^* g(t)$ satisfy Ricci flow.

$(*)$ gives $F(r) = \dfrac{2}{1 + W\left[\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right]}$

then cone angle at infinity: $\dfrac{2\pi F(0)}{F(+\infty)} = \pi F(0) \in (0, 2\pi)$

Prop. $g(t) \longrightarrow$ flat cone $(t \to \infty)$ $\quad K(t) \longrightarrow d(x,0)$ exponentially

3. Brayant Soliton

On $S^{n-1} \times (0, \infty)$. $\quad g = dr^2 + \phi(r)^2 g_{S^{n-1}}$

$$\Rightarrow Rc(g) = -(n-1)\frac{\phi''}{\phi}dr^2 + \left((n-2)(1 - \phi'^2) - \phi\phi''\right) g_{S^{n-1}}$$

$$\nabla\nabla f = f''(r)dr^2 + \phi\phi' f' g_{S^{n-1}}$$

$$\Rightarrow R(g) + \nabla\nabla f = 0 \quad \Leftrightarrow \quad \begin{cases} f'' = (n-1)\dfrac{\phi''}{\phi} \\ \phi\phi' f' = -(n-2)(1 - \phi'^2) + \phi\phi'' \end{cases}$$

substitute $\begin{cases} x = \phi' \\ y = \phi f' + (n-1)\phi' \\ dt = \dfrac{dr}{\phi} \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = x(x - y) + n - 2 \\ \dfrac{dy}{dt} = x(y - (n-1)x) \end{cases}$

Take a solution $\exists t$ $(x(t), y(t)) \to (0,\infty)$ $(t\to\infty)$ $\longrightarrow$ Bryant Soliton

Thm (Bryant) $\forall n\geq 3$ $\exists!$ steady gradient soliton on $\mathbb{R}^n$ with positive curvature operator. In this case the sectional curvature

$K_{rad} = O(r^{-2})$. $\qquad K_{sph} = O(r^{-1})$

and $V(B(0,s)) = \omega_n \int_0^s \varphi(r)^{n-1} dr \simeq s^{\frac{n+1}{2}}$ for $s\geq 1$

3 Geometry at Spatial Infinity

Asymptotic scalar curvature ratio (AScR)

$$\mathrm{AScR}(g) = \limsup_{d(x,O)\to\infty} S(x)\, d(x,O)^2 \quad \text{(independent of } O)$$

Asymptotic volume ratio

$$\mathrm{AVR}(g) = \lim_{r\to\infty} \frac{\mathrm{Vol}(B(O,r))}{r^n}$$

Thm. $(M^n, g(t))$ ancient solution with bounded nonnegative curvature operator. then $\mathrm{AScR}(g(t))$ is independent of t.

Thm. $(M^{n\geq 3}, g_0, f_0)$ complete steady soliton, if $K(g_0)>0$ and S attains maximum at $O\in M$. then $\mathrm{AScR}(g_0)=\infty$.

pf. By condition, $\exists$ $\frac{\partial}{\partial t}\varphi_t = (\nabla_{g_0} f_0)\varphi_t$ $\quad g(t) = \varphi_t^* g_0$

$$f(t) = f(\varphi_t) \qquad Rc(g(t)) + \mathrm{Hess}_{g(t)} f(t) = 0$$

$$\frac{\partial f}{\partial t} = |\nabla f|^2 = \Delta f$$

If $ASCR(g(t)) = ASCR(g_0) < \infty \Rightarrow \lim_{d_t(x,0)\to\infty} S(x,t) = 0$

since $S + |\nabla f|^2 = S(0) \Rightarrow \lim_{d(x,0)\to\infty} |\nabla f(x,t)| = \sqrt{S(0)} > 0$

also $0 \le -\frac{\partial}{\partial t} g(x,t) = 2Ric(x,t) \le 2S(x,t)\, g(x,t)$

$$\le \frac{C}{1+d_t(x,0)^2}\, g(t) = \frac{C}{1+d_0(\varphi_t(x),0)^2}\, g(t)$$

$\forall \varepsilon > 0.\ \exists c$ s.t. $d_{g_0}(\varphi_t(x), 0) \ge c|t|$ for any $x \in M - B_{g_0}(0,\varepsilon)$

then $0 \le -\frac{\partial}{\partial t} g(x,t) = \frac{C}{1+|t|^2}\, g(x,t)$

$\Rightarrow \forall \varepsilon > 0.\ \exists C$ s.t. $C g(x,0) \ge g(x,t_1) \ge g(x,t_2) \ge g(x,0)$ $\quad \forall x \in M - B_{g_0}(0,\varepsilon)$, $-\infty < t_1 \le t_2 \le 0$

$\Rightarrow$ pointwise limit $\lim_{t\to-\infty} g(x,t) = g_\infty$ exists on $M - \{0\}$

Also. $\forall x \in M - \{0\}$ $r < d_t(x,0)$. $(B_t(x,r), g(t))$ is complete

and $\forall x \in M - \{0\}$ $\quad \lim_{t\to-\infty} d_t(x,0) = \lim_{t\to-\infty} d_0(\varphi_t(x), 0) = \infty$

$\Rightarrow (M^n - \{0\}, g_\infty)$ is complete.

$S_\infty(x) = \lim_{t\to-\infty} S(x,t) = \lim_{t\to-\infty} S_0(\varphi_t(x)) = 0 \quad (\forall x \in M - \{0\})$

But $K(t) > 0 \Rightarrow K(g_\infty) \ge 0 \Rightarrow g_\infty$ is flat and complete

However. $Rc(g_0) > 0 \Rightarrow M^n \cong \mathbb{R}^n \Rightarrow M - \{0\} \cong S^{n-1} \times \mathbb{R}$ $\uparrow$ contradiction!

Lemma If $(M^n, g(t))$ $t \in [0,T)$ RF with bounded curvature and $\lim_{d_0(x,0)\to\infty} |Rm(x,0)| = 0$, then $\lim_{d_t(x,0)\to\infty} |Rm(x,t)| = 0$ $(\forall t \ge 0)$

Prop. $(M^n, g(t))$ RF with bounded nonnegative Ricci curvature. and $\lim_{d_t(x,0)\to\infty} |Rm(x,t)| = 0$ $(\forall t)$. then $AVR(g(t))$ is independent of t.

(See Hamilton's «The formation of singularities in Ricci flow»)