QUANTITIES FROM S^1-EQUIVARIANT RABINOWITZ FLOER HOMOLOGY

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Abstract. In this paper we develop the quantitative theory of S^1-equivariant Rabinowitz Floer homology. We will define spectral invariant on the universal covering of contactomorphism group of Liouville fillable contact manifolds and give a new criterion of orderability on such manifolds. We will also construct integral-valued contact capacity on periodic contact manifolds and obtain some contact non-squeezing type results.

Contents

1. Introduction and Results

Throughout the introduction we assume that (Σ, ξ) fits condition (H):

Assumption (H): (Σ, ξ) admits a Liouville filling $(X, d\lambda)$ so that the S^1-equivariant Rabinowitz Floer homology $\mathrm{RFH}_*^{S^1}(\Sigma, X)$ is non-zero.

The precise definitions involved in the above condition will be presented in Section 2.

Date: April 18, 2026.

The purpose of this article is to define some basics objects in quantitative contact geometry from the S^1-equivariant Rabinowitz homology. We construct a contact spectral invariant first.

Theorem 1.1. *For (Σ,ξ) satisfying Assumption (H), for any $0\neq\theta\in \mathrm{RFH}_*^{S^1}(\Sigma,X)$ there is a map $c(\cdot,\theta):\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)\to\mathbb{R}$ that is C^2-continuous, so that*

1. *$c(\cdot,\theta)$ descends to a map $c(\cdot,\theta):\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)\to\mathbb{R}$,*
2. *If $\varphi,\psi\in\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$ with contact Hamiltonians h_φ,h_ψ respectively satisfies $h_\varphi\geq h_\psi$, then $c(\varphi,\theta)\leq c(\psi,\theta)$,*
3. *Let ϕ_t^α be the Reeb flow of a 1-form α defining ξ, then for any $m\in\mathbb{R}$, let φ_α^m be the path given by*
$$t\mapsto\phi_{mt}^\alpha,$$
one has
$$c(\varphi_\alpha^m,\theta)=c(\mathrm{Id}_\Sigma,\theta)-m,$$

The above spectral invariant will be constructed in Section 3. We observe that this quantity has connection with the orderability question in contact geometry.

Remark 1.2. [AFM15] is the first to adapt Rabinowitz Floer homology as a tool to study orderability. Since the Rabinowitz Floer homology is closely related to the translated point of a contactomorphism, this method can be seen as a special route from translated point to orderability, which is discovered first by [San11].

Proposition 1.3. *For (Σ,ξ) satisfying Assumption (H) then $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is orderable.*

Proof. If $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is not orderable, then there is a contractible loop φ of contactomorphism with contact Hamiltonian $h_t>0$. Then since φ is contractible, one has $c(\varphi,\theta)=c(\mathrm{Id}_\Sigma,\theta)$.

On the other hand since Σ is closed there exists $\delta>0$ so that $h_t\geq\delta$, then by Theorem 1.1(3) one has
$$c(\varphi,\theta)\leq-\delta+c(\mathrm{Id}_\Sigma,\theta)<c(\mathrm{Id}_\Sigma,\theta),$$
which leads to a contradiction. □

The above can be rewritten into the following.

Corollary 1.4. *If (Σ,ξ) is a closed contact manifold with $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ not orderable, then $\mathrm{RFH}_*^{S^1}(\Sigma,X)$ vanishes for all Liouville filling X of (Σ,ξ).*

Combining this with the known orderability result, one can derive some examples of the contact manifolds with vanishing S^1-equivariant Rabinowitz Floer homology.

Example 1.5. The standard contact structure on S^{2n+1} is not orderable, so the vanishing of S^1-equivariant Rabinowitz Floer homology of the sphere follows as a consequence of Corollary 1.4.

The second quantities is the contact capacity. In the rest of this section we assume that $\alpha = \lambda|_\Sigma$ is periodic, that is, the Reeb flow of α is 1-periodic. Our main result is the construction of a contact capacity that is invariant under contact transformation.

Theorem 1.6. *For $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ we define $c_{\mathbb{Z}}(\varphi, \theta) = \lceil c(\varphi, \theta) \rceil$ and*

$$c(U, \theta) = \sup\{c_{\mathbb{Z}}(\varphi, \theta) \mid \mathrm{Supp}(\varphi) \subseteq U\} \in \mathbb{Z} \cup \{\infty\}. \tag{1.1}$$

For any $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ one has

$$c(\psi(U), \theta) = c(U, \theta).$$

This result will be proved in Proposition 4.5. The key step is to show that the integral spectral invariant $c_{\mathbb{Z}}(\cdot, \theta)$ enjoys a conjugation invariance property, see Corollary 4.3.

Having constructed the invariant capacity, we can obtain a contact non-squeezing type result.

Theorem 1.7. *(Corollary 1.7) If $U \subseteq V \subseteq \Sigma$ with $c(U, \theta) < c(V, \theta)$ there is no contact isotopy mapping V into U.*

Acknowledgements. To be added.

2. Rabinowitz Floer Theory

2.1. **Introduction and non-equivariant Rabinowitz Floer homology.** We recall the construction of non-equivariant Rabinowitz Floer theory and lay out some of its applications.

Let (Σ, ξ) be a connected coorientable closed contact manifold. To define Rabinowitz Floer theory, we require this manifold to be Liouville fillable.

Definition 2.1. A **Liouville domain** (W, λ_W) is a compact exact symplectic manifold so that $\lambda_W|_{\partial W}$ is a positive contact form on ∂W.

We say a connected coorientable closed contact manifold (Σ, ξ) is **Liouville fillable** if there is a Liouville domain (W, λ_W) so that $\Sigma = \partial W$ and $\alpha = (\lambda_W)_\Sigma$ is a positive contact form on Σ satisfying $\ker \alpha = \xi$.

Given a Liouville domain, one can complete it into a Liouville manifold. By the definition of Liouville domain, the Liouville vector field Z defined by

$$\iota_Z \lambda_W = d\lambda_W$$

is guaranteed to be transverse to ∂W, pointing outwards. Let ϕ_Z^t be the flow of Z defined for $t \leq 0$, then we define the "completion" as

$$X := W \cup_{\partial W} (\partial W \times [1, \infty)),$$

with an embedding $\partial W \times (0, 1] \to W$ defined by $(x, r) \mapsto \phi_Z^{\log r}(x)$.

λ_W can be extended to a 1-form on X by defining $\lambda = r(\lambda_W)|_{\partial W}$ on $\partial W \times [1, \infty)$. Then $(X, d\lambda)$ is a **Liouville manifold** and called the filling of Σ.

The **symplectization** $S\Sigma$ of a contact manifold $(\Sigma, \xi = \ker, \alpha)$ is the symplectic manifold $\Sigma \times (0, \infty)$ equipped with the symplectic form $d(r\alpha)$. If Σ is Liouville fillable with filling $(X, d\lambda)$ then one can embed $S\Sigma \hookrightarrow X$ by using the flow of the Liouville vector field.

Let $\mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ be the set of smoothly parametrized paths $\{\varphi_t\}_{0 \leq t \leq 1}$ with $\varphi_0 = \mathrm{Id}_\Sigma$. For a path $\varphi = \{\varphi_t\}_{0 \leq t \leq 1}$, let $\varphi_t^* \alpha = \rho_t \varphi_t$, then define $\Phi_t : S\Sigma \to S\Sigma$ by

$$\Phi_t(x, r) := \left(\varphi_t(x), \frac{r}{\rho_t(x)} \right).$$

Let W be the infinitesimal generator defined by

$$W(x) := \frac{\partial}{\partial t}\Big|_{t=0} \varphi_t(x)$$

and let

$$h_t(x) = \alpha_{\varphi_t(x)} W(\varphi_t(x)).$$

Then the path Φ_t is Hamiltonian with Hamiltonian function

$$H_t(x, r) := r h_t(x) : S\Sigma \to \mathbb{R}.$$

We define $F_0 : S\Sigma \to \mathbb{R}$ by $F_0(x, r) := f(r)$, where

$$f(r) := \frac{1}{2}(r^2 - 1) \quad \text{on } (\tfrac{1}{2}, \infty),$$

$$f''(r) \geq 0 \quad \text{for all } r \in \mathbb{R}^+,$$

$$\lim_{r\to 0} f(r) = -\frac{1}{2} + \varepsilon$$

for some sufficiently small $\varepsilon > 0$.

Now we can define the perturbed Rabinowitz functional.

Definition 2.2. Fix a path $\varphi \in \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ and let H_t denote the Hamiltonian as above. Let $\mathcal{L}(S\Sigma) = C^\infty(S^1, S\Sigma)$ be the component containing the contractible loops. We define the *perturbed Rabinowitz action functional*

$$\mathcal{A}_\varphi : \mathcal{L}(S\Sigma) \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}_\varphi(v, \eta) := \int_0^1 v^*\lambda - \eta \int_0^1 \chi(t) F_0(v(t))dt - \int_0^1 \dot{\theta}(t) H(v(t), \theta(t))dt, \tag{2.1}$$

where $\chi : \mathbb{R}/\mathbb{Z} \to [0, \infty)$ and $\theta : [0, 1] \to [0, 1]$ are smooth functions. The pair (χ, θ) is called a *perturbation pair*. Moreover, if χ satisfies

$$\chi(t) = 0 \ \forall t \in [\tfrac{1}{2}, 1], \quad \text{and} \quad \int_0^1 \chi(t)dt = 1,$$

and θ satisfies that $\theta([0, \frac{1}{2}]) = 0, \theta(1) = 1$ with $\dot{\theta}(t) \in [0, 4]$ for any $0 \le t \le 1$, the pair (χ, θ) is said to be of *Moser type*.

Moreover, if the functional $\mathcal{A}_\varphi$ is Morse-Bott, we say that φ is *non-degenerate*.

We consider the critical points of the above functional. From a straight computation one has

Lemma 2.3. *For* $(v, \eta) = (x(t), r(t), \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$*, one has*

$$\begin{cases} \dot{v}(t) = \eta\chi(t) X_{F_0}(v) + \dot{\theta}(t) X_H(\theta(t), v), \\ \int_0^1 \chi(t) F_0(v)dt = 0. \end{cases} \tag{2.2}$$

and

$$\mathcal{A}_\varphi(v, \eta) = \eta.$$

The above result exhibits the connection between our Rabinowitz functional and leaf-wise intersection points and translated points. We first give the definitions here.

Definition 2.4. Let R_α be the Reeb vector field of α and ϕ_t^α be the Reeb flow.

(1) For $\varphi \in \mathrm{Symp}(X, \omega)$, a point $x \in \Sigma$ is a **leaf-wise intersection point** for (Σ, φ) if there exists $\tau \in \mathbb{R}$ so that

$$\varphi(x, 1) = (\phi_\tau^\alpha(x), 1).$$

(2) Fix $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ be a contactomorphism of (Σ, ξ). If $\psi^*\alpha = \rho\alpha$, then we say $x \in \Sigma$ is a **translated point** for ψ if there is $\tau \in \mathbb{R}$ satisfying

$$\psi(x) = \phi^\alpha_\tau(x), \rho(x) = 1.$$

When x is not a periodic point of the Reeb flow, then the real number τ is uniquely determined from the above equations. Then we call τ be the **time shift** of x.

The following observation from [San12] gives the relation of leaf-wise intersection point and translated point.

Lemma 2.5. *For $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$, let $\varphi \in \mathrm{Symp}(S\Sigma, d(r\alpha))$ be the symplectization given by*

$$\varphi(x, r) = \left(\psi(x), \frac{r}{\rho_t(x)}\right).$$

Then $x \in \Sigma$ is a translated point of ψ if and only if $(x, 1)$ is a leaf-wise intersection point for φ.

Using Lemma 2.3 we can obtain the following.

Proposition 2.6. *Let $\psi = \varphi_1$, then if ψ has no translated points on the closed leaves of R_α, then there is a bijiection between $\mathrm{Crit}(\mathcal{A}_\varphi)$ and the set of translated points of ψ given by*

$$(v, \eta) = (x, r, \eta) \mapsto x(\frac{1}{2}).$$

Moreover, if $x(\frac{1}{2})$ is not a periodic point of ϕ^α_t then it has time shift $-\eta$.

Proof. Firstly if $(v, \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$. Let $\psi' : S\Sigma \to S\Sigma$ be the symplectization of ψ. For $t \in [0, \frac{1}{2}]$ by (2.2) one has

$$\dot{v}(t) = \eta\chi(t)X_{F_0}(v(t)).$$

F_0 is constant on flow lines of $\eta\chi X_{F_0}$, so $f(r(t))$ is constant for $t \in [0, \frac{1}{2}]$, which is 0 by the condition. Then $r(t) = 1$ for $t \in [0, \frac{1}{2}]$. Thus $v(\frac{1}{2}) = (\phi^\alpha_\eta(x(0)), 1)$.

On the other hand for $t \in [\frac{1}{2}, 1]$ we have by (2.2)

$$\dot{v}(t) = \dot{\theta}(t)X_H(\theta(t), v(t)).$$

So $\psi'(v(\frac{1}{2})) = (\phi^\alpha_{-\eta}(v(\frac{1}{2})), 1)$, and thus $v(\frac{1}{2})$ is a leaf-wise intersection point of φ, so one direction of the proposition holds.

Conversely, if x is a translated point of ψ with time shift τ then construct $v : S^1 \to X$ by

$$v(t) := \begin{cases} \left(\phi^\alpha_{-\tau\chi(t)}(\phi^\alpha_{-\tau}(x)), 1\right), & t \in [0, \frac{1}{2}], \\ \varphi_{\theta(t)}(x), & t \in [\frac{1}{2}, 1], \end{cases}$$

and define $\eta = -\tau$, we have that $(v, \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$. Since there are no translated point lying on closed leaves of R_α, it is direct to check that these two operations are mutually inverse, thus give a bijection. □

We need to impose the following conditions to simplify the study of Rabinowitz functional.

Definition 2.7. A contact 1-form α generating ξ is said to be *Morse-Bott* if it satisfies the following:

For all positive T, define

$$P_T = \{x \in \Sigma \mid \phi^\alpha_T(x) = x\}.$$

where ϕ^α_t is the Reeb flow with respect to α, then P_T is a closed submanifold of Σ with $\mathrm{rank} d\alpha_{P_T}$ is locally constant. Moreover we require that

$$T_x P_T = \ker(D\phi^\alpha_T(x) - \mathrm{Id}_{T_x\Sigma}), \quad \forall x \in P_T.$$

One has to extend the definition of Rabinowitz functional to $\mathcal{L}X \times \mathbb{R}$ in order to define the Rabinowitz Floer homology. We need to extend the map F_0 and H.

Firstly let $F : X \to \mathbb{R}$ be

$$F = \begin{cases} -\frac{1}{2} + \varepsilon, \text{ on } X \backslash S\Sigma, \\ F_0, \text{ on } S\Sigma. \end{cases}$$

Then for $\kappa > 0$, let $\beta_\kappa \in C^\infty([0, \infty), [0, 1])$ be so that

$$\beta_\kappa(r) = \begin{cases} 1, & r \in [e^{-\kappa}, e^\kappa], \\ 0, & r \in [0, e^{-2\kappa}] \cup [e^\kappa + 1, \infty), \end{cases}$$

and such that

$$0 \leq \dot{\beta}_\kappa(r) \leq 2e^{2\kappa} \quad \text{for } r \in [e^{-2\kappa}, e^{-\kappa}],$$

$$-2 \leq \dot{\beta}_\kappa(r) \leq 0 \quad \text{for } r \in [e^\kappa, e^\kappa + 1].$$

We define $H_\kappa(\varphi) : [0,1] \times X \to \mathbb{R}$ by

$$H_\kappa(\varphi) = \begin{cases} 0, \text{ on } [0,1] \times (X \backslash S\Sigma), \\ \beta_\kappa(r) r \dot{\theta}(t) h_{\theta(t)}(x), \text{ for } (t,x,r) \in [0,1] \times S\Sigma. \end{cases} \tag{2.3}$$

We write $H_\kappa = H_\kappa(\varphi)$ if φ is clear.

Then the perturbed Rabinowitz functional can be extended to

$$\mathcal{A}_\varphi^\kappa : \mathcal{L}X \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}_\varphi^\kappa(v,\eta) := \int_0^1 v^*\lambda - \eta \int_0^1 \chi(t) F(v(t)) dt - \int_0^1 \dot{\theta}(t) H_\kappa(v(t), \theta(t)) dt$$

Using the arguments in [AM13], there exists $\kappa_0(\varphi)$ so that for any $\kappa > \kappa_0(\varphi)$ the critical points and critical values of $\mathcal{A}_\varphi^\kappa$ does not depend on κ and actually agree with those of $\mathcal{A}_\varphi$.

Choose a family $J = (J_t)_{t \in S^1}$ of ω-compatible almost complex structure on X so that $J_t\big|_{\Sigma \times [1,\infty)}$ is of SFT-type, we can define a L^2-inner product on $\mathcal{L}X \times \mathbb{R}$ by

$$\langle\langle (\zeta, l), (\zeta', l') \rangle\rangle_{\mathbf{J}} := \int_0^1 \omega(J_t \zeta(t), \zeta'(t)) dt + ll'.$$

For any non-degenerate φ and $-\infty < a < b < \infty$ with $a, b \notin \operatorname{Spec}(\varphi) := \mathcal{A}_\varphi(\operatorname{Crit}(\mathcal{A}_\varphi))$, the perturbed Rabinowitz functional is Morse-Bott. Choose a Morse function $h : \operatorname{Crit}(\mathcal{A}_\varphi^\kappa) \to \mathbb{R}$ and a Riemannian metric g on $\operatorname{Crit}(\mathcal{A}_\varphi^\kappa)$ so that (h,g) is a Morse-Smale pair. Then for $c^-, c^+ \in \operatorname{Crit}(h)$, let $\hat{\mathcal{M}}(c^+, c^-)$ be the moduli space of gradient flow lines with cascades of $-\nabla_J \mathcal{A}_\varphi^\kappa$ and $-\nabla_g h$ from c^+ to c^-. Then the counting of this moduli space induces a boundary operator over the chain complex $\operatorname{RFC}_*(\mathcal{A}_\varphi^\kappa)_a^b = \operatorname{Crit}_*(h)_a^b \otimes \mathbb{Z}_2$. The induced homology will be denoted $\operatorname{RFH}_*(\mathcal{A}_\varphi^\kappa)_a^b$. Details about defining the homology will be presented in Section 2.3 and we temporarily omit them here to avoid repetition.

Let c_0 be as in [AM13, Proposition 2.5], then $\operatorname{RFH}_*(\mathcal{A}_\varphi^\kappa)_a^b$ is independent of κ if $\kappa > c_0$. Then it makes sense to define

$$\operatorname{RFH}_*(\mathcal{A}_\varphi, X) := \operatorname{RFH}_*(\{\varphi_t\}, \Sigma, X) := \varinjlim_{a \downarrow -\infty} \varprojlim_{b \uparrow \infty} \operatorname{RFH}_*(\mathcal{A}_\varphi^\kappa)_a^b.$$

Actually the above defined Rabinowitz Floer homology is independent of the choice of the auxiliary perturbation data and $\{\varphi_t\} \in \mathcal{P}\operatorname{Cont}_0(\Sigma, \xi)$.

Proposition 2.8. [AF10, Theorem 2.16] *There is a canonical isomorphism*

$$\zeta_\varphi : \mathrm{RFH}_*(\Sigma, X) \to \mathrm{RFH}_*(\mathcal{A}_\varphi, X).$$

Moreover, given two non-degenerate paths φ, ψ *there is a map*

$$\zeta_{\varphi,\psi} : \mathrm{RFH}_*(\mathcal{A}_\varphi, X) \to \mathrm{RFH}_*(\mathcal{A}_\psi, X)$$

satisfying $\zeta_\psi = \zeta_{\varphi,\psi} \circ \zeta_\varphi$.

2.2. **S^1-equivariant Rabinowitz functional.** Noe we follow the path of [FS16] to give a Borel-type construction of S^1-equivariant Rabinowitz functional. We still assume that φ is non-degenerate.

For integer $n \geq 1$, let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$, then the complex projective space $\mathbb{C}P^n$ is the quotient of S^{2n+1} of the S^1-action

$$\tau \cdot (z_1, \dots, z_{n+1}) = (\tau z_1, \dots, \tau z_{n+1}).$$

Also there is S^1-action on $\mathcal{L}X \times \mathbb{R}$ be

$$\tau \cdot (v(\cdot), \eta) \mapsto (v(\cdot - \tau), \eta).$$

Then S^1 acts on $\mathcal{L}X \times \mathbb{R} \times S^{2n+1}$ diagonally and we denote the quotient of this actino by $\mathcal{L}X \times \mathbb{R} \times_{S^1} S^{2n+1}$.

Moreover, S^1 acts on $S^1 \times S^{2n+1}$ by $\tau(\cdot, z) = (\cdot - \tau, \tau z)$ and we write the quotient under this action by $S^1 \times_{S^1} S^{2n+1}$. The non-perturbed equivariant Rabinowitz functional is defined as

$$\mathcal{A}_\varphi^{\kappa,n,S^1}([v, \eta, z]) = \int_0^1 v^*\lambda - \eta \int_0^1 F_0(v(t))dt.$$

Then we consider the perturbed case. We define a *perturbation triple* to be a triple

$$(\Psi, \Theta, k) \in C^\infty(S^1 \times_{S^1} S^{2n+1}, [0, \infty)) \times C^\infty(S^1 \times_{S^1} S^{2n+1}, \mathbb{R}) \times C^\infty(\mathbb{C}P^n, \mathbb{R}),$$

where for any $z \in S^{2n+1}$, one has

$$\int_{S^1} \Psi([t, z]) = 1.$$

Definition 2.9. The **equivariant perturbed Rabinowitz functional** with respect to perturbation triple (Ψ, Θ, k) is a map

$$\mathcal{A}_{\varphi,\Psi,\Theta,k}^{\kappa,n,S^1} : \mathcal{L}X \times \mathbb{R} \times_{S^1} S^{2n+1} \to \mathbb{R}$$

given by

$$\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}([v,\eta,z]) = \int_0^1 v^*\lambda - \eta\int_0^1 \Psi([t,z])F_0(v(t))dt - \int_0^1 \dot{\Theta}([t,z])H_\kappa(v(t),\Theta([t,z]))dt - k([z]).$$

We also denote

$$\mathcal{A}^{\kappa,n,S^1}_{0,\varphi,\Psi,\Theta,k}([v,\eta,z]) + k([z]).$$

We can also consider the liftings of the above ingredients. Let

$$(\tilde{\Psi},\tilde{\Theta},\tilde{k}) \in C^\infty(S^1\times S^{2n+1},[0,\infty)) \times C^\infty(S^1\times S^{2n+1},\mathbb{R}) \times C^\infty(S^{2n+1},\mathbb{R})$$

be the lifting of a perturbation triple and $\tilde{\mathcal{A}}^{\kappa,n,S^1}_{\varphi,\tilde{\Psi},\tilde{\Theta},\tilde{k}}$ be the lifted functional, then again a direct computation shows the following result in comparison with Lemma 2.3.

Lemma 2.10. *If* $(v,\eta,z) \in \mathrm{Crit}(\tilde{\mathcal{A}}^{\kappa,n,S^1}_{\varphi,\tilde{\Psi},\tilde{\Theta},\tilde{k}})$, *then*

$$\begin{cases} \dot{v}(t) = \eta\tilde{\Psi}(t,z)X_{F_0}(v) + \dot{\tilde{\Theta}}(t,z)X_{H_\kappa}(v(t),\widetilde{\Theta}(t,z)), \\ \int_0^1 \tilde{\Psi}(t,z)F_0(v)dt = 0, \\ 0 = \eta\int_0^1 F_0(v(t))\partial_z\tilde{\Psi}(t,z)dt - d\tilde{k}(z) - \int_0^1 \partial_z\left(\dot{\tilde{\Theta}}(t,z)H_\kappa(v(t),\tilde{\Theta}(t,z))\right)dt. \end{cases} \tag{2.4}$$

In order to connect the critical points of the equivariant perturbed functional to the non-equivariant ones, we need to impose the following conditions on the perturbation triple.

Definition 2.11. A perturbation triple (Ψ,Θ,k) is said to be of *equivariant Moser type* if it satisfies the following:

(1) For any $z \in S^{2n+1}$ and (v,η) a solution to (2.2) with respect to the perturbation pair $(\tilde{\Psi}_z,\tilde{\Theta}_z)$, there holds

$$F_0(v(t))d\tilde{\Psi}_t(z) = 0, \quad \forall t \in S^1.$$

(2) For any $z \notin \mathrm{Crit}(\tilde{k}), V \in T_zS^{2n+1}$ and $(x,t) \in X\times S^1$, one has

$$|d\tilde{k}(z)V| > |\partial_z\left(\dot{\tilde{\Theta}}(t,z)H_\kappa(v(t),\tilde{\Theta}(t,z))\right)|. \tag{2.5}$$

(3) The critical values of k lies in $\mathbb{Z}$.

The following result shows that given any perturbation pair that is of Moser type, one can construct a perturbation triple that is of equivariant Moser type out of it.

Proposition 2.12. *Given (χ, θ) a perturbation pair that is of Moser type, there exists a perturbation triple (Ψ, Θ, k) that is of equivariant Moser type, so that, for any $z \in \mathrm{Crit}(\tilde{k})$ there is $t_z \in S^1$ with*

$$\tilde{\Theta}(t,z) = \theta(t+t_z), \tilde{\Psi}(t,z) = \chi(t+t_z), \quad \forall t \in S^1.$$

Proof. The condition (1) and (2) of equivariant Moser type perturbation triple can be satisfied using the same argument as in [FS16, Proposition 4.6] and the condition (3) can be obtained from rescaling and perturbing a fixed given k. □

Critical points of the perturbed eqiuvariant functional with respect to perturbation triple of equivariant Moser type can correspond to the critical points of the non-equivariant functional.

Lemma 2.13. *If (Ψ, Θ, k) is a perturbation triple of equivariant Moser type, then for any $[(v, \eta, z)] \in \mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k})$, one has*

(1) $[z] \in \mathrm{Crit}(k)$.
(2) For any $z \in S^{2n+1}$ over $[z]$, (v, η) is contained in $\mathrm{Crit}(\mathcal{A}^{\kappa}_{\varphi})$, where the non-equivariant functional is with respect to the perturbation pair $(\tilde{\Psi}_z, \tilde{\Theta}_z)$.

We say that (Ψ, Θ, k) is induced from (χ, θ).

Proof. By comparing (2.2) and (2.4) the assertion (2) follows immediately, so it suffices to show that $[z] \in \mathrm{Crit}(k)$. By (2.4), for any $V \in T_z S^{2n+1}$ one has

$$0 = \eta \int_0^1 F_0(v(t)) d\tilde{\Psi}(t,z)(V) dt - d\tilde{k}(z)(V) - \int_0^1 d\left(\dot{\tilde{\Theta}}(t,z) H_\kappa(v(t), \tilde{\Theta}(t,z))\right)(V) dt.$$

Then by the conditions imposed in 2.11, the result follows. □

2.3. **S^1-equivariant Rabinowitz Floer homology.** Having constructed the equivariant Rabinowitz functinoal, we can define the equivariant Rabinowitz Floer homology.

Some technical lemmas are necessary. First we show that the κ-truncation of X does not affect our construction.

Lemma 2.14. *(cf.[AM13, Proposition 2.5]) Let (Ψ, Θ, k) be a There is a real number $\kappa(\varphi) > 0$ so that for any $\kappa > \kappa(\varphi)$ and $([v, \eta, z]) \in \mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k})$, then $v(S^1) \subseteq S\Sigma$ and $r(S^1) \subseteq (e^{-\frac{\kappa}{2}}, e^{\frac{\kappa}{2}})$ if $v(t) = (x(t), r(t))$.*

Proof. From the critical point equation (2.4) of $\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}$, for any $v(t) \in \Sigma \times (e^{-\kappa}, e^{\kappa})$ and $t \in [\frac{1}{2} + t_z, 1 + t_z] (\mod 1)$, $r(t)$ satisfies

$$\dot{r}(t) = -\dot{\theta}(t + t_z) \frac{\dot{\rho}_{\theta(t+t_z)}(x(t))}{\rho^2_{\theta(t+t_z)}(x(t))} \cdot r(t).$$

Then the same argument of [AM13, Proposition 2.5] applies. The constant $\kappa(\varphi)$ can be chosen as

$$\kappa(\varphi) = 8 \max \left\{ \left| \frac{\dot{\rho}_t(x)}{\rho^2_t(x)} \right| : (t, x) \in [0, 1] \times \Sigma \right\}, \tag{2.6}$$

where ρ_t satisfies $\varphi^*_t \alpha = \rho_t \alpha$. □

Then for $\kappa > \kappa(\varphi)$, the critical points and critical values of $\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}$ is independent of κ.

Since φ is non-degenerate, the arguments in [AM13] show that the functional $\tilde{\mathcal{A}}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}$ is a Morse-Bott. Fix $\kappa > \kappa(\varphi)$ and $n \in \mathbb{N}$, then since the S^1-action on $\mathcal{L}X \times \mathbb{R} \times S^{2n+1}$ is free, $\mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k})$ is a closed manifold. Fix a Riemannian metric g_n on it and choose a Morse function

$$h^n_\varphi : \mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}) \to \mathbb{R}$$

so that (h^n_φ, g_n) is a Morse-Smale pair.

The chain group $\mathrm{RCF}_*(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}, h^n_\varphi)$ consists of Novikov sum $\sum \xi_c c$ for $\xi_c \in \mathbb{Z}_2, c \in \mathrm{Crit}(h^n_\varphi)$ so that

$$\#\{c \in \mathrm{Crit}(h^n_\varphi) \mid \xi_c \neq 0, \mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}(c) \geq \varepsilon\} < \infty, \quad \forall \varepsilon > 0.$$

Then we define the boundary operators. We need to fix a family of almost complex structures first. Let $\mathcal{J}_{\mathrm{con}}(S\Sigma)$ be the set of almost complex structures J on $S\Sigma$ that is convex-at-infinity and $d(r\alpha)$-compatible, then we denote $\mathcal{J}_{\mathrm{con}}(X)$ be the set of almost complex structure J so that $J|_{S\Sigma} \in \mathcal{J}_{\mathrm{con}}(S\Sigma)$.

Fix a $J_0 \in \mathcal{J}_{\mathrm{con}}(X)$ and consider a S^1-invariant smooth $S^1 \times S^{2n+1} \times \mathbb{R}$ family $\mathbb{J}_n = \{J_{n,t,z}(\cdot, \eta)\} \subseteq \mathcal{J}_{\mathrm{con}}(X)$ with

$$\sup_{t,z,\eta} \|J_{n,t,z}(\cdot, \eta)\|_{C^\ell} < \infty, \quad \forall \ell \in \mathbb{N}.$$

Moreover we assume that there is a constant $c > 1$ so that

$$\frac{1}{c}\|J_0(x)\| \leq \|J_{n,t,z}(x, \eta)\| \leq c\|J_0(x)\|, \quad \forall (x, t, z, \eta) \in X \times S^1 \times S^{2n+1} \times \mathbb{R}. \tag{2.7}$$

The space $\mathcal{J}^{S^1}$ of such families $\mathbb{J}_n$ is non-empty and contractible.

In the following we fix a family $\mathbb{J}_n$ as above. Then we can consider the moduli space of gradient flowlines with cascades. Given two critical points c^+, c^- of h^n_φ, let C^+, C^- be the corresponding critical circles of the lifted function $\tilde{h}^n_\varphi$, which is Morse-Bott.

For $-\infty < a < b < \infty$, let $\hat{\mathcal{M}}^n(c^+, c^-)^b_a$ be the space of flowlines with cascades of $-\nabla_{\mathbb{J}_n}\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}$ and $-\nabla_g h^n_\varphi$ from a point in C^+ to a point in C^- with $\mathcal{A}^{\kappa,n,S^1}_{0,\varphi,\Psi,\Theta,k}([v,\eta,z]) \in (a,b)$ for any $s \in \mathbb{R}$.

There is a free S^1-action on $\hat{\mathcal{M}}^n(c^+, c^-)^b_a$ and the quotient of this action is equal to $\hat{\mathcal{M}}^n(c, c^-)^b_a$, the space of gradient flowlines with cascades from $c \in C^+$ to an arbitrary point in C^-.

The real number set $\mathbb{R}$ also acts freely on these moduli spaces by time shift and we write the spaces modulo the $\mathbb{R}$-action as

$$\mathcal{M}^n(c^+, c^-)^b_a = \sqcup_{c\in C^+}\mathcal{M}^n(c, c^-)^b_a.$$

Given the upper bound estimate of η from the gradient flowline equation, which is proved in [CF09, Corollary 3.3] under (2.7), the fact that the space $\mathcal{M}(c^+, c^-)^b_a$ is compact follows from a standard compactness argument. Then the boundary operator on $\mathrm{RCF}_*(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}, h^n_\varphi, \mathbb{J}_n)$ is given by

$$\partial c^+ = \sum_{c^-} |\mathcal{M}^n(c^+, c^-)^b_a / S^1| \cdot c^-,$$

where the sum is taken among c^- so that $\mathcal{M}(c^+, c^-)^b_a$ is 0-dimensional. The resulting homology is denoted by $\mathrm{RFH}^{n,S^1}_*(\varphi, X, h^n_\varphi, \mathbb{J}_n)^b_a$. And we let

$$\mathrm{RFH}^{S^1}_*(\varphi, X)^b_a = \varinjlim_{n,h,\mathbb{J}} \mathrm{RFH}^{n,S^1}_*(\varphi, X, h^n_\varphi, \mathbb{J}_n)^b_a.$$

The direct limit is taken in the following sense. Firstly note that the inclusion $S^{2n+1} \subseteq S^{2n+3}$ is S^1-equivariant, then if we choose a S^1-equivariant extension of the perturbation triple (Ψ_n, Θ_n, k_n) to $(\Psi_{n+1}, \Theta_{n+1}, k_{n+1})$ on S^{2n+3}, one has $\mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}) \subseteq \mathrm{Crit}(\mathcal{A}^{\kappa,n+1,S^1}_{\varphi,\Psi,\Theta,k})$.

Now we choose

(1) an extension of (h^n_φ, g_n) to $(h^{n+1}_\varphi, g_{n+1})$ on $\mathrm{Crit}(\mathcal{A}^{\kappa,n+1,S^1}_{\varphi,\Psi,\Theta,k})$ that is Morse-Smale,
(2) A family $\mathbb{J}_{n+1}$ satisfying (2.7) for S^{2n+3} extending $\mathbb{J}_n$,

then there is a natural inclusion of chain subcomplex $\mathrm{RCF}_*(\mathcal{A}^{\kappa,n,S^1}_{\varphi,\Psi,\Theta,k}, h^n_\varphi, \mathbb{J}_n)^b_a \subseteq \mathrm{RCF}_*(\mathcal{A}^{\kappa,n+1,S^1}_{\varphi,\Psi,\Theta,k}, h^{n+1}_\varphi, \mathbb{J}_{n+1})^b_a$ and it induces a inclusion

$$\iota_n : \mathrm{RFH}^{n,S^1}_*(\varphi, X, h^n_\varphi, \mathbb{J}_n)^b_a \to \mathrm{RFH}^{n+1,S^1}_*(\varphi, X, h^{n+1}_\varphi, \mathbb{J}_{n+1})^b_a.$$

The homology group $\mathrm{RFH}_*^{n,S^1}(\varphi, X, h_\varphi^n, \mathbb{J}_n)_a^b$ does not depend on any auxiliary choice that we made so we will fix these auxiliary data $(\Phi, \Theta, k, \kappa, h, \mathbb{J})$ and omit them in the notations. This is shown by Floer continuation arguments as in [CF09] and the continuation isomorphisms commute with ι_n, so the direct limit makes sense.

Finally we let the S^1-equivariant Rabinowitz Floer homology be

$$\mathrm{RFH}_*^{S^1}(\varphi, X) = \varinjlim_{a\downarrow-\infty} \varprojlim_{b\uparrow\infty} \mathrm{RFH}_*^{S^1}(\varphi, X)_a^b.$$

Now we spell out the properties of the above homology, which can be obtained from standard arguments.

Proposition 2.15. *(1) The S^1-equivariant Rabinowitz Floer homology is independent of φ. More precisely, for any non-degenerate $\varphi \in \mathcal{P}\mathrm{Cont}(\Sigma, \xi)$, there is an isomorphism*

$$\zeta_\varphi^{S^1} : \mathrm{RFH}_*^{S^1}(\varphi, X) \to \mathrm{RFH}_*^{S^1}(\Sigma, X) := \mathrm{RFH}_*^{S^1}(\{\mathrm{Id}_\Sigma\}, X).$$

Moreover, for any non-degenerate paths φ, ψ, there is a map

$$\zeta_{\varphi,\psi}^{S^1} : \mathrm{RFH}_*^{S^1}(\varphi, X) \to \mathrm{RFH}_*^{S^1}(\psi, X)$$

satisfying $\zeta_\psi^{S^1} = \zeta_{\varphi,\psi}^{S^1} \circ \zeta_\varphi^{S^1}$.

(2) For a non-degenerate path φ and $[a,b] \cap \mathrm{Spec}(\varphi) = \emptyset$, where $\mathrm{Spec}(\varphi) = \lim_{n\to\infty} \mathcal{A}_{0,\varphi}^{n,S^1}(\mathrm{Crit}(\mathcal{A}_\varphi^{n,S^1}))$, there is an isomorphism

$$\iota_\varphi^{a,b,S^1} : \mathrm{RFH}_*^{S^1}(\varphi, X)_a \to \mathrm{RFH}_*^{S^1}(\varphi, X)_b.$$

3. Spectral Invariants

We assume that (Σ, ξ) satisfies condition (H). Then for a non-degenerate path φ and any $0 \neq \theta \in \mathrm{RFH}_*^{S^1}(\Sigma, X)$, define the *spectral number* as

$$c(\varphi, \theta) = \inf\{a \in \mathbb{R} \mid \theta_\varphi \in \mathrm{Im}(\iota_\varphi^a : \mathrm{RFH}_*^{S^1}(\varphi, X)_a \to \mathrm{RFH}_*^{S^1}(\varphi, X))\}, \tag{3.1}$$

where θ_φ is the image of θ under the isomorphism $\mathrm{RFH}_*^{S^1}(\Sigma, X) \simeq \mathrm{RFH}_*^{S^1}(\varphi, X)$ and ι_φ^a is the natural inclusion.

Similar with the various spectral invariants in Hamiltonian Floer theory, the spectrality is satisfied.

Proposition 3.1. *For any non-degenerate path φ, one has $c(\varphi, \theta) \in \mathrm{Spec}(\varphi)$.*

Proof. This is a direct result of Proposition 2.15(2). ∎

We establish a certain kind of continuity estimate of the spectral number to extend the above definition to probably degenerate paths.

Lemma 3.2. *Write* $\mathrm{RFH}_*^{S^1}(\varphi, X)_a = \mathrm{RFH}_*^{S^1}(\varphi, X)_a^\infty$*, then for any two non-degenerate paths* φ, ψ*, there is a constant* $K(\varphi,\psi) \geq 0$ *so that there is an isomorphism*

$$\zeta_{\varphi,\psi}^{a,S^1} : \mathrm{RFH}_*^{S^1}(\varphi, X)_a \to \mathrm{RFH}_*^{S^1}(\psi, X)_{a+K(\varphi,\psi)}.$$

And the constant $K(\varphi,\psi)$ *satisfies the following estimate*

$$K(\varphi,\psi) \leq \kappa(\varphi,\psi) := 4e^{\kappa} \max\left\{\|h_\varphi - h_\psi\|_+, 0\right\}, \tag{3.2}$$

where κ *is defined as in Lemma 2.14 and* $\|\cdot\|_+$ *is defined by*

$$\|h\|_+ := \int_0^1 \max_{x\in\Sigma} h_{\Theta([t,z])}(x)dt$$

Remark 3.3. Under the above notations, we have that if φ_k converges to φ in the C^2-norm, then $\kappa(\varphi_k, \varphi)$ converges to 0.

Proof of Lemma 3.2. Let $H_\varphi := H_\kappa(\varphi)$ and $H_\psi := H_\kappa(\psi)$ be as in (2.3), where $\kappa > \max\{\kappa(\varphi), \kappa(\psi)\}$, κ defined as in Lemma 2.14.

$$\Phi_s = \nu(s)H_\varphi + (1-\nu(s))H_\psi,$$

where $\nu : \mathbb{R} \to [0,1]$ is a smooth non-increasing function with $\nu(s) = 1$ for $s \leq -1$, $\nu(s) = 0$ for $s \geq 1$. Then we define

$$\mathcal{A}_s^{n,S^1}([v,\eta,z]) = \int_0^1 v^*\lambda - \eta\int_0^1 \Psi([t,z])F_0(v(t))dt - k([z]) - \int_0^1 \dot{\Theta}([t,z])\Phi_s(v(t), \Theta([t,z]))dt$$

and corresponding $\mathcal{A}_{0,s}^{n,S^1}$.

Then by counting the moduli space of the flowlines with cascades of $-\nabla_{\mathbb{J}}\mathcal{A}_s$ and $-\nabla_g h$ we can define the continuation homomorphism. We estimate

$$\begin{aligned}
0 &\le \int_{-\infty}^{\infty}\int_0^1 |\partial_s([v,\eta,z])|^2_{\mathbb{J}} dtds \\
&= -\int_{-\infty}^{\infty}\int_0^1 \left\langle\left\langle \nabla \mathcal{A}^{n,S^1}_{0,s}([v,\eta,z]), \partial_s([v,\eta,z])\right\rangle\right\rangle_{\mathbb{J}} dtds \\
&= -\int_{-\infty}^{\infty}\int_0^1 \frac{d}{ds}\mathcal{A}^{n,S^1}_{0,s}([v,\eta,z])dtds + \int_{-\infty}^{\infty}\int_0^1 \frac{\partial \mathcal{A}^{n,S^1}_{0,s}}{\partial s}([v,\eta,z])dtds \\
&= \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) - \int_{-\infty}^{\infty}\int_0^1 \dot{\Theta}([t,z])\frac{\partial \Phi_s}{\partial s}(v(t),\Theta([t,z]))dtds \\
&\le \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) \\
&\qquad - \int_{-\infty}^{\infty}\int_0^1 \dot{\Theta}([t,z])\nu'(s)\beta_\kappa(r)r(t)\big(h_{\Theta([t,z]),\varphi}(v(t)) - h_{\Theta([t,z]),\psi}(v(t))\big)dtds.
\end{aligned}$$

So we have

$$\begin{aligned}
0 &\le \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) \\
&\qquad - \int_{-\infty}^{\infty}\int_0^1 \dot{\Theta}([t,z])\nu'(s)\beta_\kappa(r)r(t)\max\Big\{\max_{x\in\Sigma}\Big(h_{\Theta([t,z]),\varphi}(x) - h_{\Theta([t,z]),\psi}(x)\Big),0\Big\}dtds \\
&\le \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) \\
&\qquad - e^\kappa\int_{-\infty}^{\infty}\int_0^1 \nu'(s)\dot{\Theta}([t,z])\max\Big\{\max_{x\in\Sigma}\Big(h_{\Theta([t,z]),\varphi}(x) - h_{\Theta([t,z]),\psi}(x)\Big),0\Big\}dtds \\
&= \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) \\
&\qquad - 4e^\kappa\int_{-\infty}^{\infty}\nu'(s)ds\int_0^1 \max\Big\{\max_{x\in\Sigma}\Big(h_{\Theta([t,z]),\varphi}(x) - h_{\Theta([t,z]),\psi}(x)\Big),0\Big\}dt \\
&= \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) + 4e^\kappa\int_0^1 \max\Big\{\max_{x\in\Sigma}\Big(h_{\Theta([t,z]),\varphi}(x) - h_{\Theta([t,z]),\psi}(x)\Big),0\Big\}dt \\
&\le \mathcal{A}^{n,S^1}_{0,\varphi}([v_-,\eta_-,z_-]) - A^{n,S^1}_{0,\psi}([v_+,\eta_+,z_+]) + 4e^\kappa\max\big\{\|h_\varphi - h_\psi\|_+,0\big\},
\end{aligned}$$

then the result follows. □

Corollary 3.4. *For any two non-degenerate paths φ,ψ, there is an inequality*

$$c(\psi,\theta) \le c(\varphi,\theta) + \kappa(\varphi,\psi).$$

Especially, if h_φ, h_ψ be the contact Hamiltonians with respect to φ and ψ with

$$h_\varphi \le h_\psi \text{ on } \Sigma\times[0,1],$$

then $c(\varphi, \theta) \geq c(\psi, \theta)$.

Now we can show the following.

Proposition 3.5. *For any nonzero class* θ*, there is an extension of* $c(\cdot, \theta)$ *to* $\mathcal{P}\mathrm{Cont}(\Sigma, \xi)$*: for any degenerate path* φ*, choose non-degenerate paths* φ_k *so that*

$$\varphi_k \xrightarrow{C^2} \varphi,$$

then define

$$c(\varphi, \theta) = \lim_{k \to \infty} c(\varphi_k, \theta).$$

The extended spectral invariant

$$c(\cdot, \theta) : \mathcal{P}\mathrm{Cont}(\Sigma, \xi) \to \mathbb{R}$$

is a C^2*-continuous function. In particular, this descends to a function*

$$c(\cdot, \theta) : \widetilde{\mathrm{Cont}}_0(\Sigma, \xi) \to \mathbb{R},$$

where $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ *is the universal cover of* $\mathrm{Cont}_0(\Sigma, \xi)$*.*

Proof. Let h_k be the contact Hamiltonians with respect to φ_k and h be the contact Hamiltonian of φ, then if $\varphi_k \xrightarrow{C^2} \varphi$, we have

$$\kappa(\varphi_k, \varphi) \to 0, h_k \to h.$$

Then by Proposition 3.1, $c(\varphi_k, \theta)$ converges. The independence of the sequence $\{\varphi_k\}$ follows from a similar argument.

For the second part of the proposition, note that $\mathrm{Spec}(\varphi)$ only depends on the terminal map φ_1, so if the path φ is varied with φ_1 fixed, then the image of continuous map $c(\cdot, \theta)$ varies in a fixed set $\mathrm{Spec}(\varphi)$ which is nowhere dense, thus constant. □

Clearly the spectrality and estimates in Proposition 3.1 still holds for the extended spectral invariants. Lastly we compute the spectral invariant of a special path of contactomorphism, given by the Reeb flow.

Proposition 3.6. *Let* φ_α^m *be the path given by*

$$t \mapsto \phi_{mt}^\alpha,$$

then one has

$$c(\varphi_\alpha^m, \theta) = c(\mathrm{Id}_\Sigma, \theta) - m,$$

Proof. Note that $\mathrm{Spec}(\varphi_\alpha^m) = -m + \mathrm{Spec}(\mathrm{Id}_\Sigma)$, then the assertion holds since $\mathrm{Spec}(\mathrm{Id}_\Sigma)$ is nowhere dense and the spectral invariant is continuous. □

4. Contact Capacity and Non-squeezing

We assume that (Σ, ξ) satisfies condition (H) and $\alpha = \lambda|_\Sigma$ is periodic, that is, the Reeb flow ϕ_t^α is a 1-periodic loop. Then for $0 \neq \theta \in \mathrm{RFH}_*^{S^1}(\Sigma, X)$, define the *integral spectral invariants* as

$$c_{\mathbb{Z}}(\varphi, \theta) = \lceil c(\varphi, \theta) \rceil \in \mathbb{Z}.$$

The key property of this spectral invariant is the conjugation invariance.

Lemma 4.1. *If* $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ *satisfies* $\mathrm{Spec}(\varphi) \subseteq \mathbb{R} \backslash \mathbb{Z}$*, then for any* $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$*, one has*

$$c_{\mathbb{Z}}(\varphi, \theta) = c_{\mathbb{Z}}(\psi\varphi\psi^{-1}, \theta)$$

Proof. Let $\{\psi_s\} \subseteq \mathrm{Cont}_0(\Sigma, \xi)$ be a path connecting Id_Σ to Ψ, since the map $s \mapsto c(\psi\varphi\psi^{-1}, \theta)$ is continuous, it suffices to show that

$$\mathrm{Spec}(\psi_s \varphi \psi_s^{-1}) \cap \mathbb{Z} = \emptyset, \quad \forall s \in [0, 1].$$

By contrast we assume that $\mathrm{Spec}(\psi_s\varphi\psi_s^{-1}) \cap \mathbb{Z} \neq \emptyset$ for some fixed $s \in [0, 1]$. We write $\varphi' = \psi_s \varphi \psi_s'^{-1}$, then $\varphi = \psi_s^{-1} \varphi' \psi_s$.

If $([v, \eta, z]) \in \mathrm{Crit}(\mathcal{A}^{\kappa,n,S^1}_{\varphi',\Psi,\Theta,k})$ for some n with $\mathcal{A}^{\kappa,n,S^1}_{\varphi',\Psi,\Theta,k}([v, \eta, z]) \in \mathbb{Z}$, then since

$$\mathcal{A}^{\kappa,n,S^1}_{\varphi',\Psi,\Theta,k}([v, \eta, z]) = \mathcal{A}'^{\kappa}_{\varphi}(v, \eta) - k([z]) = \eta - k([z]),$$

where $\mathcal{A}'^{\kappa}_{\varphi}$ is the perturbed non-equivariant Rabinowitz functional with respect to $(\tilde{\Psi}_z, \tilde{\Theta}_z)$ for some $z \in S^{2n+1}$ over $[z]$, we have $\eta \in \mathbb{Z}$.

Then since α is periodic, by Proposition 2.6 and [AM18, Lemma 2.7], $x(\frac{1}{2} - t_z)$ is a fixed point of φ'. Then $\psi_s^{-1}(x(\frac{1}{2} - t_z))$ is a fixed point of φ.

Again by Proposition 2.6 there is

$$(x'(t), r'(t), \eta) \in \mathrm{Crit}(\mathcal{A}^{\kappa}_{\varphi})$$

with $x'(\frac{1}{2} - t_z) = \psi_s^{-1}(x(\frac{1}{2} - t_z))$. In particular, $\eta \in \mathrm{Spec}(\varphi)$, which leads to a contradiction. □

In order to extend the above property to all the elements in $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$, we need the following technical lemma.

Lemma 4.2. *([AM18, Lemma 4.2]) If φ is degenerate with $c(\varphi,\theta)\in\mathbb{Z}$, then there exists non-degenerate φ_k with* $\mathrm{Spec}(\varphi)\cap\mathbb{Z}\neq\emptyset$, *so that $c(\varphi_k,\theta)=c(\varphi,\theta)$ for k sufficiently large.*

Corollary 4.3. *For any $\varphi\in\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ and $\psi\in\mathrm{Cont}_0(\Sigma,\xi)$, one has*

$$c_{\mathbb{Z}}(\varphi,\theta)=c_{\mathbb{Z}}(\psi\varphi\psi^{-1},\theta).$$

Proof. It suffices to show the equality when $\mathrm{Spec}(\varphi)\cap\mathbb{Z}\neq\emptyset$. We follow the argument in [AM18, Proposition 4.3]. The three cases are listed as below.

(1) If φ satisfies $c(\varphi,\theta)\notin\mathbb{Z}$, then there is φ' sufficiently close to φ with $\mathrm{Spec}(\varphi')\subseteq\mathbb{R}\backslash\mathbb{Z}$. Since the spectral invariant is C^2-continuous and $c(\varphi,\theta)\notin\mathbb{Z}$, one has

$$c_{\mathbb{Z}}(\varphi',\theta)=c_{\mathbb{Z}}(\varphi,\theta).$$

Then the assertion comes from Lemma 4.1.

(2) If φ is non-degenerate with $c(\varphi,\theta)\in\mathbb{Z}$, then we choose a path ψ_s from Id_Σ to ψ. By the conjugation invariance of the linearized equation of critical equation, there exists $\varepsilon>0$ so that

$$\mathrm{Spec}(\psi_s\varphi\psi_s^{-1})\cap[c(\varphi,\theta)-\varepsilon,c(\varphi,\theta)+\varepsilon]=\{c(\varphi,\theta)\},$$

then the desired property is obtained.

(3) Lastly, if φ is degenerate with $c(\varphi,\theta)\in\mathbb{Z}$, then we choose φ_k as in Lemma 4.2. Since

$$c(\psi\varphi_k\psi^{-1})\to c(\psi\varphi\psi^{-1}),\quad k\to\infty,$$

the result follows from the last case.

□

Now we can define a contact capacity in the sense of [San11].

Definition 4.4. For any open subset $U\subseteq\Sigma$, we define the S^1-*equivariant contact capacity* by

$$c(U,\theta)=\sup\{c_{\mathbb{Z}}(\varphi,\theta)\mid\mathrm{Supp}(\varphi)\subseteq U\}\in\mathbb{Z}\cup\{\infty\}.\tag{4.1}$$

Here for $\varphi\in\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$, the support of φ is

$$\mathrm{Supp}\varphi=\bigcup_{0\le t\le1}\mathrm{supp}(\varphi_t).$$

The following is a direct result from the conjugation invariance of integral spectral invariants.

Proposition 4.5. *For any* $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ *one has*

$$c(\psi(U), \theta) = c(U, \theta).$$

Proof. Note that

$$\mathrm{Supp}(\varphi) \subseteq U \iff \mathrm{Supp}(\psi\varphi\psi^{-1}) \subseteq \psi(U).$$

The rest is a simple verification. □

Using this we establish the contact non-squeezing result.

Corollary 4.6. *If* $U \subseteq V \subseteq \Sigma$ *with* $c(U, \theta) < c(V, \theta)$ *there is no contact isotopy mapping* V *into* U*.*

Proof. Assume otherwise that there is $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$ with $\varphi(V) \subseteq U$, then $c(U, \theta) \leq c(V, \theta)$ and $c(\varphi(V), \theta) = c(U, \theta)$, by Proposition 4.5 one has $c(U, \theta) = c(V, \theta)$, which is a contradiction. □

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