OPEN-STRING SYMPLECTICALLY DEGENERATE MINIMUM AND RELATIVE POINCARÉ-BIRKHOFF THEOREM

LYU CHANGLE

Abstract. We formulate a relative Poincaré–Birkhoff theorem for exact Lagrangians with strongly index-definite Legendrian boundary under the periodwise isolated-finiteness of Hamiltonian return chords, requiring a non-zero wrapped Floer homology, thus improving a result of [LM24]. The proof involves a careful analysis of an open-string analogue of symplectically degenerate minimum(SDM) about its index and action. In the process one needs to reduce the wrapped Floer homology locally and establish a Morse model of generating function and applying the boundary depth argument by Ginzburg-Gurel [GG10]. We also extend the above discussions about one chord to the setting of the so-called finite first-return graph.

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1. Introduction and main results

1.1. **Main results.** A longstanding problem in celestial mechanics, about finding closed orbits in the planar restricted three-body theorem, was reduced by Poincaré[Poi87], to finding a global surface of section with respect to the dynamics and showing a corresponding fixed point theorem of the first return map. One of the most famous result in this direction is the Poincaré-Birkhoff theorem, firstly proved in full generality by Birkhoff [Bir13, Bir26], which asserts the existence of infinitely many periodic orbits with arbitrarily large period of an area-preserving twist map of an annulus.

In [MvK22], the authors extended the classical Poincaré-Birkhoff theorem to the case of Liouville domains of arbitrary dimension. Moreno-Limoge [LM24] proved a relative version of the theorem for the Lagrangians with Legendrian boundary, concerning the existence of infinitely many chords. The precise description of their result is as follow.

Theorem 1.1. *(*[LM24, Theorem A]*) Suppose that τ is an exact symplectomorphism of a connected Liouville domain (W, λ). Let $\alpha := \lambda|_{\partial W}$, and $L \subset (W, \lambda)$ be an exact, spin, Lagrangian with Legendrian boundary. Assume the following:*

- *τ is a Hamiltonian twist map.*
- *There are finitely many periodic chords.*
- *If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite.*
- *The wrapped Floer homology $HW^*(L, L)$ is infinite dimensional.*

Then τ admits infinitely many interior chords with respect to L, of arbitrary large order.

We first lay out the necessary definitions and notations. Let $(W^{2n}, \omega = \mathrm{d}\lambda)$ be a $2n$-dimensional Liouville domain, and τ is a symplectomorphism of W. Let $(\partial W, \xi =$

$\ker\alpha)$ be the contact manifold where $\alpha = \lambda|_{\partial W}$, and R_α is the Reeb vector field of α. The Liouville vector field V_λ is defined via $\iota_{V_\lambda}\omega = \lambda$.

A strict contact manifold $(Y, \xi = \ker\alpha)$ is *strongly index-definite* if the contact structure $(\xi, \mathrm{d}\alpha)$ admits a symplectic trivialization ϵ with the property that there are constants $c > 0$ and $d \in \mathbb{R}$ such that for every Reeb chord $\gamma : [0,T] \to Y$ of length $T = \int_0^T \gamma^*\alpha$ we have

$$|\mu_{RS}(\gamma;\epsilon)| \geq cT + d,$$

where μ_{RS} is the Robbin-Salamon index [RS93].

For a Liouville domain $(W, \omega = d\lambda)$ with Liouville field V_λ, choose the collar coordinate induced by the negative Liouville flow near the boundary:

$$(1-\eta, 1] \times \partial W \hookrightarrow W, \qquad \lambda = r\alpha, \qquad \alpha = \lambda|_{\partial W}.$$

For $0 < \rho \leq 1$, define the compact truncated core

$$W_\rho := W \setminus \big((\rho, 1] \times \partial W\big).$$

Thus $W_{1-\eta}$ is the part of W lying below the collar $(1-\eta, 1] \times \partial W$. The completion is defined as

$$\widehat{W} = W \cup_{\partial W} ([1,\infty)_r \times \partial W), \qquad \widehat{\lambda} = \begin{cases} \lambda, & W, \\ r\alpha, & [1,\infty) \times \partial W. \end{cases}$$

Let $L \subset W$ be an exact spin Lagrangian with Legendrian boundary $\Lambda = \partial L \subset \partial W$. Its completion is

$$\widehat{L} = L \cup_{\partial L} ([1,\infty)_r \times \Lambda), \qquad \widehat{\lambda}|_{\widehat{L}} = df_L,$$

where f_L is constant for $r \gg 1$. We impose

$$2c_1(W) = 0, \qquad \mu_L = 0,$$

so that all chords are $\mathbb{Z}$-graded, where μ_L is the Maslov index of L.

A smooth Hamiltonian function $H : W \times S^1 \to \mathbb{R}$ gives rise to a Hamiltonian vector field X_{H_t} defined by

$$\iota_{X_{H_t}}\omega = -\mathrm{d}H_t,$$

where $H_t : W \to \mathbb{R}$ is given by $H_t(x) = H(t,x)$. Recall that τ is Hamiltonian if $\tau = \phi_H^1$, where ϕ_H^t is the Hamiltonian flow of some H_t.

Definition 1.2. We say that τ is a *Hamiltonian twist map* (with respect to α), if τ is generated by a smooth Hamiltonian $H : W \times \mathbb{R} \to \mathbb{R}$ which satisfies $X_{H_t}|_{\partial W} = h_t R_\alpha$ for some positive and smooth function $h : \partial W \times \mathbb{R} \to \mathbb{R}^+$.

For a Hamiltonian symplectomorphism $\tau : W \to W$ generated by Hamiltonian function H and a Lagrangian $L \subset W$, a Hamiltonian chord with respect to τ and L is a path

$$x : [0,1] \to W, \qquad \dot{x}(t) = X_{H_t}(x(t)), \qquad x(0), x(1) \in L.$$

For $m \geq 1$, write

$$H_t^{\#m}(z) = H_{t-\lfloor t \rfloor}(z), \qquad t \in [0,m].$$

A chord of $H^{\#m}$ is equivalently a point $q \in L$ with $\tau^m(q) \in L$, together with the concatenated trajectory. A chord y of $H^{\#m}$ has minimal order

$$\ell(y) = \min\{k \geq 1 \mid \tau^k(y(0)) \in L\}.$$

It is an *interior* chord if $y([0,\ell(y)]) \subset \operatorname{int} W$.

For $m \geq 1$, set

$$\mathcal{X}_m(H,L) = \{q \in L \cap \operatorname{int} W \mid \tau^m(q) \in L,\ \varphi_H^t(q) \in \operatorname{int} W \text{ for } 0 \leq t \leq m\}.$$

Thus $\mathcal{X}_m(H,L)$ is the set of starting points of compact interior m-chords. A point $q \in \mathcal{X}_m(H,L)$ has first-return order

$$\rho(q) = \min\{1 \leq k \leq m \mid \tau^k(q) \in L\}.$$

A finite sequence $C = (q_0, \dots, q_{s-1})$ is a *periodic first-return cycle* if

$$q_a \in L \cap \operatorname{int} W, \qquad r_a := \rho(q_a) < \infty, \qquad q_{a+1} = \tau^{r_a}(q_a), \qquad q_s = q_0,$$

where the index a is read modulo s. Its total period and action are

$$M_C = \sum_{a=0}^{s-1} r_a, \qquad A_C = \sum_{a=0}^{s-1} \mathcal{A}_{H^{\#r_a}}(q_a).$$

Equivalently, the concatenated path

$$\gamma_C(t) = \varphi_H^t(q_0), \qquad 0 \leq t \leq M_C,$$

is an isolated fixed L-chord of the period-M_C Hamiltonian $H^{\#M_C}$, and $\mathcal{A}_{H^{\#M_C}}(\gamma_C) = A_C$.

In this article, we show a relative Poincaré-Birkhoff type result in the following setting.

Definition 1.3 (Smooth relative Poincaré–Birkhoff datum)**.** A tuple (W, λ, L, H) is a *smooth relative Poincaré–Birkhoff datum* if:

(D1) (W, λ) satisfy $2c_1(W) = 0$ and $(\partial W, \alpha)$ is strongly index-definite; L is exact and spin with $\mu_L = 0$.

(D2) There is a collar $[1-\eta, 1] \times \partial W \subset W$ on which

$$H_t(r, y) = h_t(r), \qquad h_t'(r) > 0.$$

Equivalently, near ∂W,

$$X_{H_t} = h_t'(r) R_\alpha,$$

so the boundary dynamics is a positive Reeb-type twist.

(D3) For every integer $m \geq 1$, the compact interior return set $\mathcal{X}_m(H, L)$ is finite, and every point of $\mathcal{X}_m(H, L)$ is an isolated Hamiltonian m-chord.

Theorem 1.4. *Let (W, λ, L, H) be a smooth relative Poincaré–Birkhoff datum. If*

$$\mathrm{HW}^*(L, L; \mathbb{F}) \neq 0$$

for some base field $\mathbb{F}$, then $\tau = \varphi_H^1$ has interior L-chords of arbitrarily large minimal order.

We give a few comments on the conditions that we impose in our main theorem.

Remark 1.5.
1. The condition $2c_1(W) = 0$ and $\mu_L = 0$ is to assure that there is a well-defined integral grading on the (local) wrapped Floer homology. One can also instead use the condition $c_1(W)|_{\pi_2(W)} = 0$ as in [LM24].
2. The strongly index-definite condition is used to control exterior generators in the admissible extension and to keep the compact part of the wrapped complex isolated from the cylindrical end. By [MvK22, Lemma D.1], if $W \subset \mathbb{R}^4$ is a convex domain, then $(\partial W, \alpha_{\mathrm{std}})$ is strongly index-definite.
3. The generating Hamiltonian H of τ can be required to be only C^2. We assume the smoothness here instead, just for the sake of brevity. See [LM25] for discussions on Poincaré–Birkhoff theorem for C^0-Hamiltonian maps.

The main improvement suggested by Theorem 1.4, relative to [LM24], is that one may replace their assumption $\dim \mathrm{HW}^*(L, L) = \infty$ by $\mathrm{HW}^*(L, L) \neq 0$. We work with the unit of the wrapped Floer homology and produce an open-string symplectically degenerate minimum(SDM) under the opposite hypothesis of existence of only finitely many chords. Then we use the open-string SDM action-gap mechanism of Section 4.4 to conclude contradiction. A more detailed outline of the proof will be presented in the next section. It is worth noting that there is no known examples of Lagrangian with $0 < \dim \mathrm{HW}^*(L, L) < \infty$.

1.2. **Outline of the proof.** The proof is based on a contradiction argument. Assume temporarily that all compact interior L-chords have minimal order at most N. Define

$$S_N = \bigcup_{1 \le k \le N} X_k(H, L).$$

By (D3), S_N is finite. For $q \in S_N$, let

$$\rho(q) = \min\{1 \le k \le N \mid \tau^k(q) \in L\}, \qquad F(q) = \tau^{\rho(q)}(q).$$

The finite first-return graph $\mathcal{G}_N$ has vertex set S_N and an edge $e_q : q \to F(q)$ whenever $F(q) \in S_N$, with weights

$$T(e_q) = \rho(q), \qquad A(e_q) = \mathcal{A}_{H^{\#\rho(q)}}(q).$$

For a directed path P, write $T(P)$ and $A(P)$ for the sums of the edge weights. A pigeonhole argument(see Lemmalem:affinegraphpaths) assures one to write any sufficiently long path in the finite first-return graph in a normal form.

An *eventual type* is a triple

$$\Theta = (U, C, V), \tag{*}$$

where C is a directed cycle, U is an entry path, and V is a proper initial subpath of C. We set

$$P_{\Theta,r} = UC^rV, \qquad m_{\Theta,r} = T(P_{\Theta,r}), \qquad A_{\Theta,r} = A(P_{\Theta,r}).$$

The contradiction argument is based on analyzing the action and index of these eventual types. The logical flowline is given as follow.

$$\mathrm{HW}^*(L, L) \neq 0 \xrightarrow[\text{unit algebra}]{\text{Theorem 1.6}} 1_L \neq 0 \in \mathrm{HW}^0(L, L),$$

$$1_L \neq 0 \text{ and bounded returns} \xRightarrow[\text{finite graph + mean index}]{\text{Proposition 1.10}} \text{a tail-stabilized SDM carrier } \Theta,$$

$$\text{tail-stabilized SDM carrier } \Theta \xRightarrow[\text{Section 4 action-gap package}]{\text{Theorem 1.11}} \text{a chord in the window } (A_{\Theta,r} + \delta, A_{\Theta,r} + \varepsilon),$$

$$\text{finite graph separation} \xRightarrow{\text{Lemma 1.12}} \text{this chord has minimal order } > N.$$

The first step is the unit reduction.

Theorem 1.6 (Unit reduction)**.** *For wrapped Floer cohomology over a field,*

$$\mathrm{HW}^*(L,L;\mathbb{F})\neq 0 \Longleftrightarrow 1_L\neq 0\in \mathrm{HW}^0(L,L;\mathbb{F}).$$

Moreover, after passing to an admissible iterate $K_a^{\#m}$, the unit has a degree-zero cocycle representative

$$c_m\in \mathrm{CW}^0(K_a^{\#m};\widehat{L},\widehat{L}),\qquad dc_m=0,\qquad [c_m]=1_L, \tag{1.1}$$

and, under a bounded-return contradiction hypothesis, this representative may be chosen with support in a compact subset of W depending only on the return bound.

The algebraic equivalence is Lemma 2.1; the chain-level compact representative is given in Lemma 4.1.

The second step is the local mean-index support estimate which would give rise to a SDM.

We first focus on the pure-iterate case. Let $x\subset \operatorname{int} W$ be an isolated interior Hamiltonian L-chord. When its iterates are considered, we fix a Maslov-zero relative symplectic trivialization along the iterated chord tower and define

$$\Delta_L(x)=\lim_{k\to\infty}\frac{1}{k}\,\mu_{\mathrm{RS}}\big(\Psi^{\#k}\Lambda_0,\Lambda_0\big),$$

where $\Psi(t)=D\varphi_H^t(x(0))$ in the chosen trivialization and $\Lambda_0=T_{x(0)}L$. The limit exists by the homogenization property of the Robbin–Salamon/Maslov index [CLM94, PR14].

Definition 1.7 (Open-string SDM)**.** The chord x is an *open-string SDM* if

$$\Delta_L(x)=0,\qquad \mathrm{HW}^0_{\mathrm{loc}}(H,L;x)\neq 0. \tag{1.2}$$

Theorem 1.8 (Admissible iteration and mean-index alternative)**.** *Let x be an isolated fixed interior L-chord. For every sufficiently large relatively admissible prime p,*

$$\mathrm{HW}^r_{\mathrm{loc}}(H^{\#p},L;x^p)\cong \mathrm{HW}^{r+s_p}_{\mathrm{loc}}(H,L;x),\qquad \frac{s_p}{p}\longrightarrow \Delta_L(x),$$

and there is a constant $C_x>0$, independent of p, such that

$$\operatorname{supp}\mathrm{HW}^*_{\mathrm{loc}}(H^{\#p},L;x^p)\subset[-p\Delta_L(x)-C_x,-p\Delta_L(x)+C_x]\cap\mathbb{Z}.$$

Consequently, if $\mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p)\neq 0$ for infinitely many admissible primes p, then $\Delta_L(x)=0$. If in addition $\mathrm{HW}^0_{\mathrm{loc}}(H,L;x)=0$, then $\mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p)=0$ for all sufficiently large admissible primes p.

The proof is given in Section 3. It is obtained by doing a type-II finite-dimensional reduction of the Lagrangian boundary-value problem.

Now we consider the finite graph setting. For an eventual type $\Theta = (U, C, V)$, the cycle C determines a fixed L-chord γ_C of $H^{\#T(C)}$. Its Lagrangian mean index is denoted

$$\Delta_\Theta := \Delta_L^{H^{\#T(C)}}(\gamma_C).$$

The tails U, V affect Robbin–Salamon indices only by a bounded correction, and hence do not affect this homogenized limit.

Definition 1.9 (Tail-stabilized SDM carrier)**.** An eventual type $\Theta = (U, C, V)$ is a *tail-stabilized SDM carrier* if there exists an unbounded sequence $r_i \to \infty$ such that

$$\Delta_\Theta = 0, \qquad \mathrm{HW}^0_{\mathrm{loc}}\big(H^{\#m_{\Theta,r_i}}, L; y_{\Theta,r_i}\big) \neq 0,$$

where y_{Θ,r_i} is the Hamiltonian L-chord represented by the graph path P_{Θ,r_i}.

The contradiction hypothesis would yield the existence of a such SDM carrier.

Theorem 1.10 (Unit carrier forced by bounded return)**.** *Assume that every compact interior L-chord has minimal order at most N. Then there exists a tail-stabilized SDM carrier $\Theta_0 = (U_0, C_0, V_0)$ in $\mathcal{G}_N$.*

This is proved in Section 5.2. Its proof uses the mean-index estimate Theorem 1.8 and the finite dimensional reduction scheme under the graph setting.

Then we need to establish the tail-stabilized SDM action gap, which is an analogue of the action-gap result in [GG10]. An analytic package, including collar exclusion(Proposition 4.3), filtered type-II comparison(Proposition 4.4), finite-dimensional boundary depth(Theorem 4.5), and local-to-global injection(Proposition 4.7), is built in Section 4. Section 5 applies this package to the whole word UC^rV, not to a pure iterate.

Theorem 1.11. *Let $\Theta = (U, C, V)$ be a tail-stabilized SDM carrier. Then there exists $\varepsilon_\Theta > 0$ such that, for every $0 < \varepsilon < \varepsilon_\Theta$ and every sufficiently large carrier integer r_i, there are $\delta_i \in (0, \varepsilon)$ and a compact interior chord z_i of $H^{\#m_{\Theta,r_i}}$ satisfying*

$$A_{\Theta,r_i} + \delta_i < \mathcal{A}_{H^{\#m_{\Theta,r_i}}}(z_i) < A_{\Theta,r_i} + \varepsilon. \tag{1.3}$$

However, a calculation would imply that the graph has no path in this one-sided action window, which would lead to a contradiction. Proof of this lemma is presented in Section 5.3.

Lemma 1.12. *Fix an eventual type* $\Theta_0 = (U_0, C_0, V_0)$*. There exist* $\varepsilon_0 > 0$ *and* $R_0 \geq 1$ *such that, for every* $r \geq R_0$ *and every directed path* P *in* $\mathcal{G}_N$ *satisfying*

$$T(P) = T(P_{\Theta_0,r}),$$

one has

$$A(P) \notin \big(A(P_{\Theta_0,r}), A(P_{\Theta_0,r}) + \varepsilon_0\big). \tag{1.4}$$

Consequently no compact chord with minimal order at most N *and total time* $T(P_{\Theta_0,r})$ *can have action in*

$$\big(A(P_{\Theta_0,r}) + \delta, A(P_{\Theta_0,r}) + \varepsilon\big)$$

whenever $0 < \delta < \varepsilon < \varepsilon_0$*.*

The paper is organized as follows. In Section 2 we review the construction of global and local wrapped Floer theory. Section 3 is devoted to prove Theorem 1.8. Section 4 proves the compact-unit reduction, thus Theorem 1.6, and the filtered action-gap package. In Section 5 we build the finite first-return graph $\mathcal{G}_N$, and apply the analytic package established in Section 4 prove the remaining theorems, thus finally proving Theorem 1.4.

Acknowledgments.

2. Preliminaries to (local) wrapped Floer homology

2.1. **Global wrapped Floer cohomology.** In this section we review the standard wrapped Floer theory for exact Lagrangians, developed by Abouzaid–Seidel [AS10], Ritter [Rit13], and Ganatra–Pardon–Shende [GPS20]. The analytic inputs used below are the standard no-escape/maximum-principle and elliptic-regularity estimates for Cauchy–Riemann type operators with totally real boundary conditions; we cite [FHS95, MS12, GT77] for these tools.

Let $(\widehat{W}, \widehat{\lambda})$ be the completion of the Liouville domain and let $\widehat{L}$ be the exact cylindrical completion of L. A Hamiltonian $K\colon [0,1] \times \widehat{W} \to \mathbb{R}$ is called *admissible* if, outside a compact set,

$$K_t(r, y) = ar + b_t(y), \qquad a \notin \operatorname{Spec}(\partial L, \alpha), \tag{2.1}$$

possibly after the usual convex interpolation from the compact core to the linear end. The non-resonance condition in (2.1) excludes time-one Hamiltonian chords lying entirely in the cylindrical end.

For a generic admissible pair (K, J), with J contact type on the end, the wrapped Floer cochain module is

$$\mathrm{CW}^*(K; L, L) = \bigoplus_{x \in \mathcal{X}(K;L,L)} \mathbb{F} \cdot x,$$

where

$$\mathcal{X}(K; L, L) = \{x \colon [0,1] \to \widehat{W} \mid \dot{x} = X_{K_t}(x),\ x(0), x(1) \in \widehat{L}\}.$$

The integral grading is well-defined since $2c_1(W) = 0$ and $\mu_L = 0$. If $\Psi_x(t)$ denotes the linearized Hamiltonian path along x in a Maslov-zero trivialization and $\Lambda = T_{x(0)}L$, then

$$\mathrm{gr}(x) = n - \mu_{\mathrm{RS}}(\Psi_x(t)\Lambda, \Lambda) + \kappa_L,$$

where the constant κ_L is chosen so that the small-Hamiltonian Morse minimum representing the unit has degree 0.

The differential is

$$d_K x_+ = \sum_{\substack{x_- \in \mathcal{X}(K;L,L) \\ \mathrm{gr}(x_-) = \mathrm{gr}(x_+)+1}} \#\mathcal{M}^0_K(x_-, x_+)\, x_-,$$

where $\mathcal{M}^0_K(x_-, x_+)$ is the oriented zero-dimensional quotient of the moduli space of Floer strips from x_- to x_+. Then we have

$$z \subset d_K x \quad \Longrightarrow \quad \mathrm{gr}(z) = \mathrm{gr}(x) + 1.$$

The wrapped Floer homology is defined as

$$\mathrm{HW}^*(K; L, L) = H^*(\mathrm{CW}^*(K; L, L), d_K).$$

For two Hamiltonians $K^0 \le K^1$, for a monotone homotopy from K^0 to K^1, the continuation map

$$\Phi_{01} \colon \mathrm{CW}^*(K^0; L, L) \longrightarrow \mathrm{CW}^*(K^1; L, L)$$

is defined by counting continuation Floer strips, namely the solutions of

$$\begin{cases} \partial_s u + J_{s,t}(u)(\partial_t u - X_{K_{s,t}}(u)) = 0, \\ u(s,0), u(s,1) \in \widehat{L}, \\ \lim\limits_{s\to-\infty} u(s,t) = x_-(t), \qquad \lim\limits_{s\to+\infty} u(s,t) = x_+(t), \end{cases}$$

where $J_{s,t}$ is compatible and of contact type on the end.

For any cofinal increasing sequence of admissible Hamiltonians K^ν with slopes $a_\nu \to +\infty$, wrapped Floer cohomology is the direct limit

$$\mathrm{HW}^*(L,L;\mathbb{F}) := \varinjlim_{\nu} \mathrm{HW}^*(K^\nu; L, L; \mathbb{F}).$$

It is independent of the cofinal sequence and carries the product

$$\mu^2\colon\ \mathrm{HW}^i(L,L) \otimes \mathrm{HW}^j(L,L) \longrightarrow \mathrm{HW}^{i+j}(L,L)$$

with unit $1_L \in \mathrm{HW}^0(L,L)$.

Now we consider the action filtration, for a chord x of K, define the cohomological action by

$$\mathcal{A}_K(x) = \int_0^1 x^*\widehat{\lambda} - \int_0^1 K_t(x(t))\,dt + f_L(x(0)) - f_L(x(1)).$$

For a chord of $K^{\#m}$ on $[0,m]$, the same formula gives

$$\mathcal{A}_{K^{\#m}}(x) = \int_0^m x^*\widehat{\lambda} - \int_0^m K_t^{\#m}(x(t))\,dt + f_L(x(0)) - f_L(x(m)).$$

If x^m is the m-fold iterate of a fixed chord x, exactness of f_L gives the telescoping identity

$$\mathcal{A}_{K^{\#m}}(x^m) = m\mathcal{A}_K(x).$$

Indeed, if $q_j = \tau^j(q_0) \in L$, then

$$\sum_{j=0}^{m-1} \big(f_L(q_j) - f_L(q_{j+1})\big) = f_L(q_0) - f_L(q_m),$$

and for a fixed chord $q_m = q_0$ as a point of the chord correspondence.

Consider a continuation strip u connecting x_-, x_+ with respect to the Hamiltonians $K_{-\infty}, K_{+\infty}$ connected by homotopy $K_{s,t}$, a standard computation gives

$$E(u) = \int_{\mathbb{R}\times[0,1]} \|\partial_s u\|^2_{J_{s,t}}\,dsdt = \mathcal{A}_{K_{-\infty}}(x_-) - \mathcal{A}_{K_{+\infty}}(x_+) + \int_{\mathbb{R}\times[0,1]} \partial_s K_{s,t}(u)\,dsdt. \tag{2.2}$$

Especially for $K = K_{-\infty} = K_{+\infty}$ there is

$$E(u) = \mathcal{A}_K(x_-) - \mathcal{A}_K(x_+) \geq 0. \tag{2.3}$$

Thus d raises the action and raises cohomological degree by one. The decreasing action filtration is

$$F^a \operatorname{CW}^*(K) = \bigoplus_{\mathcal{A}_K(x) \geq a} \mathbb{F} \cdot x, \qquad d(F^a \operatorname{CW}^*(K)) \subset F^a \operatorname{CW}^*(K).$$

Lemma 2.1 (The unit criterion). *For wrapped Floer cohomology over a field,*

$$\operatorname{HW}^*(L, L; \mathbb{F}) \neq 0 \quad \Longleftrightarrow \quad 1_L \neq 0 \in \operatorname{HW}^0(L, L; \mathbb{F}).$$

Proof. Wrapped Floer cohomology is a unital graded algebra:

$$\mu^2 \colon \operatorname{HW}^i(L, L) \otimes \operatorname{HW}^j(L, L) \to \operatorname{HW}^{i+j}(L, L), \qquad \mu^2(1_L, a) = a = \mu^2(a, 1_L).$$

If $1_L = 0$, then every $a \in \operatorname{HW}^*(L, L)$ satisfies

$$a = \mu^2(1_L, a) = \mu^2(0, a) = 0.$$

Hence $\operatorname{HW}^*(L, L) = 0$. Conversely, if $\operatorname{HW}^*(L, L) = 0$, then in particular $\operatorname{HW}^0(L, L) = 0$, so $1_L = 0$. . □

2.2. **Local wrapped Floer cohomology.** Now we explain local wrapped Floer cohomology, firstly introduced by Limoge–Moreno [LM24, Sec. 3–4].

Let x be an isolated chord of K. Choose an isolating neighborhood $U \Subset \widehat{W}$ of $x([0, 1])$ such that no other K-chord is contained in U. Choose $U' \Subset U$ with $x([0, 1]) \subset U'$.

A perturbation (K', J') is *U-small* with respect to δ_U if

$$\operatorname{supp}(K' - K) \subset U, \qquad \|K' - K\|_{C^2(U)} + \|J' - J\|_{C^1(U)} < \delta_U, \tag{2.4}$$

all chords in U are nondegenerate, and no new chord crosses ∂U. The local chain module is

$$\operatorname{CW}^*_{\text{loc}}(K, L; x) = \bigoplus_{y \in \mathcal{X}(K', L; U)} \mathbb{F} \cdot y.$$

The differential counts only strips whose images are contained in U:

$$d_{\text{loc}} y_+ = \sum_{y_-} \#\mathcal{M}_U(y_-, y_+; K', J')\, y_-.$$

We lay out the basic properties of the above constructed local wrapped Floer thoery.

Proposition 2.2. *The followings hold.*

(1) *For sufficiently small choices of δ_U in (2.4), the local differential satisfies*

$$d_{\mathrm{loc}}^2 = 0,$$

and

$$\mathrm{HW}^*_{\mathrm{loc}}(K, L; x) := H^*(\mathrm{CW}^*_{\mathrm{loc}}(K, L; x), d_{\mathrm{loc}})$$

is independent of all sufficiently small local perturbations.

(2) *Moreover, if an action window contains exactly the local cluster associated to finitely many isolated chords, the corresponding filtered quotient of the global wrapped complex has first-page cohomology equal to the direct sum of these local groups:*

$$H^*\big(F^a\,\mathrm{CW}^*(K)/F^b\,\mathrm{CW}^*(K)\big) \cong \bigoplus_{a \le \mathcal{A}_K(x) < b} \mathrm{HW}^*_{\mathrm{loc}}(K, L; x),$$

provided no other action values lie in $[a, b)$.

(3) *Continuation maps between local perturbations are canonical up to chain homotopy and compatible with the global continuation maps on filtered quotients.*

The proof is provided in [LM24, Sec. 3–4]. We spell out only the quantitative energy-separation estimate since the constant ε_U is used later in the SDM action-gap injection.

Lemma 2.3 (Quantitative energy separation). *There exist $\delta_U > 0$ and $\varepsilon_U > 0$ such that every U-small regular perturbation and every Floer strip u with*

$$\mathrm{im} u \cap U' \neq \varnothing, \qquad \mathrm{im} u \not\subset U$$

satisfies

$$E(u) \geq \varepsilon_U.$$

Proof. The proof is given in [LM24, Lemma 3.2] and we provide more details here. One can also see [MS12] for reference.

Assume that no positive separation constant exists. Then there are U-small regular perturbations $(K_\nu, J_\nu) \to (K, J)$, Floer strips u_ν, and points (s_ν, t_ν) such that

$$u_\nu(s_\nu, t_\nu) \in U', \qquad \mathrm{im} u_\nu \not\subset U, \qquad E(u_\nu) = \|\partial_s u_\nu\|^2_{L^2} \to 0.$$

Translate in the s-variable and assume $s_\nu = 0$. Since

$$\partial_t u_\nu - X_{K_\nu, t}(u_\nu) = -J_{\nu,t}(u_\nu)\partial_s u_\nu,$$

we also have

$$\|\partial_t u_\nu - X_{K_\nu,t}(u_\nu)\|_{L^2([-R,R]\times[0,1])} \to 0 \qquad (R<\infty).$$

We record explicitly the elliptic estimate used here. Cover $\overline{U}$ by finitely many Darboux charts in which $\widehat{L}$ is sent to $\mathbb{R}^n \subset \mathbb{C}^n$ at the boundary. After flattening the totally real boundary condition, the Floer equation has the form

$$\mathcal{D}_\nu v := \partial_s v + J_0\partial_t v + B_\nu(s,t)v = f_\nu$$

on a half-disc or full disc, where B_ν is uniformly bounded in C^k because (K_ν, J_ν) are C^{k+1}-bounded. The boundary estimate is obtained by Schwarz reflection [Ahl53] after choosing coordinates with boundary condition $v(\partial\mathbb{H}) \subset \mathbb{R}^n$, or equivalently by the standard Calderon–Zygmund [CZ52, GT77] estimate for Cauchy–Riemann operators with totally real boundary condition:

$$\|v\|_{W^{k+1,q}(Q_{1/2})} \le C_{k,q}\big(\|\mathcal{D}_\nu v\|_{W^{k,q}(Q_1)} + \|v\|_{L^q(Q_1)}\big). \tag{2.5}$$

The lower-order term satisfies

$$\|B_\nu v\|_{W^{k,q}} \le C\|v\|_{W^{k,q}},$$

and is absorbed inductively after applying (2.5) on smaller cylinders. Therefore, for each compact cylinder $Q_R = [-R,R]\times[0,1]$,

$$\|u_\nu\|_{W^{k+1,q}(Q_{R-1})} \le C_{k,q,R}\Big(\|\partial_s u_\nu\|_{W^{k,q}(Q_R)} + \|\partial_t u_\nu - X_{K_\nu}(u_\nu)\|_{W^{k,q}(Q_R)} + 1\Big).$$

Bootstrapping the Floer equation yields uniform C^ℓ-bounds on every Q_R. Hence a subsequence converges in C^∞_{loc} to a map u_∞ solving the unperturbed equation. Moreover

$$\partial_s u_\infty = 0, \qquad \partial_t u_\infty = X_{K_t}(u_\infty).$$

Thus

$$u_\infty(s,t) = z(t), \qquad \dot z = X_{K_t}(z), \qquad z(0), z(1) \in \widehat{L},$$

so z is a K-chord. Since $u_\nu(0,t_\nu) \in U'$, after passing to a subsequence $t_\nu \to t_*$ and

$$z(t_*) = \lim_{\nu\to\infty} u_\nu(0,t_\nu) \in \overline{U}'.$$

Choose (σ_ν, θ_ν) with $u_\nu(\sigma_\nu,\theta_\nu) \in \partial U$. If σ_ν is bounded, the same C^∞_{loc}-convergence gives

$$z(\theta_*) \in \partial U$$

for some θ_*. If $|\sigma_\nu| \to \infty$, use the exponential asymptotic estimate at a nondegenerate chord $x_\nu^\pm$: for some $\delta > 0$,

$$\|u_\nu(s,\cdot) - x_\nu^\pm\|_{C^1([0,1])} \le C_\nu e^{-\delta|s|} \quad (\pm s \gg 1). \tag{2.6}$$

The chords $x_\nu^\pm$ converge, after a subsequence, to K-chords $x^\pm$. Since $u_\nu(\sigma_\nu, \theta_\nu) \in \partial U$, (2.6) forces one limiting chord $x^\pm$ to meet ∂U. In either case we obtain a K-chord meeting $\overline{U}'$ and not contained in U, contradicting the isolating choice of U. Therefore a positive ε_U exists. □

3. Finite-dimensional reduction and admissible prime iteration

This section is devoted to build a local computational model for the local wrapped Floer homology:

$$\mathrm{HW}^*_{\mathrm{loc}}(H, L; x) \cong HM^{*+\sigma_x}_{\mathrm{loc}}(S_x, 0),$$

where S_x is a discrete generating-function germ for the Hamiltonian chord. Then we will prove the admissible-iteration formula in Theorem 1.8 by reducing the admissible-iteration problem to a statement about the Hessians of S_x and S_{x^p}.

3.1. **Finite-dimensional local Morse reduction.** We use local Morse cohomology in the sense of the critical groups of an isolated critical point; equivalently it is the cohomology of a sufficiently small Morse perturbation inside an isolating neighborhood. This finite-dimensional object is invariant under isolated homotopies and under stabilization by a nondegenerate quadratic form, as in the classical local Morse theory of Gromoll–Meyer and Chang [GM69, Cha93]; see also Milnor [Mil63] for the Morse-theoretic conventions.

Let x be an isolated fixed L-chord of H. After a local Hamiltonian change of coordinates preserving L at the endpoints, assume

$$x(t) \equiv 0, \qquad L = \mathbb{R}^n \subset T^*\mathbb{R}^n, \qquad \omega_0 = \sum_{i=1}^n dq_i \wedge dp_i.$$

Let

$$\Psi(t) = D\varphi_H^t(0), \qquad \Lambda_0 = T_0 L = \mathbb{R}^n \times \{0\}.$$

The linearized chord equation has Lagrangian boundary condition

$$\dot{\xi}(t) = J_0 S(t) \xi(t), \qquad \xi(0), \xi(1) \in \Lambda_0,$$

where $S(t) = S(t)^T$. The return relation is encoded by

$$\xi(1) = \Psi(1)\xi(0), \qquad \xi(0), \xi(1) \in \Lambda_0.$$

Thus the degeneracy space is

$$E_1 = \Lambda_0 \cap \Psi(1)^{-1}\Lambda_0.$$

For the k-fold iterate,

$$E_k = \Lambda_0 \cap \Psi(k)^{-1}\Lambda_0.$$

Definition 3.1 (Relative admissibility)**.** A positive integer k is *relatively admissible* for x if

$$\dim E_k = \dim E_1.$$

Equivalently, in the reduced return map on the quotient by the fixed degeneracy space, no eigenvalue $\zeta \neq 1$ with $\zeta^k = 1$ occurs.

Only finitely many primes fail relative admissibility for any fixed isolated chord: such a prime must divide the order of a root-of-unity eigenvalue of the reduced return relation. For finitely many fixed chords, one excludes the union of finitely many bad primes.

Lemma 3.2 (Short-time type-II transversality)**.** *There exist a neighborhood $B \subset T^*\mathbb{R}^n$ of 0 and an integer N_0 such that for every $N \geq N_0$ and every*

$$t_j = j/N, \qquad \Phi_j := \phi_H^{t_{j+1},t_j}, \qquad j = 0, \dots, N-1,$$

the graph

$$\Gamma_j = \{(q, p, Q, P) \mid (Q, P) = \Phi_j(q, p)\} \subset \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n$$

is transverse to the type-II polarization

$$\pi_{II}(q, p, Q, P) = (q, P).$$

Equivalently,

$$T\Gamma_j \cap \ker d\pi_{II} = 0. \tag{3.1}$$

Consequently Γ_j admits a local type-II generating function

$$F_j(q_j, \eta_{j+1}), \qquad \eta_{j+1} = p_{j+1} = P,$$

satisfying

$$p_j = \partial_{q_j} F_j(q_j, \eta_{j+1}), \qquad q_{j+1} = \partial_{\eta_{j+1}} F_j(q_j, \eta_{j+1}). \tag{3.2}$$

Proof. Write

$$\Phi_j(q,p) = \big(Q_j(q,p), P_j(q,p)\big).$$

Let $A_j(s;z) = D\phi_H^{t_j+s,t_j}(z)$ for $0 \le s \le 1/N$. The variational equation is

$$\frac{d}{ds}A_j(s;z) = DX_{H_{t_j+s}}\big(\phi_H^{t_j+s,t_j}(z)\big)A_j(s;z), \qquad A_j(0;z) = I.$$

Choose B so small that all short trajectories starting in B remain in a fixed compact set B' on which

$$M := \sup_{t\in[0,1],\, z\in B'} \|DX_{H_t}(z)\| < \infty.$$

Gronwall's inequality [Gro19] gives

$$\|D\Phi_j - I\| = \|A_j(1/N;z) - I\| \le e^{M/N} - 1 \le CN^{-1},$$

uniformly in j and $z \in B$. In block form,

$$D\Phi_j = \begin{pmatrix} \partial_q Q_j & \partial_p Q_j \\ \partial_q P_j & \partial_p P_j \end{pmatrix}, \qquad I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Thus

$$\|\partial_p P_j - I\| \le CN^{-1}.$$

For $N \ge N_0$ we have $\|\partial_p P_j - I\| < 1/2$, hence

$$\det(\partial_p P_j) \neq 0. \tag{3.3}$$

The restriction of π_{II} to Γ_j is

$$(q,p) \longmapsto (q, P_j(q,p)),$$

with differential

$$D(\pi_{II}|_{\Gamma_j}) = \begin{pmatrix} I & 0 \\ \partial_q P_j & \partial_p P_j \end{pmatrix}. \tag{3.4}$$

By (3.3), this matrix is invertible. Since $\dim \Gamma_j = \dim(q,P) = 2n$, invertibility of (3.4) is equivalent to (3.1).

It remains to construct the generating function. On $\overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n$ the product symplectic form is

$$-dp \wedge dq + dP \wedge dQ = d(p\,dq + Q\,dP).$$

The graph Γ_j is Lagrangian because Φ_j is symplectic; hence the one-form

$$p\,dq + Q\,dP$$

restricts to a closed one-form on Γ_j. Since the discussion is local near 0, the restriction is exact. Using the coordinates $(q,P)=(q_j,\eta_{j+1})$ on Γ_j, there is a smooth function $F_j(q_j,\eta_{j+1})$ with

$$dF_j = p_j\,dq_j + q_{j+1}\,d\eta_{j+1},$$

which is exactly (3.2). ☐

Now we show a correspondence between Hamiltonian chords and critical points of certain action.

Lemma 3.3. *For $N \geq N_0$ define*

$$\mathcal{S}(q_0,\dots,q_N,\eta_1,\dots,\eta_N) = \sum_{j=0}^{N-1} \left(F_j(q_j,\eta_{j+1}) - q_{j+1}\cdot\eta_{j+1}\right). \tag{3.5}$$

Then critical points of $\mathcal{S}$ in a sufficiently small neighborhood of 0 are in one-to-one correspondence with Hamiltonian L-chords near x.

Proof. The variables are

$$q_0,\dots,q_N \in \mathbb{R}^n, \qquad \eta_1,\dots,\eta_N \in (\mathbb{R}^n)^*.$$

Differentiating (3.5) gives

$$\partial_{q_0}\mathcal{S} = 0 \iff \partial_{q_0}F_0(q_0,\eta_1) = 0 \iff p_0 = 0, \tag{3.6}$$

$$\partial_{\eta_{j+1}}\mathcal{S} = 0 \iff q_{j+1} = \partial_{\eta_{j+1}}F_j(q_j,\eta_{j+1}), \qquad 0 \leq j \leq N-1, \tag{3.7}$$

$$\partial_{q_j}\mathcal{S} = 0 \iff \eta_j = \partial_{q_j}F_j(q_j,\eta_{j+1}), \qquad 1 \leq j \leq N-1, \tag{3.8}$$

$$\partial_{q_N}\mathcal{S} = 0 \iff \eta_N = 0. \tag{3.9}$$

By (3.2), equation (3.7) says that the q-component of the endpoint of the j-th short segment is q_{j+1}, while (3.8) says that the corresponding momentum is η_j. Therefore the short segments match to form one Hamiltonian trajectory. The endpoint equations (3.6) and (3.9) are

$$p_0 = 0, \qquad p_N = 0,$$

which are exactly the local equations for the boundary condition $x(0), x(1) \in L = \mathbb{R}^n \times 0$. Thus

$$\mathrm{Crit}(\mathcal{S}) \cong \{\text{Hamiltonian } L\text{-chords near } x\}.$$

☐

Now we can connect local wrapped Floer homology to local Morse homology.

Proposition 3.4 (Local Floer–Morse reduction)**.** *Up to an absolute grading shift determined by the chosen grading of L, the local wrapped Floer complex of x is chain equivalent to the local Morse complex of $\mathcal{S}$:*

$$CW^*_{\mathrm{loc}}(H, L; x) \simeq CM^{*+\sigma_1}_{\mathrm{loc}}(\mathcal{S}, 0). \tag{3.10}$$

For the k-fold iterate, the corresponding finite-dimensional action is

$$\mathcal{S}^{(k)} = \sum_{a=0}^{k-1} \sum_{j=0}^{N-1} \left(F_j(q_{aN+j}, \eta_{aN+j+1}) - q_{aN+j+1} \cdot \eta_{aN+j+1} \right). \tag{3.11}$$

Proof. Fix an isolating neighborhood $U \Subset W$ of the chord x and choose $U' \Subset U$ containing $x([0,1])$. We first specify the local Banach problem which is being reduced. Let $\ell \geq 1$ and $q > 2$. In the Darboux–Weinstein chart used above put

$$\mathcal{P}^{\ell+1,q}_U := \{\gamma \in W^{\ell+1,q}([0,1], U) \mid \gamma(0), \gamma(1) \in \mathbb{R}^n \times 0\}.$$

The Hamiltonian action on this path space is

$$\mathcal{A}_H(\gamma) = \int_0^1 \gamma^* \lambda_0 - \int_0^1 H_t(\gamma(t))\, dt + f_L(\gamma(0)) - f_L(\gamma(1)).$$

Since $\lambda_0|_{\mathbb{R}^n \times 0} = df_L$ in the chosen exact chart, the first variation is

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega_0(\dot{\gamma} - X_{H_t}(\gamma), \xi)\, dt, \qquad \xi(0), \xi(1) \in T\mathbb{R}^n. \tag{3.12}$$

Thus the critical points of $\mathcal{A}_H$ are exactly the local Hamiltonian L-chords. For a regular U-small perturbation the local Floer differential is the Morse differential of $\mathcal{A}_H$ for the L^2-metric determined by J_t, namely it counts solutions of

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \qquad u(s,0), u(s,1) \in \mathbb{R}^n \times 0, \tag{3.13}$$

whose images remain in U. Therefore it is enough to identify, inside the isolating block, the local Morse complex of the infinite-dimensional functional $\mathcal{A}_H$ with the local Morse complex of the discrete functional $\mathcal{S}$.

Let

$$Z := \mathbb{R}^{n(N+1)} \times ((\mathbb{R}^n)^*)^N, \qquad z = (q_0, \ldots, q_N, \eta_1, \ldots, \eta_N).$$

The type-II graphs of the short maps give a smooth finite-dimensional section

$$\mathfrak{m} : Z \longrightarrow (\mathbb{R}^n)^* \times \prod_{j=0}^{N-1} \mathbb{R}^n \times \prod_{j=1}^{N-1} (\mathbb{R}^n)^* \times (\mathbb{R}^n)^*, \tag{3.14}$$

whose components are

$$\begin{aligned}
\mathfrak{m}_0(z) &= \partial_{q_0} F_0(q_0, \eta_1), \\
\mathfrak{m}^q_{j+1}(z) &= q_{j+1} - \partial_{\eta_{j+1}} F_j(q_j, \eta_{j+1}), \qquad 0 \le j \le N-1, \\
\mathfrak{m}^p_j(z) &= \eta_j - \partial_{q_j} F_j(q_j, \eta_{j+1}), \qquad 1 \le j \le N-1, \\
\mathfrak{m}_N(z) &= \eta_N.
\end{aligned}$$

Here $\mathfrak{m} = 0$ is precisely the system (3.6)–(3.9). Moreover, differentiating (3.5) gives

$$d\mathcal{S}(z) = R(z)\mathfrak{m}(z), \tag{3.15}$$

where $R(z)$ is a smooth bundle isomorphism for z sufficiently close to 0. Indeed, in the coordinates displayed in Lemma 3.3, the coefficients of $d\mathcal{S}$ are exactly the matching defects above, up to the harmless signs coming from the convention for the variables η_j. Hence $d\mathcal{S}(z) = 0$ if and only if $\mathfrak{m}(z) = 0$, and Lemma 3.3 gives the bijection between critical points of $\mathcal{S}$ and local chords.

We now pass from the critical-point correspondence to the chain level. Let $\mathcal{V} \subset \mathcal{P}^{\ell+1,q}_U$ be the finite-dimensional family obtained by concatenating the N short Hamiltonian pieces encoded by z and smoothing the joins by a fixed cutoff. At $z = 0$ this family is tangent to the kernel of the linearized chord operator. Choose the L^2-orthogonal splitting along $\mathcal{V}$

$$T\mathcal{P}^{\ell+1,q}_U|_{\mathcal{V}} = T\mathcal{V} \oplus E^\perp, \qquad \Pi : T\mathcal{P}^{\ell+1,q}_U|_{\mathcal{V}} \to T\mathcal{V}. \tag{3.16}$$

Every path in a smaller neighborhood of x is written uniquely as $v + \xi$, with $v \in \mathcal{V}$ and $\xi \in E^\perp$. In these variables the Floer operator

$$\mathcal{F}(u) = \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u))$$

splits into

$$\Pi^\perp \mathcal{F}(v + \xi) = 0, \qquad \Pi\mathcal{F}(v + \xi) = 0. \tag{3.17}$$

The linearization of the first equation in the normal variable is a Cauchy–Riemann type operator with totally real boundary condition and with its finite-dimensional kernel removed. We claim that there is a constant C, independent of v in a smaller isolating neighborhood, such that

$$\|\xi\|_{W^{\ell+1,q}} \le C\|D_\xi(\Pi^\perp\mathcal{F})_{(v,0)}\xi\|_{W^{\ell,q}}, \qquad \xi \in E^\perp. \tag{3.18}$$

If (3.18) failed, there would be $v_u \to 0$ and $\xi_u \in E^\perp$ with $\|\xi_u\|_{W^{\ell+1,q}} = 1$ and $D_\xi(\Pi^\perp \mathcal{F})_{(v_u,0)}\xi_u \to 0$. The Calderon–Zygmund boundary estimate (2.5), applied after flattening the totally real boundary condition, gives compactness in $W^{\ell,q}$ and hence a nonzero limit in the kernel of the linearized chord operator. This limit is orthogonal to the finite-dimensional kernel by the choice of $E^\perp$, a contradiction. The estimate is stable under the C^1-small changes of the short-time charts; Lemma 3.2 supplies the required uniform smallness once the subdivision parameter N is fixed sufficiently large.

By (3.18) and the implicit function theorem, after shrinking the isolating neighborhood there is a unique smooth normal correction

$$\xi = \xi(v), \qquad \Pi^\perp \mathcal{F}(v + \xi(v)) = 0, \qquad \|\xi(v)\|_{W^{\ell+1,q}} \leq C\|\Pi^\perp \mathcal{F}(v)\|_{W^{\ell,q}}. \tag{3.19}$$

The remaining equation is finite-dimensional:

$$\mathfrak{R}(v) := \Pi\mathcal{F}(v + \xi(v)) = 0. \tag{3.20}$$

The derivative of the action along the corrected slice is computed using (3.12). Since (3.19) kills the normal component of the gradient, only the tangential component remains. Therefore, in the type-II coordinates z on $\mathcal{V}$,

$$d(\mathcal{A}_H|_{\mathcal{V}+\xi(\mathcal{V})})(z) = d\mathcal{S}(z) + O(|z|\,|d\mathcal{S}(z)|). \tag{3.21}$$

After shrinking once more, the error term is absorbed into a smooth positive definite metric g_z on the finite-dimensional space. Equivalently,

$$\mathfrak{R}(z) = 0 \iff \nabla_{g_z}\mathcal{S}(z) = 0, \tag{3.22}$$

and the negative-gradient equation of the corrected action on the center manifold is

$$\dot{z} + \nabla_{g_z}\mathcal{S}(z) = 0. \tag{3.23}$$

Thus the local Floer equation is reduced, inside the isolating block, to the finite-dimensional gradient equation of $\mathcal{S}$ for a smooth metric.

Let $\mathcal{S}_\epsilon$ be a C^2-small Morse perturbation of $\mathcal{S}$ supported in the isolating ball. Choose the corresponding U-small Hamiltonian perturbation so that the above reduction gives $\mathcal{S}_\epsilon$ on the finite-dimensional slice. The moduli spaces of (3.13) and those of (3.23) are related by the one-parameter homotopy between the full operator and the reduced operator. The uniform inverse (3.18) gives compactness and gluing for this parametrized problem; Lemma 2.3 excludes breaking which leaves U. Consequently the zero-dimensional parametrized moduli spaces define chain maps

$$\Phi : CW^*_{\mathrm{loc}}(H, L; x) \to CM^{*+\sigma_1}_{\mathrm{loc}}(\mathcal{S}, 0), \qquad \Psi : CM^{*+\sigma_1}_{\mathrm{loc}}(\mathcal{S}, 0) \to CW^*_{\mathrm{loc}}(H, L; x). \tag{3.24}$$

The two-parameter homotopies obtained by concatenating the homotopy with its reverse have compact one-dimensional moduli spaces whose boundary strata are exactly

$$\Psi\Phi - \mathrm{id} = dK + Kd, \qquad \Phi\Psi - \mathrm{id} = dK' + K'd.$$

Hence Φ and Ψ are inverse up to chain homotopy. This proves the chain equivalence in (3.10).

It remains to record the grading and the iterated functional. The determinant line of the linearized Floer operator splits as

$$\det D_{\mathrm{Floer}} \cong \det D_{\mathrm{Morse}(\mathcal{S})} \otimes \det Q_{\mathrm{nor}}, \tag{3.25}$$

where Q_{nor} is the nondegenerate quadratic form obtained from the eliminated normal directions. Therefore orientations are identified by (3.25), and the absolute degree changes by a fixed integer

$$\sigma_1 = \mathrm{ind}_{\mathrm{coh}}(Q_{\mathrm{nor}}) + \text{the Maslov-normalization constant determined by the grading of } L.$$

The integer σ_1 is independent of the particular generator in the local cluster, because the isolating neighborhood is connected and the relative Maslov class of L vanishes.

For $H^{\#k}$ we subdivide the interval $[0, k]$ into the kN subintervals $[a + j/N, a + (j + 1)/N]$. Since $H^{\#k}$ is obtained by repeating H, the corresponding short-time type-II generating functions are the same F_j, repeated periodically. The endpoint conditions are $p_0 = 0$ and $p_{kN} = 0$, and the interior variables impose the same matching equations on each subinterval. Hence the discrete action is exactly

$$\mathcal{S}^{(k)} = \sum_{a=0}^{k-1} \sum_{j=0}^{N-1} \left(F_j(q_{aN+j}, \eta_{aN+j+1}) - q_{aN+j+1} \cdot \eta_{aN+j+1} \right),$$

which is (3.11). The same local reduction applies to this iterated functional. No closed-orbit iteration theorem is being used here; the argument is the boundary-value finite-dimensional reduction obtained from the type-II generating functions and the local Fredholm splitting above. □

Lemma 3.5. *(1) Let*

$$B_k = d^2\mathcal{S}^{(k)}(0).$$

Then

$$\ker B_k \cong \Lambda_0 \cap \Psi(k)^{-1}\Lambda_0 = E_k. \tag{3.26}$$

(2) Assume that k is relatively admissible and put $d = \dim E_1 = \dim E_k$. After a smooth local change of variables

$$(q_0, \dots, q_{kN}, \eta_1, \dots, \eta_{kN}) = (u, v), \qquad u \in \mathbb{R}^d,$$

one has

$$\mathcal{S}^{(k)}(u, v) = f_k(u) + Q_k(v), \qquad Q_k(v) = \frac{1}{2}\langle A_k v, v\rangle, \qquad \det A_k \neq 0.$$

Moreover the reduced germs f_k and f_1 have canonically isomorphic local Morse cohomology:

$$HM^*_{\mathrm{loc}}(f_k, 0) \cong HM^*_{\mathrm{loc}}(f_1, 0). \tag{3.27}$$

Proof. (1) The Hessian equation $B_k\xi = 0$ is the linearization of the critical equations (3.6)–(3.9) for the kN short pieces. Thus a vector in $\ker B_k$ is a sequence of variations

$$(\delta q_m, \delta\eta_m)_{m=0}^{kN}$$

with

$$\delta\eta_0 = 0, \qquad \delta\eta_{kN} = 0,$$

and satisfying the linearized matching equations

$$(\delta q_{m+1}, \delta\eta_{m+1}) = D\Phi_m(0)(\delta q_m, \delta\eta_m), \qquad m = 0, \dots, kN-1. \tag{3.28}$$

Here Φ_m denotes the corresponding short-time map, periodically extended in m. Concatenating (3.28) gives

$$(\delta q_{kN}, 0) = \Psi(k)(\delta q_0, 0).$$

Therefore $(\delta q_0, 0) \in \Lambda_0$ and $\Psi(k)(\delta q_0, 0) \in \Lambda_0$. Conversely, every vector in $\Lambda_0 \cap \Psi(k)^{-1}\Lambda_0$ determines a unique sequence by the linearized flow. This proves (3.26).

(2) By (1) and relative admissibility,

$$\dim\ker d^2\mathcal{S}^{(k)}(0) = d = \dim E_1.$$

Choose a Euclidean splitting of the finite-dimensional variable space

$$\mathbb{R}^{(2kN+1)n} = E_k \oplus V_k, \qquad B_k|_{V_k} = A_k, \qquad A_k \colon V_k \to V_k^* \text{ invertible.}$$

Write coordinates as $(u,v)\in E_k\oplus V_k$. Taylor expansion gives

$$\mathcal{S}^{(k)}(u,v)=\mathcal{S}^{(k)}(0)+\frac{1}{2}\langle A_k v,v\rangle+O(|u|^3+|u|^2|v|+|u||v|^2+|v|^3).$$

The vertical critical equation has the form

$$\partial_v\mathcal{S}^{(k)}(u,v)=A_k v+N_k(u,v)=0,\qquad |N_k(u,v)|\le C(|u|^2+|u||v|+|v|^2).$$

Since A_k is invertible, the implicit function theorem gives a unique smooth map

$$\psi_k: E_k\supset B_\rho\to V_k,\qquad \psi_k(0)=0,\qquad d\psi_k(0)=0,$$

with

$$\partial_v\mathcal{S}^{(k)}(u,\psi_k(u))=0.$$

Put

$$f_k(u)=\mathcal{S}^{(k)}(u,\psi_k(u)).$$

The fiberwise Morse lemma with parameter u gives a smooth change of the v-variable, preserving $v=0$, such that

$$\mathcal{S}^{(k)}(u,v)=f_k(u)+Q_k(v),\qquad Q_k(v)=\frac{1}{2}\langle A_k v,v\rangle.$$

This proves the splitting.

It remains to compare the reduced center germ with the one-period reduced germ. We give the finite-dimensional argument explicitly. Let

$$R_m(q_0,p_0)=\pi_p\big(\varphi_H^m(q_0,p_0)\big),\qquad (q_0,p_0)\in T^*\mathbb{R}^n,$$

where $\pi_p(q,p)=p$. In the chart $L=\{p=0\}$, the critical equation for an m-chord near 0 is

$$p_0=0,\qquad R_m(q_0,0)=0.$$

After the normal variables in the type-II action have been eliminated, the reduced equation $df_m(u)=0$ is precisely this endpoint equation written in the kernel coordinate $u\in E_m$. More precisely, there are smooth bundle isomorphisms $B_m(u)$, with $B_m(0)$ invertible, such that

$$df_m(u)=B_m(u)\,R_m(\iota_m(u),0),\tag{3.29}$$

where $\iota_m: E_m\to\Lambda_0$ sends a kernel vector of the discrete Hessian to its initial variation. Formula (3.29) follows by differentiating the type-II action: all interior variations give the matching equations, and the only remaining first variation is the terminal momentum component.

Relative admissibility says that $\iota_k : E_k \to E_1$ is an isomorphism onto the same tangent space of the endpoint equation. We use this isomorphism to regard both reduced germs as functions on the same vector space $E := E_1$. The chain rule for the iterate $\varphi_H^k = \varphi_H \circ \cdots \circ \varphi_H$ gives, on E,

$$R_k(\iota_k(u), 0) = A_k(u) R_1(\iota_1(u), 0) + O\big(|R_1(\iota_1(u), 0)|^2\big), \tag{3.30}$$

where $A_k(0)$ is invertible. The invertibility of $A_k(0)$ is exactly the assertion that no new kernel direction appears for the k-fold endpoint problem. Combining (3.29) and (3.30), after shrinking the ball in E, we obtain a smooth family of exact one-forms

$$df_{k,s}, \qquad 0 \le s \le 1,$$

with $f_{k,0} = f_k$, $f_{k,1} = f_1$, and a constant $c > 0$ such that

$$|df_{k,s}(u)| \ge c\,|df_1(u)| \qquad \text{for all } 0 \le s \le 1 \tag{3.31}$$

outside an arbitrarily small neighborhood of 0. Since x is isolated, 0 is the only zero of df_1 in a sufficiently small ball. Equation (3.31) therefore gives an isolating ball $B_0 \subset E$ satisfying

$$\mathrm{Crit}(f_{k,s}) \cap B_0 = \{0\} \qquad (0 \le s \le 1). \tag{3.32}$$

Local Morse cohomology is invariant under isolated homotopies of germs, by the standard continuation construction for critical groups [GM69, Cha93]. Applying this continuation to (3.32) gives (3.27). □

Lemma 3.6 (Split local Morse complex). *For the split germ $f_k + Q_k$ one has*

$$HM^r_{\mathrm{loc}}(f_k + Q_k, 0) \cong HM^{r-\lambda_k}_{\mathrm{loc}}(f_k, 0), \tag{3.33}$$

where λ_k is the cohomological Morse degree of the unique critical point of the nondegenerate quadratic form Q_k.

Proof. Choose product Morse perturbations of f_k and Q_k inside isolating blocks. The split negative gradient equation is

$$\dot u = -\nabla f_k(u), \qquad \dot v = -\nabla Q_k(v).$$

Thus critical points are pairs

$$(a, b) \in \mathrm{Crit}(f_k) \times \mathrm{Crit}(Q_k),$$

with cohomological Morse degree

$$|(a,b)| = |a| + |b|.$$

The local cochain module is therefore

$$CM^r_{\mathrm{loc}}(f_k + Q_k, 0) = \bigoplus_{i+j=r} CM^i_{\mathrm{loc}}(f_k, 0) \otimes CM^j_{\mathrm{loc}}(Q_k, 0). \tag{3.34}$$

The product orientation line is $o_{(a,b)} = o_a \otimes o_b$, and the compactified zero-dimensional moduli spaces split. Therefore

$$d_{f_k+Q_k} = d_{f_k} \otimes I + (-1)^i I \otimes d_{Q_k} \quad \text{on } CM^i(f_k) \otimes CM^*(Q_k).$$

Since Q_k is a nondegenerate quadratic form,

$$CM^j_{\mathrm{loc}}(Q_k, 0) = \begin{cases} \mathbb{F} \cdot o_k, & j = \lambda_k, \\ 0, & j \neq \lambda_k, \end{cases} \qquad d_{Q_k} = 0. \tag{3.35}$$

Equations (3.34)–(3.35) imply (3.33). □

3.2. **Proof of Theorem 1.8.** Now we can prove Theorem 1.8.

Proof of Theorem 1.8. Fix a sufficiently large relatively admissible integer k. All constructions take place in the Darboux–Weinstein chart fixed before Definition 1.7; hence all Robbin–Salamon indices are computed in the same Maslov-zero relative trivialization. Proposition 3.4 gives finite-dimensional chain models

$$\mathrm{HW}^r_{\mathrm{loc}}(H^{\#k}, L; x^k) \cong HM^{r+\sigma_k}_{\mathrm{loc}}(\mathcal{S}^{(k)}, 0), \qquad \mathrm{HW}^r_{\mathrm{loc}}(H, L; x) \cong HM^{r+\sigma_1}_{\mathrm{loc}}(\mathcal{S}, 0), \tag{3.36}$$

where $\sigma_1, \sigma_k \in \mathbb{Z}$ are the grading shifts determined by the Fredholm determinant-line identification in Proposition 3.4.

By Lemma 3.5, relative admissibility gives a splitting of the k-fold discrete action germ

$$\mathcal{S}^{(k)}(u, v) = f_k(u) + Q_k(v), \qquad Q_k(v) = \frac{1}{2}\langle A_k v, v\rangle, \tag{3.37}$$

with Q_k nondegenerate. The same lemma gives an isolated continuation from f_k to f_1; this is the finite-dimensional assertion proved in (3.29)–(3.32), not a black-box appeal to a closed-orbit local Floer theorem. Lemma 3.6 then gives

$$\begin{aligned}\mathrm{HW}^r_{\mathrm{loc}}(H^{\#k}, L; x^k) &\cong HM^{r+\sigma_k}_{\mathrm{loc}}(\mathcal{S}^{(k)}, 0)\\ &\cong HM^{r+\sigma_k-\lambda_k}_{\mathrm{loc}}(f_k, 0)\\ &\cong HM^{r+\sigma_k-\lambda_k}_{\mathrm{loc}}(f_1, 0)\\ &\cong \mathrm{HW}^{r+s_k}_{\mathrm{loc}}(H, L; x),\end{aligned}$$

where

$$s_k = \sigma_k - \lambda_k - \sigma_1. \tag{3.38}$$

This proves the local iteration isomorphism.

It remains to compute the asymptotic behavior of s_k and the possible degrees of the iterated local group. Let

$$\Lambda(t) = D\varphi^t_H(T_{x(0)}L), \qquad \Lambda_0 = T_{x(0)}L, \qquad \mu_k = \mu_{\mathrm{RS}}(\Lambda^{\#k}, \Lambda_0).$$

The grading convention of Section 2 is

$$\mathrm{gr}(y) = n - \mu_{\mathrm{RS}}(y) + \kappa_L. \tag{3.39}$$

For the finite-dimensional model, the determinant-line comparison gives the following shift-index estimate. If a is a critical point of a sufficiently small Morse perturbation of the reduced germ f_1, and (a, o_k) denotes the corresponding critical point of $f_k + Q_k$ under the isolated continuation above, then

$$\mathrm{gr}_k(a, o_k) + s_k = \mathrm{gr}_1(a), \qquad |s_k - k\Delta_L(x)| \le C'_x. \tag{3.40}$$

Here C'_x is independent of k. Indeed, the left equality is the definition of s_k. The difference between s_k and the Robbin–Salamon index μ_k is obtained by comparing the determinant line of the Cauchy–Riemann operator with the determinant line of the Hessian of the type-II action. This comparison is a spectral-flow computation: the crossing forms of the Hessian are the crossing forms of the Lagrangian path $\Lambda^{\#k}$ relative to Λ_0. Hence the shift differs from μ_k by the fixed one-period normalization and by a bounded endpoint correction, the latter being the Hörmander–Kashiwara term for changing the reference Lagrangian; see [RS93, CLM94] and the Maslov-type Lagrangian iteration estimate [Liu07]. Since

$$\Delta_L(x) = \lim_{k\to\infty} \frac{\mu_k}{k},$$

(3.40) follows. In particular,

$$\lim_{k\to\infty}\frac{s_k}{k}=\Delta_L(x). \tag{3.41}$$

Let

$$J_x=\{j\in\mathbb{Z}\mid \mathrm{HW}^j_{\mathrm{loc}}(H,L;x)\neq 0\}.$$

The set J_x is finite because the local Morse model is finite-dimensional and has an isolated critical point. Put

$$B_x=\max\{|j|\mid j\in J_x\}+C'_x+1.$$

If $\mathrm{HW}^r_{\mathrm{loc}}(H^{\#k},L;x^k)\neq 0$, then by the isomorphism already proved one has $r+s_k\in J_x$. Therefore (3.40) gives

$$\operatorname{supp}\mathrm{HW}^*_{\mathrm{loc}}(H^{\#k},L;x^k)\subset[-k\Delta_L(x)-B_x,\,-k\Delta_L(x)+B_x]\cap\mathbb{Z}. \tag{3.42}$$

This is the support estimate in the cohomological grading convention $\mathrm{gr}=n-\mu_{\mathrm{RS}}+\kappa_L$. Taking $C_x=B_x$ proves the support assertion of the theorem.

If $\mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p)\neq 0$ for infinitely many relatively admissible primes p, then (3.42) gives

$$|p\Delta_L(x)|\leq B_x$$

for infinitely many p. Hence $\Delta_L(x)=0$.

Finally assume $\Delta_L(x)=0$ and $\mathrm{HW}^0_{\mathrm{loc}}(H,L;x)=0$. Since μ_p is bounded along relatively admissible primes and the nullity $\dim E_p$ is constant, the spectral flow of the nondegenerate normal Hessian block cannot cross zero for all sufficiently large relatively admissible primes. Therefore the cohomological degree λ_p of the stabilization form Q_p is eventually constant. With the normalization used in Definition 1.7, the extremal local degree is degree 0; hence the eventual constant is the same as in the one-period model, and $s_p=0$ for all sufficiently large relatively admissible primes. The iteration isomorphism therefore becomes

$$\mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p)\cong\mathrm{HW}^0_{\mathrm{loc}}(H,L;x),$$

and the assumed vanishing of the right-hand side gives

$$\mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p)=0$$

for every sufficiently large relatively admissible prime p. This proves the final assertion and completes the proof. □

4. Filtered algebra and the reusable SDM action-gap package

This section has two logically separate roles. First, Sections 4.1–4.3 construct compact representatives of the wrapped unit, the local-to-global spectral sequence, and the collar/end action estimates. These are used in Section 5 before any SDM argument enters. Second, Section 4.4 proves the analytic SDM window mechanism. The local computations are written first in the clean notation of a normalized local germ, and the proof is deliberately decomposed into reusable pieces–filtered type-II comparison, finite-dimensional boundary depth, local-to-global injection, and collar exclusion. In Section 5, Proposition 5.3 verifies these hypotheses for the tail-stabilized germs $G_{\Theta,r}$, and the proof of Theorem 1.11 applies the package to UC^rV.

4.1. **Compact representatives of the wrapped unit.**

The filtered contradiction in the proof of the main theorem needs a representative of the unit which is supported in the compact interior part of the wrapped complex.

Lemma 4.1 (Compact representative for the unit)**.** *Fix $p \geq 1$. For all sufficiently large admissible slopes a, the image of 1_L in*

$$H^0(\mathrm{CW}^*((K_a)^{\#p}; L, L))$$

has a degree-zero cocycle representative generated by Hamiltonian chords contained in a compact subset $C_p \Subset W$. More precisely, if for some N and some unbounded set $\mathcal{P} \subset \mathbb{N}$ every compact interior chord of $H^{\#p}$, $p \in \mathcal{P}$, has minimal order at most N, then there is a single compact set $C_N \Subset W$, independent of $p \in \mathcal{P}$, supporting such representatives for all sufficiently large $p \in \mathcal{P}$.

Proof. Choose a Weinstein neighborhood

$$\mathcal{N}(L) \cong D^*L, \qquad \widehat{\lambda}|_{\mathcal{N}(L)} = \lambda_{\mathrm{can}} + df_L,$$

and a Morse function $f : L \to \mathbb{R}$ with a unique minimum $q_{\min}$. For $0 < \epsilon \ll 1$, take

$$K_0(q,p) = \epsilon f(q) + \frac{\kappa}{2}|p|^2, \qquad (q,p) \in D^*L,$$

extended by zero outside $\mathcal{N}(L)$. The Hamiltonian chords of K_0 are identified with critical points of f. With the cohomological grading convention used in this paper,

$$\mathrm{CW}^r(K_0; L, L) \cong CM^r(f; \mathbb{F}), \qquad d_{K_0} = d_{\mathrm{Morse}} + O(\epsilon^2),$$

and for ϵ sufficiently small the continuation/PSS map

$$\mathrm{PSS}_{K_0} : CM^*(f;\mathbb{F}) \longrightarrow \mathrm{CW}^*(K_0; L, L)$$

is a chain isomorphism whose leading term sends a critical point q to the corresponding short chord x_q. The unique minimum $q_{\min}$ represents the ordinary cohomological unit

$$[q_{\min}] = 1 \in H^0(L;\mathbb{F}).$$

Let

$$e_L(t) \equiv q_{\min}.$$

Then

$$d_{K_0} e_L = 0, \qquad \mathrm{PSS}_{K_0}(q_{\min}) = e_L + d_{K_0}\theta$$

for some $\theta \in \mathrm{CW}^{-1}(K_0; L, L)$ after possibly replacing e_L by the cohomologous PSS representative. Hence

$$[e_L] = \mathrm{PSS}_{K_0}([q_{\min}]) = \mathrm{PSS}_{K_0}(1) = 1_L \in H^0(\mathrm{CW}^*(K_0; L, L)).$$

The last equality is the defining compatibility of the open-string PSS map with the product unit. At the chain level this says that for every cocycle $\alpha \in \mathrm{CW}^*(K_0; L, L)$,

$$\mu^2(e_L, \alpha) - \alpha = dh_L(\alpha) + h_L(d\alpha), \qquad \mu^2(\alpha, e_L) - \alpha = dh_R(\alpha) + h_R(d\alpha), \tag{4.1}$$

where the homotopies h_L, h_R count one-dimensional pearly triangles. In the limit $\epsilon \to 0$, these triangles collapse to constant Morse trees at $q_{\min}$, so (4.1) is precisely the Morse identity operation. Therefore $[e_L]$ is the wrapped Floer unit.

We now define the intermediate Hamiltonian $K_{\mathrm{core},p}$. Let $C_p^{\mathrm{int}} \Subset W$ contain all compact interior chords of $(K_a)^{\#p}$ which can occur in the argument. Choose radii

$$1 < R_0 < R_1 < R_a, \qquad C_p^{\mathrm{int}} \subset W_{R_0} := W \cup ([1, R_0] \times \partial W),$$

and a cut-off $\beta \in C^\infty([1,\infty), [0,1])$ with

$$\beta(r) = 1 \ (r \le R_0), \qquad \beta(r) = 0 \ (r \ge R_1), \qquad \beta'(r) \le 0.$$

On the cylindrical end, write $(K_a)_t^{\#p}(r, y) = h_{a,p,t}(r, y)$. Define

$$(K_{\mathrm{core},p})_t(r, y) = \beta(r) h_{a,p,t}(r, y) + (1 - \beta(r)) h_{a,p,t}(R_0, y), \qquad r \ge 1, \tag{4.2}$$

and set $K_{\mathrm{core},p} = (K_a)^{\#p}$ on W_{R_0}. After the usual smoothing of (4.2),

$$K_{\mathrm{core},p} = (K_a)^{\#p} \text{ on } W_{R_0}, \qquad \partial_r K_{\mathrm{core},p} = 0 \text{ for } r \ge R_1, \qquad K_{\mathrm{core},p} \le (K_a)^{\#p}.$$

The continuation is split as

$$K_0 \xrightarrow{K_s^0} K_{\mathrm{core},p} \xrightarrow{K_s^1} (K_a)^{\#p}, \qquad \partial_s K_s^0 \geq 0, \quad \partial_s\big(\partial_r K_s^1\big) \leq 0 \quad \text{on the radial barrier collar.} \tag{4.3}$$

Let

$$c_{\mathrm{core},p} := \Phi_{K_0,K_{\mathrm{core},p}}(e_L), \qquad c_{a,p} := \Phi_{K_{\mathrm{core},p},(K_a)^{\#p}}(c_{\mathrm{core},p}). \tag{4.4}$$

At chain level,

$$c_{a,p} = \sum_{y\in\mathcal{X}((K_a)^{\#p},L)} n_y y, \qquad n_y = \sum_{u\in\mathcal{M}^0_{\mathrm{cont}}(e_L,y;K_s^0\#K_s^1)} \epsilon(u), \tag{4.5}$$

where $\mathcal{M}^0_{\mathrm{cont}}$ denotes the zero-dimensional continuation moduli space. Since continuation is a chain map,

$$d_{(K_a)^{\#p}} c_{a,p} = \Phi_{K_{\mathrm{core},p},(K_a)^{\#p}} \Phi_{K_0,K_{\mathrm{core},p}}(d_{K_0} e_L) = 0,$$

and, because continuation maps preserve the wrapped unit,

$$[c_{a,p}] = \Phi_{K_{\mathrm{core},p},(K_a)^{\#p},*} \Phi_{K_0,K_{\mathrm{core},p},*}([e_L]) = 1_L \in H^0(\mathrm{CW}^*((K_a)^{\#p}; L, L)).$$

Thus (4.4) is the requested representative.

It remains to prove that no coefficient n_y with y outside a compact core is needed. For the first homotopy in (4.3), every asymptotic chord lies in W_{R_0}. On $r \geq R_1$, K_s^0 is independent of r. If $\rho = r \circ u$, the contact-type computation gives, in the weak sense,

$$\Delta\rho = |\partial_s u|^2 + \partial_r^2 K_s^0(\rho, y)|\partial_s \rho|^2 + \partial_s(\partial_r K_s^0)(\rho, y) \geq 0, \qquad \partial_t\rho(s, 0) = \partial_t\rho(s, 1) = 0.$$

The weak maximum principle [PW84, Chapter 2] on $\mathbb{R} \times [0, 1]$ therefore gives

$$\sup_{\mathbb{R}\times[0,1]} r(u) \leq \max\{\sup r(x_-), \sup r(x_+)\} \leq R_0.$$

Hence every term of $c_{\mathrm{core},p}$ is generated by chords in W_{R_0}.

For the second homotopy, assume that a continuation strip from a compact generator to a generator y with $\operatorname{im} y \cap \{r \geq R_1\} \neq \varnothing$ exists. Let

$$\Sigma_{r_0} := u^{-1}([r_0, \infty) \times \partial W), \qquad R_0 < r_0 < R_1$$

be a regular superlevel domain. Stokes' theorem, exactness $\widehat{\lambda}|_{\widehat{L}} = df_L$, and convex radial interpolation give

$$\begin{aligned}
0 &\le \int_{\Sigma_{r_0}} |\partial_s u|^2 \\
&= \int_{\partial\Sigma_{r_0}} u^*\widehat{\lambda} - \int_{\Sigma_{r_0}} dK^1_s(\partial_s u) + \int_{\Sigma_{r_0}} \partial_s K^1_s(u) \\
&\le -\int_{u^{-1}(r=r_0)} r_0\,\alpha(\partial_t u - X_{K^1_s})\,dt - \int_{\Sigma_{r_0}} \partial_r K^1_s\,\partial_s(r\circ u) + \int_{\Sigma_{r_0}} \partial_s K^1_s \\
&\le 0. \qquad (4.6)
\end{aligned}$$

We now spell out the last inequality in (4.6). On the barrier collar the homotopy is chosen radial up to a uniformly bounded tangential term which is constant on the level used for the barrier; write, on this collar,

$$K^1_s(r,y) = h_s(r), \qquad a_s(r) := \partial_r h_s(r), \qquad \partial_r a_s(r) \ge 0, \qquad \partial_s a_s(r) \le 0.$$

The last inequality is first proved for regular values of $R := r\circ u$ and then for arbitrary levels by Sard's theorem and monotone convergence. Put

$$\Gamma_{r_0} := u^{-1}(r=r_0) = \partial\Sigma_{r_0} \setminus \partial(\mathbb{R}\times[0,1]).$$

The boundary pieces on $t = 0,1$ have zero contribution because $u(s,0), u(s,1) \in \widehat{L}$ and $\widehat{\lambda}|_{\widehat{L}} = df_L$; on each compact component their contributions telescope. For the remaining level-boundary contribution, the contact-type convention on the end gives

$$\widehat{\lambda}(J\xi) = -dr(\xi), \qquad \partial_t u - X_{K^1_s} = J\partial_s u, \qquad r_0\alpha(\partial_t u - X_{K^1_s}) = -\partial_s R \quad \text{on } \Gamma_{r_0}.$$

Orient Γ_{r_0} as the boundary of $\Sigma_{r_0} = \{R \ge r_0\}$. If $d\ell$ is arclength on Γ_{r_0}, then the positive tangent vector is

$$T = \frac{1}{|\nabla R|}(R_t, -R_s), \qquad dt(T) = -\frac{R_s}{|\nabla R|}.$$

Hence

$$\begin{aligned} -\int_{\Gamma_{r_0}} r_0\alpha(\partial_t u - X_{K_s^1})\,dt &= \int_{\Gamma_{r_0}} R_s\,dt \\ &= \int_{\Gamma_{r_0}} R_s\,dt(T)\,d\ell \\ &= -\int_{\Gamma_{r_0}} \frac{R_s^2}{|\nabla R|}\,d\ell \\ &\le 0. \end{aligned} \tag{4.7}$$

Thus the level-boundary term is non-positive.

For the interior radial term, decompose for a.e. t

$$\{s \mid (s,t) \in \Sigma_{r_0}\} = \coprod_{\nu} [A_\nu(t), B_\nu(t)], \qquad R(A_\nu(t),t) = R(B_\nu(t),t) = r_0.$$

Normalize $h_s(r_0) = 0$, which does not change $X_{K_s^1}$. Since

$$h_s(R) = \int_{r_0}^{R} a_s(\rho)\,d\rho, \qquad \partial_s h_s(R) = \int_{r_0}^{R} \partial_s a_s(\rho)\,d\rho,$$

we get on each component $[A_\nu, B_\nu]$

$$\begin{aligned} &\int_{A_\nu}^{B_\nu} \left[-\partial_r K_s^1(u)\,\partial_s R + \partial_s K_s^1(u)\right] ds \\ &= \int_{A_\nu}^{B_\nu} \left[-a_s(R)\partial_s R + \int_{r_0}^{R} \partial_s a_s(\rho) d\rho\right] ds \\ &= -\left[\int_{r_0}^{R(s,t)} a_s(\rho) d\rho\right]_{s=A_\nu}^{s=B_\nu} + 2\int_{A_\nu}^{B_\nu} \int_{r_0}^{R(s,t)} \partial_s a_s(\rho) d\rho\,ds \\ &= 2\int_{A_\nu}^{B_\nu} \int_{r_0}^{R(s,t)} \partial_s a_s(\rho) d\rho\,ds \le 0, \end{aligned} \tag{4.8}$$

where the boundary term vanishes because $R = r_0$ at the endpoints, and the final inequality follows from $R \ge r_0$ on Σ_{r_0} and $\partial_s a_s \le 0$. Summing (4.8) over ν and integrating in t gives

$$-\int_{\Sigma_{r_0}} \partial_r K_s^1\,\partial_s(r \circ u) + \int_{\Sigma_{r_0}} \partial_s K_s^1 \le 0. \tag{4.9}$$

Equations (4.7) and (4.9) prove the last inequality in (4.6). Hence equality holds everywhere in (4.6); in particular,

$$\int_{\Sigma_{r_0}} |\partial_s u|^2 = 0, \qquad \partial_s u \equiv 0 \quad \text{on } \Sigma_{r_0}.$$

A nonconstant continuation trajectory cannot cross $r = r_0$. Thus a nonzero coefficient n_y in (4.5) has $\operatorname{im} y \subset W_{R_1}$, except for transition/end generators excluded from the compact action quotient by Proposition 4.3.

Assume now the uniform bounded-return condition in the second sentence of the lemma. Put

$$S_N = \bigcup_{1 \le k \le N} \mathcal{X}_k(H, L), \qquad C_N^0 = \bigcup_{q \in S_N} \{\varphi_H^t(q) \mid 0 \le t \le \rho(q)\}.$$

The set S_N is finite by (D3), hence $C_N^0 \Subset W$. If a compact chord of $H^{\#p}$, $p \in \mathcal{P}$, has minimal order at most N, its trajectory decomposes greedily into first-return segments whose starting points lie in S_N; consequently its image is contained in C_N^0. Enlarging C_N^0 slightly to absorb the fixed continuation collars and local perturbation supports gives a compact set $C_N \Subset W$, independent of p, which supports the unit representatives. This proves the lemma. □

Proof of Theorem 1.6. The equivalence $\mathrm{HW}^*(L, L; \mathbb{F}) \neq 0 \iff 1_L \neq 0$ is Lemma 2.1. Fix an admissible iterate p and choose the cofinal slope a large enough for Lemma 4.1. The cocycle

$$c_p := c_{a,p} = \Phi_{K_{\mathrm{core},p},(K_a)^{\#p}} \Phi_{K_0,K_{\mathrm{core},p}}(e_L)$$

constructed in (4.4) has degree zero because the continuation maps preserve the absolute grading and e_L is the small-Morse representative of the ordinary unit in degree zero. The chain-map identity gives

$$d_{(K_a)^{\#p}} c_p = 0,$$

and functoriality of continuation maps gives

$$[c_p] = \Phi_{K_{\mathrm{core},p},(K_a)^{\#p},*} \Phi_{K_0,K_{\mathrm{core},p},*}([e_L]) = 1_L.$$

Lemma 4.1 proves the asserted compact support. This is exactly the representative denoted by c_p in (1.1). □

4.2. **Filtered local-to-global algebra.** Let K be admissible and assume that all relevant chords are isolated. Pick regular values of the action

$$+\infty = a_0 > a_1 > a_2 > \cdots, \qquad a_j \to -\infty, \qquad a_j \notin \mathcal{A}_K(\mathcal{X}(K,L)).$$

Define

$$F^j \operatorname{CW}^*(K) = F^{a_j} \operatorname{CW}^*(K).$$

We have the following local-to-global spectral sequence.

Proposition 4.2. [LM24, Proposition 4.1] *There is a spectral sequence converging to* $\operatorname{HW}^*(L,L;K)$ *with first page*

$$E_1^{j,q}(K) \cong \bigoplus_{a_{j+1}<\mathcal{A}_K(x)<a_j} \operatorname{HW}^{j+q}_{\mathrm{loc}}(K,L;x).$$

Its differentials have bidegree

$$d_r : E_r^{j,q} \to E_r^{j+r,q-r+1}.$$

4.3. **Collar action estimate.** We next ensure that transition or end chords of the admissible extension do not replace the compact unit contribution.

Proposition 4.3 (Collar action domination)**.** *For every compact action scale* $B_p = O(p)$*, there is a slope* $a(p)$ *such that every chord* z *of* $(K_{a(p)})^{\#p}$ *meeting the cylindrical end satisfies*

$$\mathcal{A}_{(K_{a(p)})^{\#p}}(z) < -B_p - p. \tag{4.10}$$

In particular, such chords lie below any action window containing the actions of the finitely many p-fold iterates of interior fixed chords.

Proof. Write the admissible Hamiltonian on the end in the form

$$K_a(t,r,y) = h_a(r) + b_a(t,y), \qquad h_a'(r) = a \text{ for } r \geq R_a.$$

Adding a time-dependent constant to K_a does not change X_{K_a}, but shifts all end actions. We choose the standard cofinal normalization so that on the transition annulus

$$\mathcal{T}_a = [R_a^-, R_a^+] \times \partial W$$

one has

$$r\partial_r h_a(r) - h_a(r) \leq -\gamma(a), \qquad \gamma(a) \xrightarrow[a\to\infty]{} +\infty.$$

For a general y-dependent perturbation $b_a(t,y)$, the Hamiltonian vector field decomposes as

$$X_{K_a} = h_a'(r)R_\alpha + X_{b_a}^{\xi} + Y_a^r,$$

where $X_{b_a}^{\xi} \in \xi = \ker\alpha$ and the radial term $Y_a^r\partial_r$ are uniformly bounded after the usual rescaling of the transition. Since $\widehat{\lambda} = r\alpha$,

$$\begin{aligned}\widehat{\lambda}(X_{K_a}) - K_a &= r\alpha(h_a'(r)R_\alpha + X_{b_a}^{\xi} + Y_a^r) - h_a(r) - b_a(t,y)\\ &= rh_a'(r) - h_a(r) - b_a(t,y) + O(1).\end{aligned}$$

Thus there is $C_1 > 0$, independent of a, such that

$$\widehat{\lambda}(X_{K_a}) - K_a \le -\gamma(a) + C_1 \quad \text{on } \mathcal{T}_a, \tag{4.11}$$

and

$$|\widehat{\lambda}(X_{K_a}) - K_a| \le C_1 \quad \text{outside } \mathcal{T}_a$$

for the part of the cylindrical end relevant to chords in the chosen compact action scale.

Because $a \notin \operatorname{Spec}(\Lambda,\alpha)$, no Hamiltonian chord with endpoints on $\widehat{L}$ is entirely contained in the pure linear region $r \ge R_a^+$: such a chord would project to a Reeb chord of length a. Hence every chord z meeting the cylindrical end either meets the transition annulus or crosses it. The rescaled vector fields on $\mathcal{T}_a$ are uniformly C^1-bounded and the annulus has fixed rescaled width; therefore there exists $\theta_0 > 0$ such that

$$\operatorname{meas}\big\{t \in [0,p] \mid z(t) \in \mathcal{T}_a\big\} \ge \theta_0 \tag{4.12}$$

whenever z meets the end. The complement of this time set has length at most p. Combining (4.11)–(4.12),

$$\begin{aligned}\int_0^p (\widehat{\lambda}(X_{K_a}) - K_a)(z(t))\,dt &\le -\theta_0\gamma(a) + C_1\theta_0 + C_1 p\\ &\le -\theta_0\gamma(a) + C_2 p.\end{aligned}$$

The primitive contribution satisfies

$$|f_L(z(0)) - f_L(z(p))| \le D_0,$$

because f_L is constant on the cylindrical end and bounded on the compact transition. Hence

$$\mathcal{A}_{(K_a)^{\#p}}(z) \le -\theta_0\gamma(a) + C_2 p + D_0. \tag{4.13}$$

Given $B_p = O(p)$, choose $a(p)$ so large that

$$\theta_0\gamma(a(p)) - C_2p - D_0 > B_p + p.$$

Then (4.13) implies (4.10). This is precisely the end-exclusion estimate used in the Liouville-domain Poincaré–Birkhoff argument: the end slope is nonresonant, and the additive normalization of the cofinal Hamiltonian pushes all transition/end generators below the compact action window. □

4.4. **The open-string SDM action-gap theorem as a model case.** The pure-iterate notation in this subsection is intentional but should not be read as a restriction on the final proof. The symbols pc and G_p are placeholders for a center action and a normalized finite-dimensional local germ. In Section 5, the same proof is reused after replacing pc by $A_{\Theta,r}$ and G_p by the eventual-type germ $G_{\Theta,r}$.

Let x be an open-string SDM and put

$$c = \mathcal{A}_H(x).$$

The point of this subsection is to separate three logically different assertions. First, the Lagrangian Floer problem near x^p is converted, by the type-II finite-dimensional reduction of Section 3, to a filtered Morse complex of a finite-dimensional germ. Second, one needs a one-sided boundary-depth statement for this finite-dimensional SDM germ. Third, this finite-dimensional class is transported back to the local Floer complex and then injected into the global filtered wrapped complex by the energy-separation estimate of Lemma 2.3.

4.4.1. *Filtered local model and sign convention.* For $a < b$ define

$$\mathrm{CW}^*_{(a,b)}(K; L, L) := F^a\,\mathrm{CW}^*(K; L, L)/F^b\,\mathrm{CW}^*(K; L, L), \qquad F^a\,\mathrm{CW}^*(K) = \bigoplus_{\mathcal{A}_K(y)\geq a} \mathbb{F}\cdot y. \tag{4.14}$$

Thus $\mathrm{CW}^*_{(a,b)}$ is generated by chords satisfying $a \leq \mathcal{A}_K(y) < b$. If z occurs in dy, then by (2.3)

$$\mathcal{A}_K(z) - \mathcal{A}_K(y) = E(u) \geq 0, \qquad \mathrm{gr}(z) = \mathrm{gr}(y) + 1. \tag{4.15}$$

Consequently $d(F^a) \subset F^a$, and the quotient differential in (4.14) is well-defined.

Let $U' \Subset U \Subset W$ be an isolating pair for x, with energy-separation constant ε_U from Lemma 2.3. For every sufficiently large relatively admissible prime p, Section 3 gives a type-II generating-function germ

$$\mathcal{A}_{H^{\#p}} = pc + G_p \tag{4.16}$$

on the finite-dimensional local model. More explicitly, after the normal directions have been split off as in Lemma 3.6, the germ has the form

$$G_p(u,v) = f_p(u) + Q_p(v),$$

where Q_p is nondegenerate and the local Morse cohomology of f_p is canonically identified with that of the one-period center germ. Since x is an open-string SDM, Theorem 1.8 gives

$$H^0(CM^*_{\mathrm{loc}}(G_p,0)) \cong \mathrm{HW}^0_{\mathrm{loc}}(H^{\#p},L;x^p) = 0 \tag{4.17}$$

for all sufficiently large relatively admissible primes. The grading shift in Theorem 1.8 has already been accounted for in (4.16); in particular the class in (4.17) is in degree zero in the conventions of this paper.

The following filtered form of the finite-dimensional reduction is the only analytic fact about the reduction used below.

Proposition 4.4 (Filtered type-II reduction)**.** *For every $\eta > 0$, after shrinking $U' \Subset U$ and choosing the local perturbation sufficiently small, there are chain maps*

$$\Phi_p : \mathrm{CW}^*_{\mathrm{loc}}(H^{\#p},L;x^p) \longrightarrow CM^*_{\mathrm{loc}}(G_p,0), \qquad \Psi_p : CM^*_{\mathrm{loc}}(G_p,0) \longrightarrow \mathrm{CW}^*_{\mathrm{loc}}(H^{\#p},L;x^p), \tag{4.18}$$

and chain homotopies

$$\Psi_p\Phi_p - \mathrm{id} = dP_p + P_p d, \qquad \Phi_p\Psi_p - \mathrm{id} = dQ_p + Q_p d, \tag{4.19}$$

such that the maps and homotopies have action loss at most η. Namely, if a generator ζ occurs with nonzero coefficient in $\Phi_p\xi$, $\Psi_p\xi$, $P_p\xi$, or $Q_p\xi$, then

$$\bigl|\, \mathcal{A}_{H^{\#p}}(\zeta) - pc - G_p(\xi) \,\bigr| \le \eta. \tag{4.20}$$

Proof. This is the filtered version of Proposition 3.4. We recall the estimates because they are needed later. In the notation of that proposition, a point in the Floer Banach chart is written as $v + w$, where v lies in the finite-dimensional type-II slice and w lies in the L^2-orthogonal complement. The range equation is solved by a smooth map $w = w(v)$ satisfying

$$\|w(v)\|_{W^{m+1,q}} \le C\|\nabla G_p(v)\|, \qquad \|dw(v)\| \le C\|d^2G_p(v)\|.$$

The constants are uniform for all sufficiently large admissible p because the normal operator has a uniformly bounded inverse after the admissible kernel has been removed. This uniform inverse is exactly the one used in Proposition 3.4; it follows from the short-time type-II transversality, the Calderon–Zygmund boundary estimate (2.5), and the stability of bounded right inverses.

The action identity on the slice gives

$$\mathcal{A}_{H^{\#p}}(v) = pc + G_p(v).$$

Taylor expansion in the normal direction and the range equation give

$$\left|\mathcal{A}_{H^{\#p}}(v + w(v)) - pc - G_p(v)\right| \leq C\|w(v)\|^2_{W^{m+1,q}}.$$

By shrinking the isolating ball, the right hand side is made $< \eta$. The same estimate holds for the parametrized equations defining the comparison maps and homotopies, because their normal components satisfy the same inverse estimate with uniformly bounded inhomogeneous terms. This proves (4.18)–(4.20). □

4.4.2. *The finite-dimensional boundary-depth input.* For a finite-dimensional germ G_p choose a small closed isolating ball B and define

$$C_p^a := CM^*\big(\{G_p \geq a\} \cap B, \{G_p \geq a\} \cap \partial B; \mathbb{F}\big), \qquad C^*_{p,(a,b)} := C_p^a / C_p^b.$$

The differential preserves C_p^a, because the Morse differential is written in the same cohomological convention as (4.15): it raises the value of G_p.

Theorem 4.5. *Let G_p be the finite-dimensional type-II germ arising from an open-string SDM as above. Then there is $\varepsilon_0 > 0$, independent of all sufficiently large relatively admissible primes p, such that for every $0 < \varepsilon < \varepsilon_0$ and every sufficiently large such p, there exists a regular value $\bar{\delta}_p \in (0, \varepsilon)$ for G_p and a class*

$$0 \neq [\beta_p^M] \in H^1(C^*_{p,(\bar{\delta}_p,\varepsilon)}).$$

Equivalently, if $e_p \in H^0(C_p^0/C_p^{\bar{\delta}_p})$ is the degree-zero extremal SDM class, then the connecting homomorphism of

$$0 \to C_p^{\bar{\delta}_p}/C_p^{\varepsilon} \longrightarrow C_p^0/C_p^{\varepsilon} \longrightarrow C_p^0/C_p^{\bar{\delta}_p} \to 0$$

satisfies

$$\partial_{\bar{\delta}_p,\varepsilon} e_p \neq 0 \in H^1(C_p^{\bar{\delta}_p}/C_p^{\varepsilon}). \tag{4.21}$$

The finite-dimensional proof is the generating-function boundary-depth argument used in [GG10, Sections 3–5] and [Maz13]; after the reduction above, the objects are ordinary finite-dimensional filtered Morse complexes. The argument does not rely on the wrapped Floer theory setting and the proof is omitted here.

Theorem 4.6 (Local open-string SDM action gap)**.** *Let x be an open string SDM. There is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every sufficiently large relatively admissible prime p, there exists $\delta_p \in (0, \varepsilon)$ with*

$$H^1\Big(\mathrm{CW}^*_{\mathrm{loc},(pc+\delta_p,\,pc+\varepsilon)}(H^{\#p}, L; x^p)\Big) \neq 0. \tag{4.22}$$

Here the local filtered quotient is generated only by perturbed chords contained in U.

Proof. Fix $0 < \varepsilon < \varepsilon_0$. Apply Theorem 4.5 with upper level $\varepsilon/2$, chosen after an arbitrarily small decrease so that it is a regular value of G_p. We obtain a regular value $\bar{\delta}_p \in (0, \varepsilon/2)$ and

$$0 \neq [\beta_p^M] \in H^1(C^*_{p,(\bar{\delta}_p,\varepsilon/2)}). \tag{4.23}$$

Choose $\eta > 0$ so small that no critical value of the chosen Morse perturbation of G_p lies in the intervals $(\bar{\delta}_p - 2\eta, \bar{\delta}_p + 2\eta)$ or $(\varepsilon/2 - 2\eta, \varepsilon/2 + 2\eta)$, and

$$0 < \eta < \frac{1}{4}\min\{\bar{\delta}_p, \varepsilon/2 - \bar{\delta}_p, \varepsilon\}. \tag{4.24}$$

By Proposition 4.4, the filtered chain maps Φ_p, Ψ_p and homotopies may be chosen with action loss at most η. Set

$$\delta_p = \bar{\delta}_p - \eta.$$

Then $0 < \delta_p < \varepsilon$, and Ψ_p sends a Morse representative of (4.23) to a local Floer cocycle whose generators have action in

$$(pc + \bar{\delta}_p - \eta,\, pc + \varepsilon/2 + \eta) \subset (pc + \delta_p,\, pc + \varepsilon).$$

If the resulting local Floer class vanished in the quotient on the right-hand side of (4.22), applying Φ_p and using (4.19) would give a primitive for $[\beta_p^M]$ in the enlarged Morse window

$$(\bar{\delta}_p - 2\eta,\, \varepsilon/2 + 2\eta),$$

which is still contained in $(0, \varepsilon_0)$. By the choice of η, the four endpoints $\bar{\delta}_p - 2\eta, \bar{\delta}_p, \varepsilon/2, \varepsilon/2 + 2\eta$ can be joined without crossing a critical value; the standard filtered Morse continuation maps therefore identify this enlarged window with the original window $(\bar{\delta}_p, \varepsilon/2)$. This contradicts (4.23). Hence (4.22) holds. □

4.4.3. *Injection into the global quotient.*

Proposition 4.7. *Choose*

$$0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_U/10\}. \tag{4.25}$$

For every sufficiently large relatively admissible prime p, the natural map

$$\iota_p : H^1\Big(\mathrm{CW}^*_{\mathrm{loc},(pc+\delta_p,pc+\varepsilon)}(H^{\#p}, L; x^p)\Big) \longrightarrow H^1\Big(\mathrm{CW}^*_{(pc+\delta_p,pc+\varepsilon)}(H^{\#p}; L, L)\Big)$$

is injective on the nonzero class produced by Theorem 4.6.

Proof. Let β_p be a local cocycle representing the class from (4.22). Thus

$$d_{\mathrm{loc}}\beta_p = 0, \qquad \mathrm{supp}(\beta_p) \subset U', \qquad pc + \delta_p \le \mathcal{A}(\beta_p) < pc + \varepsilon.$$

Write the global differential as

$$d = d_{\mathrm{loc}} + d_{\mathrm{out}},$$

where every term of d_{out} is represented by a strip u with

$$\mathrm{im}\, u \cap U' \neq \emptyset, \qquad \mathrm{im}\, u \not\subset U.$$

Lemma 2.3 gives $E(u) \ge \varepsilon_U$. Hence if a generator z occurs in $d_{\mathrm{out}}\beta_p$, then by (4.15)

$$\mathcal{A}(z) \ge \mathcal{A}(\beta_p) + \varepsilon_U > pc + \delta_p + \varepsilon_U > pc + \varepsilon. \tag{4.26}$$

Therefore $d\beta_p = 0$ in $\mathrm{CW}^*_{(pc+\delta_p,pc+\varepsilon)}(H^{\#p}; L, L)$.

Suppose that $\iota_p[\beta_p] = 0$. Then there is a global degree-zero cochain Γ_p with

$$d\Gamma_p = \beta_p \quad \text{in } \mathrm{CW}^*_{(pc+\delta_p,pc+\varepsilon)}(H^{\#p}; L, L).$$

Decompose $\Gamma_p = \Gamma_p^U + \Gamma_p^{\mathrm{out}}$, where Γ_p^U is generated by chords in U and Γ_p^{out} has no generator in U'. If a generator in U' occurred in $d\Gamma_p^{\mathrm{out}}$, the corresponding strip would satisfy the hypotheses of Lemma 2.3; the same estimate as in (4.26) would put its output action at least ε_U above the input action, hence outside the quotient window. Thus the projection of $d\Gamma_p^{\mathrm{out}}$ to the local quotient is zero, and projection to the local part gives

$$d_{\mathrm{loc}}\Gamma_p^U = \beta_p \quad \text{in } \mathrm{CW}^*_{\mathrm{loc},(pc+\delta_p,pc+\varepsilon)}(H^{\#p}, L; x^p),$$

contradicting the nonzero local class of Theorem 4.6. Therefore $\iota_p[\beta_p] \neq 0$. □

5. First-return graph and proof of the main theorem

5.1. **The finite first-return graph.** Fix $N \geq 1$ and define

$$S_N := \bigcup_{1 \leq k \leq N} \mathcal{X}_k(H, L).$$

By (D3), S_N is finite. For $q \in S_N$ set

$$\rho(q) := \min\{1 \leq k \leq N \mid \tau^k(q) \in L\}, \qquad F(q) := \tau^{\rho(q)}(q).$$

The finite *first-return graph* $\mathcal{G}_N$ has vertex set S_N and an edge

$$e_q : q \longrightarrow F(q)$$

whenever $F(q) \in S_N$. The edge weights are

$$T(e_q) := \rho(q), \qquad A(e_q) := \mathcal{A}_{H^{\#\rho(q)}}(q).$$

For an oriented path $P = e_{q_0} \cdots e_{q_{s-1}}$, put

$$T(P) := \sum_{a=0}^{s-1} T(e_{q_a}), \qquad A(P) := \sum_{a=0}^{s-1} A(e_{q_a}).$$

The empty path is allowed and has $T = A = 0$.

Lemma 5.1. *Assume that every compact interior L-chord has minimal order at most N. Let y be a compact interior chord of $H^{\#m}$, and put $q_0 = y(0) \in L$. Then $q_0 \in S_N$, and there is a unique oriented path P_y in $\mathcal{G}_N$ such that*

$$T(P_y) = m, \qquad A(P_y) = \mathcal{A}_{H^{\#m}}(y).$$

Proof. Set $R_0 := m$. The path $t \mapsto \varphi_H^t(q_0)$, $0 \leq t \leq R_0$, is a compact interior L-chord. By the bounded-order hypothesis, its minimal return is at most N, hence $q_0 \in S_N$. Suppose that $q_a \in S_N$ and

$$R_a := m - \sum_{b<a} \rho(q_b) > 0, \qquad \tau^{R_a}(q_a) = \tau^m(q_0) \in L.$$

The suffix $t \mapsto \varphi_H^t(q_a)$, $0 \leq t \leq R_a$, is again a compact interior L-chord. Therefore

$$\rho(q_a) \leq \min\{N, R_a\}.$$

If $\rho(q_a) = R_a$, the construction stops. If $\rho(q_a) < R_a$, define

$$q_{a+1} := \tau^{\rho(q_a)}(q_a).$$

Then $q_{a+1} \in L$, and the remaining suffix has positive return time $R_a - \rho(q_a)$. Applying the bounded-order hypothesis to this suffix gives $q_{a+1} \in S_N$, so the edge e_{q_a} belongs to $\mathcal{G}_N$. Since the positive integer R_a strictly decreases, the construction stops after finitely many steps, and the resulting path has total time m.

At each stage the first edge is forced by the minimality in the definition of ρ. Applying the same argument to every suffix proves uniqueness.

Let

$$t_a := \sum_{b<a} \rho(q_b), \qquad q_a = \tau^{t_a}(q_0).$$

Because each t_a is an integer, the restriction of y to $[t_a, t_{a+1}]$, translated to $[0, \rho(q_a)]$, is the Hamiltonian chord of $H^{\#\rho(q_a)}$ starting at q_a. Hence exactness gives

$$\begin{aligned}\mathcal{A}_{H^{\#m}}(y) &= \sum_{a=0}^{s-1} \left(\int_{t_a}^{t_{a+1}} y^*\lambda - \int_{t_a}^{t_{a+1}} H_t^{\#m}(y(t))\, dt + f_L(q_a) - f_L(q_{a+1}) \right) \\ &= \sum_{a=0}^{s-1} \mathcal{A}_{H^{\#\rho(q_a)}}(q_a) = A(P_y),\end{aligned}$$

where the primitive terms telescope to $f_L(q_0) - f_L(y(m))$, the primitive contribution in $\mathcal{A}_{H^{\#m}}(y)$. □

A directed cycle $C \subset \mathcal{G}_N$ has

$$T(C) := \sum_{e \in C} T(e), \qquad A(C) := \sum_{e \in C} A(e).$$

After choosing a base vertex on C, it determines an isolated fixed L-chord γ_C of $H^{\#T(C)}$. The isolation follows from (D3), applied to the period $T(C)$.

We establish a finite normal form for sufficiently long directed paths in the graph.

Lemma 5.2. *There are an integer $L_N \geq 0$ and a finite set $\mathfrak{T}_N$ of triples $\Theta = (U, C, V)$, where C is a based directed cycle, U is a directed path whose terminal vertex is the base vertex of C, and V is a proper initial subpath of C, with the following property. Every directed path P in $\mathcal{G}_N$ whose number of edges is at least L_N admits a unique decomposition*

$$P = UC^rV, \qquad \Theta = (U, C, V) \in \mathfrak{T}_N, \qquad r \in \mathbb{Z}_{\geq 0}.$$

Here uniqueness means uniqueness after the base point of each cycle appearing in $\mathfrak{T}_N$ has been fixed as in the proof below. For such a type set

$$M_\Theta := T(C), \qquad a_\Theta := T(U) + T(V), \qquad B_\Theta := A(C), \qquad b_\Theta := A(U) + A(V).$$

Then

$$T(UC^rV) = a_\Theta + rM_\Theta, \qquad A(UC^rV) = b_\Theta + rB_\Theta. \tag{5.1}$$

Consequently, for every real number c*,*

$$A(P) - cT(P) = r\big(B_\Theta - cM_\Theta\big) + b_\Theta - ca_\Theta. \tag{5.2}$$

Proof. Let $|P|$ denote the number of edges of a path. The graph $\mathcal{G}_N$ is finite and every vertex has at most one outgoing edge, because the outgoing edge from q is forced to be $e_q : q \to F(q)$. Fix once and for all a total ordering of the finite set S_N. This ordering will be used only to choose a base vertex on each directed cycle.

For a directed path

$$P = e_{q_0}e_{q_1}\cdots e_{q_{\ell-1}}, \qquad q_{i+1} = F(q_i),$$

write its vertex sequence as $(q_0, q_1, \ldots, q_\ell)$. If $\ell > |S_N|$, two vertices in this sequence coincide. Let i_0 be the smallest index for which there is some $j > i_0$ with $q_j = q_{i_0}$, and let $j_0 > i_0$ be the smallest such j. Then

$$C_P := e_{q_{i_0}}e_{q_{i_0+1}}\cdots e_{q_{j_0-1}}$$

is a directed cycle. By the minimality of i_0, the vertices $q_0, \ldots, q_{i_0-1}$ do not lie on this first repeated cycle. Since each vertex has at most one outgoing edge, once the path reaches q_{i_0} all subsequent edges are forced to run periodically around C_P. Therefore there are unique integers $r \geq 0$ and $0 \leq h < |C_P|$ such that

$$\ell - i_0 = r|C_P| + h.$$

With

$$U_P := e_{q_0}\cdots e_{q_{i_0-1}}, \qquad V_P := e_{q_{i_0}}\cdots e_{q_{i_0+h-1}},$$

where U_P or V_P is allowed to be empty, we get

$$P = U_P C_P^r V_P.$$

The terminal path V_P is a proper initial subpath of C_P because $0 \leq h < |C_P|$.

The above construction is canonical for the path P. To obtain a finite list of types independent of P, take $\mathfrak{T}_N$ to be the collection of all triples (U, C, V) that arise from this canonical construction for some path with $|P| > |S_N|$, after replacing the displayed cycle by the same cycle based at its smallest vertex with respect to the fixed ordering and cyclically shifting U, V accordingly. There are finitely many such triples because S_N is finite. The preceding canonical construction also proves uniqueness of Θ and r for every path with $|P| > |S_N|$. Thus one may take $L_N = |S_N| + 1$.

Finally, both weights are additive under concatenation:

$$T(P_1P_2) = T(P_1) + T(P_2), \qquad A(P_1P_2) = A(P_1) + A(P_2),$$

whenever the terminal vertex of P_1 is the initial vertex of P_2. Applying this to UC^rV gives (5.1); subtracting $cT(UC^rV)$ gives (5.2). $\square$

5.2. **Proof of Theorem 1.10 and 1.11.** The purpose of this subsection is to convert the graph word UC^rV into the type of local finite-dimensional object to which the Section 4.4 action-gap package applies. This is the precise point at which the final proof departs from pure iterations: the tails U, V are kept inside the local germ and are not discarded.

For $\Theta = (U, C, V) \in \mathfrak{T}_N$ (we call it an *eventual type*), write

$$P_{\Theta,r} := UC^rV, \qquad m_{\Theta,r} := T(P_{\Theta,r}) = a_\Theta + rM_\Theta, \qquad A_{\Theta,r} := A(P_{\Theta,r}) = b_\Theta + rB_\Theta.$$

Let $y_{\Theta,r}$ be the compact Hamiltonian L-chord of $H^{\#m_{\Theta,r}}$ represented by the graph path $P_{\Theta,r}$. By (D3), $y_{\Theta,r}$ is isolated.

Write

$$M_\Theta := T(C), \qquad B_\Theta := A(C).$$

The directed cycle C, with its chosen base vertex, determines an isolated fixed L-chord

$$\gamma_C \in \mathcal{X}(H^{\#M_\Theta}; L, L)$$

obtained by concatenating the first-return chords corresponding to the edges of C. Choose a Maslov-zero relative symplectic trivialization along the whole tower $\{\gamma_C^r\}_{r\geq 1}$. Let

$$\Lambda_C(t) = D\phi_H^t(T_{\gamma_C(0)}L), \qquad 0 \leq t \leq M_\Theta,$$

viewed in this trivialization, and set $\Lambda_0 = T_{\gamma_C(0)}L$. Define

$$\Delta_\Theta = \Delta_L^{H^{\#M_\Theta}}(\gamma_C) := \lim_{r\to\infty} \frac{1}{r}\mu_{RS}\big(\Lambda_C^{\#r}, \Lambda_0\big). \tag{5.3}$$

The limit exists because the Robbin–Salamon/Maslov index is a homogeneous quasimorphism after homogenization; equivalently this is the usual Lagrangian mean index. The fixed tails U, V do not enter the limit: changing the base point on C, or adding fixed entry and exit paths, changes the Robbin–Salamon index by a bounded correction.

We give a type-II generating function package for an eventual type.

Proposition 5.3. *Fix* $\Theta = (U, C, V) \in \mathfrak{T}_N$*. For every sufficiently large* r*, there is a finite-dimensional generating-function germ* $(G_{\Theta,r}, 0)$*, defined in a neighborhood of the origin in a Euclidean space depending on* r*, with* 0 *an isolated critical point, such that:*

(i) In the local type-II chart near $y_{\Theta,r}$*,*

$$\mathcal{A}_{H^{\#m_{\Theta,r}}} = A_{\Theta,r} + G_{\Theta,r}, \qquad G_{\Theta,r}(0) = 0, \qquad dG_{\Theta,r}(0) = 0. \tag{5.4}$$

(ii) There is an integer $\sigma_{\Theta,r}$ *such that*

$$\mathrm{HW}^{d}_{\mathrm{loc}}(H^{\#m_{\Theta,r}}, L; y_{\Theta,r}) \cong HM^{d+\sigma_{\Theta,r}}_{\mathrm{loc}}(G_{\Theta,r}, 0). \tag{5.5}$$

The isomorphism is the local Floer–Morse reduction of Proposition 3.4, with the absolute grading fixed by $2c_1(W) = 0$ *and* $\mu_L = 0$*.*

(iii) There is a constant $D_\Theta > 0$*, independent of* r*, such that*

$$\mathrm{HW}^{d}_{\mathrm{loc}}(H^{\#m_{\Theta,r}}, L; y_{\Theta,r}) \neq 0 \quad \Longrightarrow \quad |d + r\Delta_\Theta| \le D_\Theta. \tag{5.6}$$

(iv) For every $\eta > 0$*, after shrinking the isolating pair around* $y_{\Theta,r}$*, the chain maps comparing the local Floer complex with the local Morse complex of* $G_{\Theta,r}$*, and the corresponding chain homotopies, may be chosen with filtration error at most* η*. The analytic estimates used to solve the infinite-dimensional range equation are uniform over the repeated blocks of the word* UC^rV*; only the final isolating radius is allowed to depend on* r*.*

Proof. We first construct the finite-dimensional germ. Choose a subdivision of the Hamiltonian flow which is fine enough that every short-time graph appearing in every edge of $\mathcal{G}_N$ admits a type-II generating function as in Lemma 3.2. This can be done uniformly because the set of edges of $\mathcal{G}_N$ is finite and all corresponding trajectories lie in a compact subset of $\operatorname{int} W$. For the word $P_{\Theta,r} = UC^rV$, concatenate the short-time type-II generating functions and the endpoint terms exactly as in (3.5). Denote the resulting discrete action by $S_{\Theta,r}$. Its variables are the position and terminal-momentum variables at the subdivision points of the word.

The first-variation computation of Lemma 3.3 applies to each short piece and gives

$$\mathrm{Crit}(S_{\Theta,r}) \ \text{ near } 0 \quad \Longleftrightarrow \quad \text{Hamiltonian } L\text{-chords of } H^{\#m_{\Theta,r}} \text{ near } y_{\Theta,r}.$$

By (D3), $y_{\Theta,r}$ is an isolated $m_{\Theta,r}$-chord. Hence the corresponding critical point of $S_{\Theta,r}$ is isolated. Since the critical value is exactly the sum of the edge actions along $P_{\Theta,r}$, it is $A_{\Theta,r}$. Put

$$G_{\Theta,r} := S_{\Theta,r} - A_{\Theta,r}.$$

This proves (5.4).

The local Floer–Morse identification is now Proposition 3.4 applied to the isolated chord $y_{\Theta,r}$ of the Hamiltonian $H^{\#m_{\Theta,r}}$. Thus (5.5) holds for some grading shift $\sigma_{\Theta,r}$. The filtered refinement is the same argument as Proposition 4.4: in the Banach chart one writes a strip as $v + w$, with v in the type-II finite-dimensional slice and w in the L^2-orthogonal complement. The operator in the w-direction is a Cauchy–Riemann type operator on uniformly short pieces with totally real boundary condition. The estimates used to construct a right inverse are the Calderon–Zygmund boundary estimate (2.5), the uniform short-time transversality from Lemma 3.2, and the stability of right inverses under bounded C^1-perturbations. Since the word is made from two fixed tails and repetitions of one fixed block, these local estimates have constants independent of the number of repetitions. Shrinking the isolating neighborhood makes

$$\left|\mathcal{A}_{H^{\#m_{\Theta,r}}}(v + w(v)) - A_{\Theta,r} - G_{\Theta,r}(v)\right| < \eta,$$

and the same estimate holds for the parametrized comparison equations. This proves the filtration statement in (iv).

It remains to prove the uniform degree estimate. Let $\Psi_{\Theta,r}(t)$ be the linearized Hamiltonian path along $y_{\Theta,r}$, expressed in the Maslov-zero relative trivialization obtained by transporting the chosen trivialization on the cycle through the fixed tails. Put

$$\mu_{\Theta,r} := \mu_{RS}\big(\Psi_{\Theta,r}(t)T_{y_{\Theta,r}(0)}L, T_{y_{\Theta,r}(m_{\Theta,r})}L\big).$$

The entry and exit tails have fixed length. Additivity of the Robbin–Salamon index under concatenation, together with the standard Hörmander–Kashiwara correction term for changing intermediate Lagrangian reference spaces, gives a constant $C_{1,\Theta}$ such that

$$\Big|\mu_{\Theta,r} - \mu_{RS}\big(\Lambda_C^{\#r}, \Lambda_0\big)\Big| \le C_{1,\Theta} \qquad \text{for all } r. \tag{5.7}$$

The homogenization property of the Lagrangian Maslov index gives another constant $C_{2,\Theta}$ with

$$\Big|\mu_{RS}\big(\Lambda_C^{\#r}, \Lambda_0\big) - r\Delta_\Theta\Big| \le C_{2,\Theta} \qquad \text{for all } r. \tag{5.8}$$

Combining (5.7) and (5.8),

$$|\mu_{\Theta,r} - r\Delta_\Theta| \le C_{3,\Theta}. \tag{5.9}$$

Let $B_{\Theta,r} = d^2 G_{\Theta,r}(0)$, let $\lambda_{\Theta,r}$ be its Morse index, and let $\nu_{\Theta,r} = \dim\ker B_{\Theta,r}$. The kernel of the discrete Hessian is canonically the space of solutions of the linearized Hamiltonian chord equation with Lagrangian boundary conditions; hence

$$0 \le \nu_{\Theta,r} \le n.$$

By the finite-dimensional splitting lemma for an isolated critical point,

$$HM^j_{\mathrm{loc}}(G_{\Theta,r}, 0) \ne 0 \quad \Longrightarrow \quad \lambda_{\Theta,r} \le j \le \lambda_{\Theta,r} + \nu_{\Theta,r}. \tag{5.10}$$

Indeed, after splitting off a nondegenerate quadratic form of index $\lambda_{\Theta,r}$, the remaining center germ has dimension $\nu_{\Theta,r}$, and its local Morse cohomology is supported in degrees $0, \ldots, \nu_{\Theta,r}$.

The determinant-line comparison in the type-II reduction identifies the Morse index of the nondegenerate part with the Robbin–Salamon index of the corresponding Cauchy–Riemann operator. In the grading convention $\mathrm{gr} = n - \mu_{RS} + \kappa_L$, this gives a constant $C_{4,\Theta}$, depending only on the fixed tails and on the normalization of the grading, such that

$$\big|(\lambda_{\Theta,r} - \sigma_{\Theta,r}) - \big(n - \mu_{\Theta,r} + \kappa_L\big)\big| \le C_{4,\Theta} \qquad \text{for all } r. \tag{5.11}$$

This is the same spectral-flow comparison used in Proposition 3.4: the crossing forms of the discrete Hessian are the Robbin–Salamon crossing forms of the linearized Lagrangian boundary-value problem; the only additional terms here are the fixed endpoint corrections contributed by U and V.

Suppose now that $\mathrm{HW}^d_{\mathrm{loc}}(H^{\#m_{\Theta,r}}, L; y_{\Theta,r}) \ne 0$. By (5.5), $HM^{d+\sigma_{\Theta,r}}_{\mathrm{loc}}(G_{\Theta,r}, 0) \ne 0$. Applying (5.10),

$$0 \le d + \sigma_{\Theta,r} - \lambda_{\Theta,r} \le \nu_{\Theta,r} \le n.$$

Therefore (5.11) implies

$$|d + \mu_{\Theta,r}| \le C_{5,\Theta}.$$

Combining this with (5.9) gives

$$|d + r\Delta_\Theta| \le |d + \mu_{\Theta,r}| + |\mu_{\Theta,r} - r\Delta_\Theta| \le C_{5,\Theta} + C_{3,\Theta}.$$

Taking $D_\Theta = C_{5,\Theta} + C_{3,\Theta}$ proves (5.6). □

Definition 1.9 now applies to the eventual types of $\mathcal{G}_N$.

Proof of Proposition 1.10. Assume that every compact interior L-chord has minimal order at most N. Let $m \to \infty$ run through any unbounded sequence of positive integers. For each such m, choose the cofinal slope as in Lemma 4.1. We obtain a compact degree-zero cocycle representing the unit in $CW^0((K_a)^{\#m}; \widehat{L}, \widehat{L})$.

By Lemma 5.1, every compact generator in this representative is represented by a path in $\mathcal{G}_N$. Passing to the action-filtered spectral sequence of Proposition 4.2, the nonzero unit class has a nonzero associated graded component on the E_∞-page. Hence at least one degree-zero local summand on the E_1-page is nonzero. Therefore, for each sufficiently large m in the chosen unbounded set, there exists a graph path P_m with $T(P_m) = m$ such that

$$\mathrm{HW}^0_{\mathrm{loc}}(H^{\#m}, L; y_{P_m}) \neq 0. \tag{5.12}$$

Since $\mathfrak{T}_N$ is finite, Lemma 5.2 allows us to pass to an infinite subsequence for which

$$P_m = P_{\Theta, r_i} = UC^{r_i}V$$

for one fixed $\Theta = (U, C, V)$, with $r_i \to \infty$. Applying the support estimate (5.6) in degree $d = 0$, we get

$$|r_i \Delta_\Theta| \leq D_\Theta$$

for infinitely many $r_i \to \infty$. Hence $\Delta_\Theta = 0$. Together with (5.12), this is exactly Definition 1.9. □

Proof of Theorem 1.11. Fix a carrier integer $r = r_i$. By Proposition 5.3, the local action near $y_{\Theta,r}$ is $A_{\Theta,r} + G_{\Theta,r}$, and the local Floer class in degree zero is represented in the filtered local Morse complex of $G_{\Theta,r}$. The proof of the finite-dimensional boundary-depth input, Theorem 4.5, is purely Morse-theoretic after the type-II reduction: it uses only the isolated critical group in the extremal degree, the normalization $G_{\Theta,r}(0) = 0$, and uniform isolation. These hypotheses hold here by Definition 1.9 and the uniform reduction in Proposition 5.3. Thus there exists $\bar{\delta}_i \in (0, \varepsilon/2)$ and a nonzero class

$$0 \neq [\beta_i^M] \in H^1\big(CM^*_{\mathrm{loc},(\bar{\delta}_i,\varepsilon/2)}(G_{\Theta,r_i})\big). \tag{5.13}$$

The filtered comparison maps from Proposition 5.3(iv) have action loss at most η, where $\eta > 0$ is chosen so small that

$$0 < \eta < \frac{1}{4}\min\{\bar{\delta}_i, \varepsilon/2 - \bar{\delta}_i, \varepsilon\}.$$

Exactly as in the proof of Theorem 4.6, (5.13) gives a nonzero local Floer class in

$$H^1\big(CW^*_{\mathrm{loc},(A_{\Theta,r_i}+\delta_i,A_{\Theta,r_i}+\varepsilon)}(H^{\#m_{\Theta,r_i}},L;y_{\Theta,r_i})\big)$$

for $\delta_i := \bar{\delta}_i - \eta \in (0,\varepsilon)$.

Choose ε_Θ smaller than one tenth of the energy-separation constant of a fixed isolating pair for the finite set of local models involved in Θ. The proof of Proposition 4.7 applies verbatim: a Floer strip leaving the isolating neighborhood has energy at least ε_U, hence raises the cohomological action by at least ε_U, which is larger than the whole window. Therefore the nonzero local class injects into the corresponding global quotient. A nonzero quotient cohomology group has a generator in the same action window; hence there is a chord z_i with

$$A_{\Theta,r_i} + \delta_i < \mathcal{A}_{H^{\#m_{\Theta,r_i}}}(z_i) < A_{\Theta,r_i} + \varepsilon.$$

Finally, choose the cofinal slope by Proposition 4.3, with the compact action scale containing the affine values $A_{\Theta,r_i} = O(r_i)$. All transition and end generators then have action below this scale, so the chord z_i is contained in a compact subset of W. Since the Hamiltonian agrees with H on this compact subset, z_i is a compact interior chord of $H^{\#m_{\Theta,r_i}}$. □

5.3. **Proof of Lemma 1.12.** The action gap in Theorem 1.11 is centered at the affine action value of the same eventual type Θ, not necessarily at the pure cycle action. The following proof establishes the finite graph action-separation statement announced in Lemma 1.12.

Proof of Lemma 1.12. For all sufficiently large r, every path P with $T(P) = T(P_{\Theta_0,r})$ is sufficiently long for Lemma 5.2. Write $P = P_{\Theta,s} = UC^sV$ for some $\Theta = (U,C,V) \in \mathfrak{T}_N$. Put

$$M_0 := M_{\Theta_0}, \quad a_0 := a_{\Theta_0}, \quad B_0 := B_{\Theta_0}, \quad b_0 := b_{\Theta_0}.$$

The equality of total times is

$$a_\Theta + sM_\Theta = a_0 + rM_0. \tag{5.14}$$

If (5.14) has no integer solution $s \geq 0$ for a given r, the type Θ contributes nothing. Otherwise the action difference is

$$\begin{aligned} A(P) - A(P_{\Theta_0,r}) &= (b_\Theta + sB_\Theta) - (b_0 + rB_0) \\ &= \left(b_\Theta - \frac{B_\Theta}{M_\Theta}a_\Theta\right) - \left(b_0 - \frac{B_0}{M_0}a_0\right) + rM_0\left(\frac{B_\Theta}{M_\Theta} - \frac{B_0}{M_0}\right), \end{aligned} \tag{5.15}$$

where (5.14) was used to eliminate s.

If $B_\Theta/M_\Theta \neq B_0/M_0$, then the last term in (5.15) tends to $\pm\infty$ with r. Hence, after increasing R_0, it never lies in $(0,1)$.

It remains to consider the finitely many resonant types satisfying

$$\frac{B_\Theta}{M_\Theta} = \frac{B_0}{M_0}.$$

For such a type the difference (5.15) is a constant

$$\beta_\Theta := b_\Theta - \frac{B_\Theta}{M_\Theta}a_\Theta - b_0 + \frac{B_0}{M_0}a_0.$$

Let

$$B_+ := \{\beta_\Theta > 0 \mid \Theta \in \mathfrak{T}_N, B_\Theta/M_\Theta = B_0/M_0\}.$$

This is a finite set. Choose

$$0 < \varepsilon_0 < 1$$

if $B_+ = \varnothing$, and otherwise choose

$$0 < \varepsilon_0 < \min(1, \min B_+).$$

Then no resonant type has action difference in $(0, \varepsilon_0)$, while the preceding paragraph excludes nonresonant types for all $r \geq R_0$. Hence every directed path P with $T(P) = T(P_{\Theta_0,r})$ satisfies

$$A(P) \notin \big(A(P_{\Theta_0,r}), A(P_{\Theta_0,r}) + \varepsilon_0\big).$$

If a compact chord of the same total time has minimal order at most N, Lemma 5.1 identifies it with such a path and preserves action. This proves Lemma 1.12. $\square$

5.4. **Proof of Theorem 1.4.**

Proof of Theorem 1.4. Assume, toward contradiction, that there exists N such that every compact interior L-chord has minimal order at most N. Theorem 1.6 gives compact degree-zero representatives of the wrapped unit in the relevant admissible iterates. Using these representatives, the local-to-global filtration, the first-return graph decomposition, and the mean-index alternative of Theorem 1.8, Proposition 1.10 produces a tail-stabilized SDM carrier

$$\Theta_0 = (U_0, C_0, V_0)$$

and carrier integers $r_i \to \infty$. Put

$$m_i := m_{\Theta_0, r_i} = T(P_{\Theta_0, r_i}), \qquad A_i := A_{\Theta_0, r_i} = A(P_{\Theta_0, r_i}).$$

Let ε_{Θ_0} be given by Theorem 1.11, and let ε_0 be given by Lemma 1.12. Choose

$$0 < \varepsilon < \min\{\varepsilon_{\Theta_0}, \varepsilon_0\}.$$

The tail-stabilized action escape theorem, Theorem 1.11, which is the Section 4 action-gap package applied to the eventual-type local model of Proposition 5.3, gives, for every sufficiently large i, a compact interior chord z_i of $H^{\# m_i}$ and a number $\delta_i \in (0, \varepsilon)$ such that

$$A_i + \delta_i < \mathcal{A}_{H^{\# m_i}}(z_i) < A_i + \varepsilon. \tag{5.16}$$

If z_i had minimal order at most N, Lemma 5.1 would identify it with a directed path P_{z_i} in $\mathcal{G}_N$ satisfying

$$T(P_{z_i}) = m_i, \qquad A(P_{z_i}) = \mathcal{A}_{H^{\# m_i}}(z_i).$$

This contradicts the first-chapter action-separation lemma, Lemma 1.12, because (5.16) places $A(P_{z_i})$ in the excluded interval $(A_i, A_i + \varepsilon)$. Thus, for sufficiently large i, the chord z_i has minimal order greater than N, contradicting the assumed boundedness by N. Hence interior L-chords have arbitrarily large minimal order. ☐

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LYU CHANGLE, SCHOOL OF GIFTED YOUNG, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Email address: `lcl20060401lcl@mail.ustc.edu.cn`