SUPERHEAVINESS RESULT ON THE SKELETON OF DIVISOR COMPLEMENT

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ABSTRACT. In a negatively monotone manifold, we show the superheaviness results for the skeleton of the complement of a simple crossings symplectic divisor satisfying certain conditions.

CONTENTS

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1. Introduction

1.1. **Setup and main results.** Let (M^{2n}, ω) be a closed symplectic manifold and $C(M)$ is the space of continuous functions on M, Entov-Polterovich [EP06, EP09] introduced the concept of *partial symplectic quasi-state*, which is a functional $\zeta : C(M) \to \mathbb{R}$ satisfying the following properties.

(1) (Additive with respect to constant) $\zeta(F+a) = \zeta(F) + a$ for any $a \in \mathbb{R}$. In particular $\zeta(1) = 1$.
(2) (Semi-homogeneity) $\zeta(\lambda F) = \lambda\zeta(F)$ for any $\lambda \in \mathbb{R}_{\geq 0}$.
(3) (Monotonicity) If $F \leq G$, then $\zeta(F) \leq \zeta(G)$.
(4) (Lipschitz continuity) $|\zeta(F) - \zeta(G)| \leq |F - G|_{C^0}$.
(5) (Partial additivity) If F, G are Poisson-commuting smooth functions on M, then $\zeta(F+G) \leq \zeta(F) + \zeta(G)$.
(6) (Invariance) $\zeta(F) = \zeta(F \circ \varphi)$ for Hamiltonian diffeomorphism φ on M.

Partial symplectic quasi-states are powerful tools to detect various rigidity phenomena in symplectic geometry.

One can construct partial symplectic quasi-states via spectral invariants and we briefly recall the construction here. The details will be given in Section 2.

Given with any idempotent e of the quantum cohomology $\mathrm{QH}(M, \omega)$ of M and a Hamiltonian function H, one can define a number $c_e(H) \in \mathbb{R}$, called the *spectral invariant* with respect to e and H. For any autonomous Hamiltonian H on M, we define

$$\zeta_e(H) = \lim_{k\to\infty} \frac{c_e(kH)}{k}$$

and ζ_e is the desired partial symplectic quasi-state. Note that the limit is well defined due to the properties of spectral invariants.

Once a partial quasi-state ζ is defined, one can have the notion of (super)heavy sets, which is the target of this paper.

Definition 1.1. Fix an idempotent e as above, then a compact subset $K \subseteq M$ is called

(1) *e-heavy* if

$$\zeta_e(H) \leq \sup_K H \qquad (\forall H \in C(M))$$

holds.

(2) *e-superheavy* if

$$\zeta_e(H) \geq \inf_K H \qquad (\forall H \in C(M))$$

holds.

When we take e to be the identity element in quantum cohomology, we omit it from the notations.

The above notations have been closely related to various phenomena in symplectic topology, including displacing and large scale geometry [KO21, Sun24, FZ25]. At the same time, the problem of deciding (super)heaviness has also been of independent research result, advanced tools such as relative symplectic cohomology. Related notions such as SH-fullness and SH-heaviness have been introduced in [DGPZ24, MSV24].

In this article we aim to deduce superheaviness of the Lagrangian skeleton of the complement with respect to some certain symplectic divisors in a closed symplectic manifold. To be specific, we mainly focus on the negatively monotone closed symplectic manifold, that is, a closed symplectic manifold (M^{2n}, ω) with some negative constant τ so that

$$2\tau c_1(TM) = [\omega]$$

in $H^2(M; \mathbb{R})$.

We address the following result on the super-rigidity of Lagrangian skeleton of certain divisor complement.

Theorem 1.2. *Let (M, ω) be a negatively monotone manifold and D be an orthogonal SC divisor satisfying Hypothesis A, then the Lagrangian skeleton L of the divisor complement $X = M \backslash D$ is superheavy in M. The Hypothesis A is given as follow.*

Hypothesis A. If $D = \cup_{i=1}^N D_i$, then there are negative rational numbers $\lambda_1, \ldots, \lambda_N$ satisfying

$$2c_1(TM) = \sum_{i=1}^N \lambda_i \cdot \mathrm{PD}(D_i)$$

in $H^2(M; \mathbb{R})$.

Let X be the divisor complement $X = M \backslash D$, then there is an “adapted” one-form θ on X so that (X, θ) is a convex symplectic manifold of finite type and satisfies fine local properties(the details will be presented in Section 3.1 and 3.2).

Let Z be the corresponding Liouville flow on X. Then one can define a function $\rho^0 : X \to \mathbb{R}$ in the following way. For any $x \in X$, if the Liouville flow starting at x is defined for $t \in (-\infty, A)$, where $A \in \mathbb{R} \cup \{+\infty\}$, we let

$$\rho^0(x) = e^{-A}.$$

ρ^0 is a continuous function, but not smooth. We extend it to M continuously by setting $\rho^0 = 1$ on D. It is direct that the Lagrangian skeleton L is the zero preimage of ρ^0. In practice we sometimes call L the Lagrangian skeleton of (M, ω) with respect to D.

Since superheavy sets are SH-full(Theorem 2.14), we have that

Corollary 1.3. *Let (M, ω) be a negatively monotone manifold and D be an orthogonal SC divisor satisfying Hypothesis A, then the Lagrangian skeleton L of (M, ω) with respect to divisor D is SH-full in M.*

Remark 1.4. In [BSV22, TV23], the above result in the positive monotone case and Calabi-Yau case are derived respectively by verifying the definition of SH-fullness and analyzing the relative symplectic cohomology.

Using the properties of heavy and superheavy sets, we can also obtain the following corollary concerning the displaceability of the skeleton.

Corollary 1.5. *Let (M, ω) be a negatively monotone manifold and D be an orthogonal SC divisor satisfying Hypothesis A, then*

(1) L is stably non-displaceable in M.
(2) L is strongly non-displaceable in M.
(3) L has an n-dimensional cell.
(4) Any Lagrangian in M with non-vanishing Floer cohomology can not be displaced from L by Hamiltonian diffeomorphism.

The first two results come from Theorem 2.12. By [BC02, Lemma 3.2], if L cannot be Hamiltonian displaced from itself, then it admits an n-dimensional cell.

1.2. **Examples.** Now we present some examples which fits the conditions in Theorem 1.2, and consequently in all the corollaries.

Example 1.6. Let $(M, \omega) = (\Sigma_2 \times \Sigma_2, \pi_1^*\omega_2 + \pi_2^*\omega_2)$, where Σ_2 is the closed oriented Riemann surface of genus 2, with ω_2 the normalized Kähler form, and π_i is the projection to the i-th factor.

Consider $D_1 = \Sigma_2 \times \{p\}$ and $D_2 = \{q\} \times \Sigma_2$ and $H_i = \mathrm{PD}([D_i])$, then $D = D_1 \cup D_2$ is an orthogonal SC divisor. A direct computation yields $c_1(TM) = -2H_1 - 2H_2$ and $[\omega] = H_1 + H_2$ so (M, ω) is negatively monotone. Besides there is

$$2c_1(TM) = -4\,\mathrm{PD}([D_1]) - 4\,\mathrm{PD}([D_2])$$

so (M, ω) and D satisfy all of our conditions.

The Lagrangian skeleton L of $X = M \backslash D$ is $B^4 \times B^4$, where B^4 is the bouquet of four circles in $\Sigma_2 \backslash \{pt\}$, so by Theorem 1.2 and Corollary 1.5, L is superheavy and stably non-displaceable in M.

Example 1.7. Fix $n \geq 2$. Let $M \subseteq \mathbb{C}P^{n+1}$ be a smooth projective hypersurface with degree $d \geq n+3$, for example

$$M = \{[Z_0 : Z_1, \ldots, Z_{n+1}] \subseteq \mathbb{C}P^{n+1} \mid Z_0^d + \cdots + Z_{n+1}^d = 0\},$$

equipped with the Fubini-Study form. Then since

$$c_1(TM) = (n+2-d)h,$$

where h is the hyperplane class restricted to M, we have that (M, ω) is negatively monotone.

Let $D_i = M \cap \{Z_i = 0\}$, then $\mathrm{PD}(D_i) = (n+2)h$, then take

$$\lambda_i = \frac{2(n+2-d)}{n+2} \in \mathbb{Q}_{<0}, \quad i = 0, \ldots, n+1.$$

we have that $D = \cup_{i=0}^{n+1} D_i$ is an orthogonal SC divisor satisfying Hypothesis A.

Then $X = M \backslash D$ is a hypersurface in the algebraic torus $(\mathbb{C}^*)^{n+1}$. By [Zho20], the Lagrangian skeleton L of X is the dual of the RSTZ skeleton[RSTZ14] associated to the Newton polytope $d\Delta$. By Theorem 1.2, L is superheavy in M, thus strongly and stably non-displaceable.

1.3. **Further discussions.**

In this paper we basically deal with the superheaviness through its definition, which involves a rather concrete computation of the index and action of the Hamiltonian orbits. Though our method is direct, a relation of index and symplectic area is required, so the negatively monotone condition seems crucial.

We expect that Theorem 1.2 can be extended to more closed symplectic manifolds and SC divisors. In [MSV24], the authors proposed the following criterion for superheaviness from SH-fullness.

Theorem 1.8. [MSV24, Theorem 1.13] *Let K be a compact subset of M and $K_0 \supset K_1 \cdots$ be compact sets that contain K in their interior with $\cap K_i = K$. If $c_1(TM)|_{\pi_2(M)} = 0$ and the integral graded relative symplectic cochain complexes $SC^*_M(K_n;\Lambda)$ have finite boundary depth in all degrees for all $n \geq 1$, then K being SH-full implies that it is superheavy.*

As pointed out in [MSV24, Remark 1.14], this result can extend to certain monotone manifolds. Since the SH-fullness of Lagrangian skeleton is proved in [BSV22, TV23] using different methods, we hope that combining these arguments and a generalized version of Theorem 1.8 would yield the superheaviness result under a broader setting than ours.

Remark 1.9. [Gat24] presented a super-rigidity result for Lagrangian skeletons in a positively monotone manifold with respect to SC divisors, where λ_i in Hypothesis A are positive rational number in $(0, 2]$ but not necessarily orthogonal.

Although his argument to deduce superheaviness is different from ours, it is expected that his method to deal with non-orthogonality works in our setting, which would extend our results to non-orthogonal divisors.

1.4. **Organization of the paper.** The article will be organized as follow.

In Section 2, we review the classical Hamiltonian Floer theory on closed manifolds and give a discussion on the rigidity results and quantitative symplectic topology.

In Section 3 we present a detailed analysis on SC divisors, introducing a system of neighborhoods of the divisor component D_i and the system of commuting Hamiltonians based on these neighborhoods. Then we construct an admissible local chart in every neighborhood.

In Section 4 firstly we complete the construction of the "radius" type Hamiltonian function, then we give a canonical capping of the radius type index using the local construction in admissible charts. Then we lay out the computation results on the index and action of the canonically capped orbits.

Finally in Section 5 we prove the main result Theorem 1.2 on superheaviness, based on a refined argument of [Sun24].

Acknowledgements.

2. Hamiltonian Floer Theory Preliminaries

2.1. **Hamiltonian Floer homology.** In this section we give a brief review of the Hamiltonian Floer theory and partial symplectic quasi-states on a general closed symplectic manifold. The discussions on virtual techniques concerning the problem of regularity of moduli spaces [LT98, FO99, Par16] are omitted here since these are quite independent of our arguments.

Fix a closed symplectic manifold (M, ω), let $H : S^1 \times M \to \mathbb{R}$ be a time-dependent non-degenerate Hamiltonian function on M. We say H is *normalized* if H_t has zero mean with respect to ω^n for any $t \in S^1$. $\mathrm{Fix}(H)$ is the set of contractible one-periodic orbits of the Hamiltonian vector field X_H determined by $-\iota_{X_H}\omega = dH$. We also fix a regular compatible almost complex structure J as in [Flo88]. A *capping* u of $\gamma \in \mathrm{Fix}(H)$ is a smooth map $u : \mathbb{D} \to M$ with

$$u(e^{2\pi i t}) = \gamma(t), \forall t \in [0, 1].$$

Two cappings u, u' of the same orbit γ are called equivalent if and only if

$$\omega(u) = \omega(u'), c_1(u) = c_1(u'),$$

where $c_1 = c_1(TM, \omega)$ is the Chern class of the triple (M, ω, J), which is independent of the choice of compatible almost structure J. And the set of equivalence classes is denoted $\widetilde{\mathrm{Fix}}(H)$.

The action of an element $[\gamma, u] \in \widetilde{\mathrm{Fix}}(H)$ is defined as

$$\mathcal{A}_H([\gamma, u]) = \int_S^1 H(t, \gamma(t))dt + \int_{\mathbb{D}} u^*\omega$$

and the degree is

$$\deg([\gamma, u]) = n + \mathrm{CZ}([\gamma, u]),$$

where $\mathrm{CZ}([\gamma, u])$ is the Conley-Zehnder index defined as in [CZ83].

For any $A \in \pi_2(M)$ and capped orbit $[\gamma, u]$, there is a capped orbit $[\gamma, u+A]$, where $u + A$ is the capping obtained by gluing a sphere representing the class A to u. The following result on the index and action changes are crucial to us.

$$\mathcal{A}_H([\gamma, u + A]) = \mathcal{A}_H([\gamma, u]) + \omega(A),$$
$$\deg([\gamma, u + A]) = \deg([\gamma, u]) + 2c_1(A).$$

The Hamiltonian Floer cochain complex is the $\mathbb{Q}$-vector space generated by formal sums of capped orbits as

$$\mathrm{CF}^k(H,J)=\{x=\sum_{i=1}^{\infty}a_i[\gamma_i,u_i]\mid a_i\in\mathbb{Q},\deg([\gamma_i,u_i])=k,\lim_{i\to\infty}\mathcal{A}_H([\gamma,u_i])=+\infty\}.$$

There is a natural index filtration on $\mathrm{CF}^*(H,J)$ and an action filtration given by

$$\mathcal{A}_H(x)=\min\{\mathcal{A}_H([\gamma,u_i])\mid a_i\neq 0\},x=\sum_{i=1}^{\infty}a_i[\gamma_i,u_i]\in\mathrm{CF}^*(H,J).$$

There is also a notion of *fractional caps* [BSV22, Section 3.3], that is, a formal expression $u+a$, where (γ,u) is a capped orbit and $a\in\mathbb{R}$. The associated index and action is given by

$$\mathcal{A}_H(\gamma,u+a)=\mathcal{A}_H(\gamma,u)+\tau a,\deg(\gamma,u+a)=\deg(\gamma,u)+a.$$

Note that the fractionally cap orbit $(\gamma,u+a)$ can be corresponded to an element $(\gamma,u)\otimes e^a\in\mathrm{CF}^*(H,J)$.

The differential d on the chain complex is given by virtual counting the moduli space of Floer cylinders and it increases the degree and action, the details can be found in [Par16]. The resulting cohomology $\mathrm{HF}^*(H,J)$ is called the Hamiltonian Floer cohomology of (H,J).

Remark 2.1. The coefficient ring is chosen to be $\mathbb{Q}$ in order to fit the virtual counting techniques. The recent effort in [AMS21, BX22, Rez22] allows one to define Hamiltonian Floer chain complex over integers on general close symplectic manifolds.

2.2. **Spectral invariants and partial symplectic quasi-states.** Now we can define the spectral invariants and thus construct partial symplectic quasi-states as in the introduction.

For any closed symplectic manifold (M,ω), its quantum cohomology is $\mathrm{QH}^*(M)=\mathrm{QH}^*(M,\omega;\Lambda_\omega)=H^*(M;\mathbb{Q})\otimes_{\mathbb{Q}}\Lambda_\omega$ and Λ_ω is the Novikov ring as in [MS12]. In [PSS96], the authors constructed the PSS isomorphism

$$PSS:\mathrm{QH}^*(M)\to\mathrm{HF}^*(H,J).$$

Then for any non-zero class $A\in\mathrm{QH}^*(M)$ of degree k, we define the spectral invariant as

$$c_A(H)=\sup\{\mathcal{A}_H(x)\mid x\in\mathrm{CF}^k(H,J),dx=0,[x]=PSS(A)\}.$$

Spectral invariants enjoy the following properties [FOOO19].

Theorem 2.2. *Let A be a non-zero pure-degree class in $QH^*(M)$, and let H be a non-degenerate Hamiltonian function. The spectral invariant $c_A(H)$ has the following properties.*

(1) (Spectrality) $c_A(H)$ is a finite number in $\mathrm{Spec}_k(H)$ where $k = \deg A$, and $\mathrm{Spec}_k(H)$ is the degree-k action spectrum

$$\mathrm{Spec}_k(H) = \{\mathcal{A}_H(x) \mid x \in \mathrm{CF}^k(H, J)\}.$$

(2) (Shift by Hamiltonian) For any function $\lambda(t)$ on S^1, one has

$$c_A(H + \lambda(t)) = c_A(H) + \int_{S^1} \lambda(t).$$

(3) (Lipschitz property) For any H_1, H_2, there is

$$\int_{S^1} \min_M (H_1 - H_2) \le c_A(H_1) - c_A(H_2) \le \int_{S^1} \max_M (H_1 - H_2).$$

In particular, if $H_1 \ge H_2$ then $c_A(H_1) \ge c_A(H_2)$.

(4) (Triangle inequality) $c_{A_1 * A_2}(H_1 \# H_2) \ge c_{A_1}(H_1) + c_{A_2}(H_2)$ where $*$ is the quantum product.

(5) (Homotopy invariance) $c_A(H_1) = c_A(H_2)$ for any two normalized Hamiltonian functions generating the same element in $\widetilde{\mathrm{Ham}}(M)$, the universal cover of the group of Hamiltonian diffeomorphisms on M.

Remark 2.3. The Lipschitz property allows us to define $c_A(H)$ for any continuous H(not necessarily non-degenerate) using approximations. However, the spectrality of extended spectral invariants might not hold.

Now for any idempotent $e \in \mathrm{QH}^*(M)$, by the triangle inequality, the following limit can be defined:

$$\zeta_e(H) = \lim_{k \to \infty} \frac{c_e(kH)}{k}.$$

One can check that the functional $\zeta_e : C(M) \to \mathbb{R}$ satisfies all the conditions for a partial symplectic quasi-state.

For e an idempotent in the quantum cohomology, recall that a compact subset $K \subseteq M$ is called $e-$heavy if

$$\zeta(H) \le \sup_K H \qquad (\forall H \in C(M))$$

and $e-$superheavy if

$$\zeta(H) \ge \inf_K H \qquad (\forall H \in C(M)).$$

Remark 2.4. Our sign conventions follow [Sun24], which is different to the original ones in [EP09]. However, the two definitions of (super)heaviness coincide, thanks to the duality of spectral invariants.

Remark 2.5. Every superheavy set is heavy, but not vice versa. The meridian of a 2-torus is a heavy but not superheavy subset.

It is worth mentioning that the superheaviness with respect to the identity $1 \in \mathrm{QH}^*(M)$ is special.

Proposition 2.6. [EP09, Theorem 1.3] *Let e be a non-zero idempotent in $\mathrm{QH}^*(M)$, then*

(1) Every set that is superheavy with respect to 1 is also superheavy with respect to e.
(2) Every set that is heavy with respect to e is also heavy with respect to 1.

There are some classical examples of (non-)superheavy subsets.

Example 2.7. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$ is superheavy for any $n \geq 1$, while the real part of the Fermat hypersurface

$$M = \{-Z_0^d + Z_1^d + \cdots + Z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}$$

is not superheavy for even $d \geq 4$ and $n > 2d - 3$ [EP09].

Example 2.8. The union of the meridian and longitude of a 2-torus is superheavy.

The following criterion in [EP09] will be applied to prove superheaviness. For the sake of completeness we include the proof here.

Lemma 2.9. *(cf.* [EP09, Proposition 4.1]*)*

(1) A compact set K in M is superheavy if and only if for any non-positive function H on M with $H|_K = 0$, one has $\zeta(H) = 0$.
(2) A compact set K in M is heavy if and only if for any non-negative function H on M with $H|_K = 0$, one has $\zeta(H) = 0$.

Proof. We only prove the superheavy case since we are mainly interested with it in this article.

If K is superheavy, then for any non-positive H with $H|_K = 0$, by definition one has $\zeta(H) \geq \inf_K H = 0$. On the other hand by monotonicity of partial symplectic quasi-state, $\zeta(H) \leq \zeta(0) = 0$, which yields $\zeta(H) = 0$.

Suppose conversely that for any non-positive function H on M with $H|_K = 0$, one has $\zeta(H) = 0$. For any $F \in C(M)$, let $H = \min(F - \inf_K F, 0)$, then F is non-positive with $H|_K = 0$. By assumption one has $\zeta(H) = 0$, so

$$0 \leq \zeta(H) \leq \zeta(F - \inf_K F) = \zeta(H) - \inf_K F,$$

which implies superheaviness since the function F is arbitrary. □

2.3. **Applications of (super)heaviness.** In this section we deduce some rigidity results from (super)heaviness. The first one is about the intersecting properties of superheavy subset.

Proposition 2.10. *For any superheavy subset X and heavy subset Y in M, one has $X \cap Y \neq \emptyset$.*

Proof. The proof is basically applying Lemma 2.9 repeatedly.

Assume conversely that $X \cap Y = \emptyset$, choose a non-negative function H with $H|_Y \equiv 0$ and $H|_X \equiv -1$.

Since Y is heavy, by Lemma 2.9(2), one has $\zeta(H) = 0$. Let $G = H + 1$, then by the shifting property one has $\zeta(G) = 1$.

Note that $G|_X \equiv 0$. By superheaviness and Lemma 2.9(1), we have $\zeta(-|G|) = 0$. Since superheavy sets are heavy, by Lemma 2.9(2) again, there is $\zeta(|G|) = 0$. By the monotone property one has

$$0 = \zeta(-|G|) \leq \zeta(G) \leq \zeta(|G|) = 0,$$

which contradicts to that $\zeta(G) = 1$. □

Next we introduce the definitions on displaceability.

Definition 2.11. Let (M, ω) be a closed symplectic manifold.

(1) A subset $K \subset M$ is called non-displaceable from another subset K' if for any Hamiltonian diffeomorphism φ we have $\varphi(K) \cap K' \neq \emptyset$. If K is non-displaceable from itself, then we say it is *non-displaceable*.
(2) A subset $K \subset M$ is called *stably-non-displaceable* if for any symplectomorphism(not necessarily Hamiltonian) φ, there is $\varphi(K) \cap K \neq \emptyset$.

(3) A subset K of M is called *stably non-displaceable* if K times the zero section 0_{S^1} of (T^*S^1, ω_{std}) is non-displaceable in $M \times T^*S^1$.

Displaceability is related to (super)heaviness in the following way.

Theorem 2.12. [EP09, Theorem 1.2] *In a closed symplectic manifold (M, ω), we have the following.*

(1) Every heavy set is stably non-displaceable.

(2) Every superheavy set is strongly non-displaceable.

To describe the last application of (super)heaviness, we introduce some definitions about relative symplectic cohomology.

Definition 2.13. Let K be a compact set in M and let $\mathrm{SH}^*(K; M)$ be the relative symplectic cohomology of K in M defined as in [Var21].

(1) K is called *SH-visible* if $\mathrm{SH}^*(K; M) \neq 0$, otherwise it is called SH-invisible.

(2) K is called *SH-full* if every compact set contained in $M - K$ is SH-invisible.

Mak-Sun-Varolgunes [MSV24] connected the above properties about relative symplectic cohomology to our (super)heaviness.

Theorem 2.14. [MSV24, Theorem 1.7] *Let K be a compact set in a closed symplectic manifold (M, ω), then the following holds.*

(1) K is SH-visible if and only if K is heavy in M.

(2) If K is superheavy, then K is SH-full.

3. Simple Crossings Symplectic Divisors

3.1. **Basic definitions.** We firstly give some basic relevant definitions about SC symplectic divisors, the main reference is [McL12, TMZ18].

Let (M^{2n}, ω) be a closed symplectic manifold, then $D = \cup_{i=1}^N D_i$ is a symplectic divisor in (M, ω) if

(1) $D_i \subset M$ is a connected smooth closed submanifold with real codimension two for each i,

(2) for each subset $I \subseteq [N]$ the intersection $\bigcap_{i \in I} D_i$ is transverse and

$$D_I := \bigcap_{i \in I} D_i \subset M$$

is a symplectic submanifold.

Since the D_i intersect transversally, for each $I \subset [N]$ there is an isomorphism of vector bundles

$$N_M D_I \xrightarrow{\sim} \bigoplus_{i \in I} N_M D_i|_{D_I} \tag{3.1}$$

over D_I.

Definition 3.1. A symplectic divisor $D \subseteq (M, \omega)$ is

(i) a *simple crossings divisor*(also written as SC divisor) if (3.1) is an isomorphism of oriented vector bundles for all I, where each normal bundle $N_M D_i$ is equipped with the orientation induced by the symplectic orientation on TM and TD_i.

(ii) *orthogonal* if for all $i \neq j$ and $x \in D_i \cap D_j$, the ω-normal bundle $(T_x D_i)^\omega \subset T_x M$ is contained in $T_x D_j$.

In this section we always consider orthogonal SC divisor D which satisfies Hypothesis A.

The classes $\mathrm{PD}(D_i)$ have canonical lifts $\mathrm{PD}^{rel}(D_i)$ to the relative cohomology group $H^2(M, X)$ and form a basis of $H^2(M, X; \mathbb{R})$.

Define

$$\boldsymbol{\lambda} := \sum_i \lambda_i \cdot \mathrm{PD}^{rel}(D_i) \in H^2(M, X; \mathbb{R}),$$

which is a lift of $2c_1(TM)$ by construction.

For a class $A \in H^2(M, X; \mathbb{R})$, an one-form $\theta \in \Omega^1(X)$ satisfying

$$\omega|_X = d\theta, \qquad \text{and} \qquad \int_u \omega - \int_{\partial u} \theta = \tau \boldsymbol{\lambda} \cdot u \qquad \text{for all } u \in H_2(M, X)$$

is said to have *wrapping numbers* A_i for D [McL16], where $(A_1, \ldots, A_N) \in \mathbb{R}^N$ is the image of A under Lefschetz isomorphism. By the relative de Rham isomorphism, such an one-form θ must exist.

Since $\tau\lambda_i > 0$ for all i, we may choose θ with wrapping numbers $\tau\lambda_i$ and arrange that (X, θ) is a finite type convex symplectic manifold. The details will be presented in the next section.

3.2. **Commuting Hamiltonians and adapted one-form.** Now following [McL12], we introduce the notion of system of commuting Hamiltonians. These functions provide a local model of the radius function constructed in Section 4 and satisfy certain compatible conditions at the overlaps.

Lemma 3.2. [McL12, Lemma 5.14] *If $D = \cup_{i=1}^N D_i$ is an orthogonal SC divisor in (M, ω). There exist small neighborhoods UD_i of D_i and projections $\pi_i : UD_i \to D_i$ such that*

(1) *For $1 \le i_1 < i_2 < \cdots < i_l \le N$,*

$$\pi_{i_l} \circ \cdots \circ \pi_{i_1} : \bigcap_{j=1}^{l} UD_{i_j} \to D_{\{i_1,\dots,i_l\}}$$

has fibers that are symplectomorphic to $\prod_{j=1}^{l} \mathbb{D}_\epsilon$ where $\mathbb{D}_\epsilon$ is the disk of radius ϵ.

(2) *If we look at a fiber $\prod_{j=1}^{l} \mathbb{D}_\epsilon$ of $\pi_{i_l} \circ \cdots \circ \pi_{i_1}$, then for $1 \le m \le l$, π_{i_m} maps this fiber to itself. It is equal to the natural projection*

$$\prod_{j=1}^{l} \mathbb{D}_\epsilon \to \prod_{j=1, j\neq m}^{l} \mathbb{D}_\epsilon$$

eliminating the m-th disk $\mathbb{D}_\epsilon$.

(3) *The symplectic structure on UD_i induces a natural connection for $\pi_{i_l} \circ \cdots \circ \pi_{i_1}$ given by the ω orthogonal vector bundles to the fibers. We may require the associated parallel transport maps to be elements of $U(1) \times \cdots \times U(1)$ where $U(1)$ acts on the disk $\mathbb{D}_\epsilon$ by rotation.*

The above results can lead the construction of the following system of commuting Hamiltonians, which will be crucial in our argument.

Proposition 3.3. *D is an orthogonal SC divisor in (M, ω), then for any $R > 0$, there exists a system of commuting Hamiltonians (scH) near D of radius R, which is a collection of open neighborhoods $UD_i \supset D_i$ and proper smooth functions $r_i : UD_i \to [0, R)$, for each i, with the following properties. For each i,*

- *r_i generates an S^1 action on UD_i, and $r_i^{-1}(0) = D_i$.*
- *The fixed point set of the $\mathbb{R}/\mathbb{Z}$ action on UD_i is D_i.*
- *The S^1 action on $UD_i \setminus D_i$ is free.*

For all pairs i, j,

- *$UD_i \cap UD_j$ is invariant under the S^1 action generated by r_i.*
- *The Hamiltonians r_i and r_j Poisson commute on $UD_i \cap UD_j$.*

Proof. For any $1 \leq i \leq N$, let r_i be the radial coordinate of the ω orthogonal symplectic disk bundle over UD_i, then it is direct to check that $\{r_i\}_i$ satisfies all the conditions above. ☐

Remark 3.4. Actually the converse also holds, that is, if a SC divisor D admits a scH, then it is orthogonal [BSV22, Proposition 2.5].

Let $\{r_i : UD_i \to [0, R)\}$ be a scH near an orthogonal SC divisor D. For subset $I \subset [N]$, define $UD_I = \cap_{i \in I} UD_i$, then one has a $(\mathbb{R}/\mathbb{Z})^I$ action on UD_I with a moment map

$$r_I : UD_I \to [0, R)^I$$

whose components are given by r_i, for $i \in I$.

Now we impose some condition on the one-form θ on $X = M \backslash D$.

Definition 3.5. Let D be an orthogonal SC divisor in a closed symplectic manifold (M, ω) and let $\{r_i : UD_i \to [0, R)\}$ be an admissible scH near D. We call a one-form $\theta \in \Omega^1(M \setminus D)$ *adapted* to $\{r_i : UD_i \to [0, R)\}$ with wrapping numbers κ_i, if it satisfies

(1) $d\theta = \omega|_X$ with wrapping numbers κ_i,
(2) the Liouville vector field Z of θ satisfies

$$Z(r_i) = r_i - \kappa_i$$

over $UD_i \setminus D$, for all i.

This technical condition will help translate conditions with respect to the Liouville vector field Z to the local Euler-type vector field $\tilde{Z}_I$ in Section 4.

Again the existence of adapted one-form is guaranteed.

Lemma 3.6. [McL12, Lemma 5.17] *Let D be an orthogonal SC divisor in a closed symplectic manifold (M, ω). Assume that*

$$[\omega] = \sum_i \kappa_i \cdot \mathrm{PD}(D_i) \quad \text{in } H_2(M; \mathbb{R}),$$

with $\kappa_i > 0$. Then there exists $\{r_i : UD_i \to [0, R)\}$ a scH near D for which there exists an adapted $\theta \in \Omega^1(M \setminus D)$ with wrapping numbers κ_i.

Note that by the negatively monotone condition and Hypothesis A, the above lemma applies to our setting in Theorem 1.2.

To conclude this section we show that (X, θ) admits the structure of finite type convex symplectic manifold in our setting.

Definition 3.7. (1) A *convex symplectic manifold* is a closed manifold M along with an one-form θ so that $\omega = d\theta$ is a symplectic form and there is an unbounded subset $A_M \subset \mathbb{R}$ and $f_M : M \to \mathbb{R}$ an exhausting function, satisfying the ω-dual X_θ of θ is a vector field satisfying $df_M(X_\theta) > 0$ in the region $f_M^{-1}(A_M)$.

(2) A convex symplectic manifold (M, θ) is said to be of *finite type* if $df_M(X_\theta) > 0$ in the region $[C, +\infty)$ for some C.

Proposition 3.8. *Let (M, ω) and D be as in Theorem 1.2 and θ be adapted to $\{r_i\}$ with wrapping numbers $\tau\lambda_i$, then (X, θ) has a structure of finite type convex symplectic manifold.*

Proof. See [Gat24, Proposition 4.8]. □

3.3. **Admissible charts.** To compute of action and index with respect to an orbit, one has to fix a capping. We work in the standard chart satisfying certain conditions concerning the uniqueness of cappings.

Definition 3.9. Let $D = \cup_{i=1}^N D_i$ be an SC divisor in a closed symplectic manifold (M, ω), and let $\{r_i : UD_i \to [0, R)\}$ be a scH near D.

(1) Given $I \subseteq [N]$, a *standard chart* (U, ϕ) in UD_I is an $(S^1)^I$-invariant open subset $U \subset UD_I$ and a $(S^1)^I$-equivariant symplectic embedding

$$\phi : U \to \mathbb{C}^I \times \mathbb{C}^{n-|I|},$$

where we use the action of $(S^1)^I$ on $\mathbb{C}^I \times \mathbb{C}^{n-|I|}$ given by

$$\theta \cdot ((z_i)_{i\in I}, w) = ((e^{2\pi i\theta_i} z_i)_{i\in I}, w) \text{ for all } \theta \in (S^1)^I \text{ and } ((z_i)_{i\in I}, w) \in \mathbb{C}^I \times \mathbb{C}^{n-|I|}.$$

(2) Fix an arbitrary Riemannian metric on M, then a standard chart (U, φ) is *admissible* if U is contractible and has diameter less than $\frac{1}{2}\mathrm{inj}(M)$, where $\mathrm{inj}(M)$ is the injectivity radius of M with respect to a chosen Riemannian metric.

(3) We call a scH near D *admissible* if for every $I \subseteq [N]$ and $y \in UD_I$, there exists an admissible standard chart (U, ϕ) in UD_I with $y \in U$.

Using equivariant Darboux theorem [GS84], for any $I \subseteq [N]$ and $x \in D_I$, there is a standard chart in UD_I containing x. And any sufficiently small shrinking of the scH is admissible.

The following is a direct result from the definition of admissibility.

Proposition 3.10. [BSV22, Lemma 2.14] *If γ is contained in an admissible standard chart, then there is a capping of γ with image contained inside an admissible chart. Moreover, such a capping is independent of the choice of admissible chart, up to homotopy rel. boundary in M.*

Proof. Since admissible charts are contractible, the existence is trivial.

As for uniqueness, for caps in two admissible charts containing γ, since the union of these two charts has a diameter less than $\mathrm{inj}(M)$, they are contained in a ball of radius less than $\mathrm{inj}(M)$. Then the two caps are homotopic rel. boundary in M since the ball is contractible. □

So there is a well-defined canonical cap of an orbit contained in an admissible chart.

4. Radius-type Hamiltonians and its orbits

4.1. **Construction of the radius.** The main goal of this section is to construct certain well-behaved function ρ^R, which will serve as a radius and help us define useful special Hamiltonians out of it for our arguments. The construction is due to [BSV22].

First we recall our geometric settings. On a negatively monotone closed symplectic manifold (M, ω), $D = \cup_{i=1}^N D_i \subseteq (M, \omega)$ is an orthogonal SC divisor and negative rational numbers $\lambda_1, \ldots, \lambda_N$ are the weights. We write $X = M \backslash D$ as the divisor complement. Since D is orthogonal we can choose an admissible system of commuting Hamiltonians $\{r_i : UD_i \to [0, R_0)\}$ near D.

We consider one-form $\theta \in \Omega^1(X)$ such that $d\theta = \omega|_X$, and the relative de Rham cohomology class of (ω, θ) is $\tau \boldsymbol{\lambda}$, where $\boldsymbol{\lambda} \in \mathbb{R}^N \cong H^2(M, X; \mathbb{R})$ is the lift of $2c_1(TM)$. By Lemma 3.6, we assume that θ is adapted to $\{r_i : UD_i \to [0, R_0)\}$. By shrinking the scH if necessary, we assume that $R_0 < \tau\lambda_i (1 \leq i \leq N)$.

Our target is to define a family of functions ρ^R(where R serves as some type of smoothing parameter) that satisfy the following conditions:

(1) ρ^R is continuous on M, and smooth on the complement of the skeleton L;
(2) $\rho^R|_L = 0$ and $\rho^R|_D \approx 1$;
(3) we have $Z(\rho^R) = \rho^R$ on $X \setminus L$, where Z is the Liouville vector field on (X, θ);
(4) $\rho^R \to \rho^0$ as $R \to 0$, where we recall that $\rho^0 : M \to \mathbb{R}$ is the function extended from X, so that the Liouville flow starting at x is exactly defined for time $t \in (-\infty, -\log(\rho^0(x)))$.

For any $R > 0$ and $\sigma \in (0,1)$, set $K^R_\sigma = \{\rho^R \leq \sigma\}$. Then from the condition $Z(\rho^R) = \rho^R$ one has that

$$K^R_\sigma \to L, \text{ when } \sigma \to 0.$$

We firstly give a sketchy outline of the construction. The idea is to use the scH to reduce the problem locally to the Euclidean space(might with corner). There are three tasks to finish.

(1) Find a good open cover of $M \backslash L$ so that on each open subset in the cover our scH is defined.
(2) Translate the conditions imposed on ρ^R to the functions in the local model.
(3) Construct our desired functions locally.

Now we consider the vector field $\tilde{Z}_I$ on $\mathbb{R}^I$ defined by

$$\left(\tilde{Z}_I\right)_r := \sum_{i \in I} (r_i - \tau\lambda_i) \frac{\partial}{\partial r_i}.$$

Let UD_i^{max} be the union as the union of UD_i with the set of points in X that enter into UD_i under the positive Liouville flow in finite time, then we can extend r_i to

$$r_i^{max} : UD_i^{max} \to \mathbb{R}_{\geq 0}$$

by first flowing into UD_i with the Liouville flow in some time $T \geq 0$, applying r_i, and then flowing with $\tilde{Z}_{\{i\}}$ for time $-T$.

It is natural to define

$$UD_I^{max} := \cap_{i \in I} UD_i^{max}$$

and

$$r_I^{max} : UD_I^{max} \to \prod_{i \in I} [0, \tau\lambda_i).$$

We claim first that the Liouville-invariant condition is well behaved under local reducing. A simple computation yields $(r_I)_* Z_x = \left(\tilde{Z}_I\right)_{r_I(x)}$ for any $x \in UD_I \setminus D$, where $\mathrm{pr}_I : \mathbb{R}^N \to \mathbb{R}^I$ is the standard projection. So we have the following.

Lemma 4.1. *The function $\rho^R := \tilde{\rho}_I^R \circ r_I^{max}$ satisfies $Z(\rho^R) = \rho^R$ if and only if $\tilde{Z}_I(\tilde{\rho}_I^R) = \tilde{\rho}_I^R$.*

Remark 4.2. Note that a function

$$f : V_I := \mathbb{R}^I \setminus \prod_{i \in I} [\tau\lambda_i, \infty) \to \mathbb{R}$$

satisfies $\tilde{Z}_I(f) = f$ if and only if it is linear along the rays emanating from $\tau\boldsymbol{\lambda}_I$, converging to 0 at that point.

Now we modify UD_I^{max} to obtain a good open covering of $M \backslash L$. Let

$$UD_i^{1/2} := \{r_i \le R/2\} \subseteq UD_i = \{r_i < R\}$$

and define $UD_i^{1/2,max}$ to be the union of $UD_i^{1/2}$ with the set of points in X that enter into $UD_i^{1/2}$ under the positive Liouville flow in finite time. For $I \subseteq [N]$, let

$$\mathring{U}D_I^{max} := UD_I^{max} \setminus \bigcup_{j \notin I} UD_j^{1/2,max}.$$

Lemma 4.3. *The sets $\left\{\mathring{U}D_I^{max}\right\}_{\emptyset \neq I \subseteq [N]}$ form an open cover of $M \setminus L$.*

Now it suffices to construct $\tilde{\rho}_I^R$. We briefly elaborate the process here. Recall that we hope that ρ^R is a smoothing of ρ^0. On UD_I^{max}, we have

$$\rho^0 = \tilde{\rho}_I^0 \circ r_I^{max},$$

where

$$\tilde{\rho}_I^0(r) = \max_{i \in I} \frac{\tau\lambda_i - r_i}{\tau\lambda_i}.$$

Note that $\tilde{\rho}_I^0$ is equal to 0 at $\tau\lambda_I$ and linear along the rays emanating from this point. Besides it is equal to 1 along $\partial\mathbb{R}^I_{\ge 0}$.

Since functions $\tilde{\rho}_I^R$ should be chosen to be smoothings of the functions $\tilde{\rho}_I^0$. We desire to construct a hypersurface

$$\tilde{Y}_I^R \subset V_I \cap \mathbb{R}^I_{\ge 0}$$

which is a smoothing of $\tilde{Y}_I^0 := \partial\mathbb{R}^I_{\geq 0}$, satisfying certain properties. Then, we can take $\tilde{\rho}_I^R$ to the function that converges to zero at $\tau\boldsymbol{\lambda}_I$, being linear along the rays emanating from it, and takes the value 1 on $\tilde{Y}_I^R$.

The desired family of hypersurfaces is constructed as follow.

Lemma 4.4. *For any $R_0 > R > 0$, let $q^R : \mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$(q^R)'(r) < 0 \textit{ for } r < R/2$$

with $q^R(0) = 1$ and $q^R(r) \equiv 0$ for $r \geq R/2$.

Let

$$Q_I^R : \mathbb{R}^I \to \mathbb{R}, Q_I^R(r) = \sum_{i \in I} q^R(r_i)$$

and

$$\tilde{Y}_I^R := \{Q_I^R = 1\}$$

Then there is a unique smooth function $\tilde{\rho}_I^R : V_I \to \mathbb{R}$ satisfying

$$\tilde{\rho}_I^R|_{\tilde{Y}_I^R} = 1 \quad \textit{and} \quad \tilde{Z}_I(\tilde{\rho}_I^R) = \tilde{\rho}_I^R,$$

satisfying the followings.

(1) For any $\emptyset \neq I, J \subseteq [N]$, there is

$$\tilde{\rho}_I^R \circ r_I^{max} = \tilde{\rho}_J^R \circ r_J^{max} \quad \textit{over} \quad \mathring{U}D_I^{max} \cap \mathring{U}D_J^{max}.$$

(2) We have

$$\tilde{\rho}_I^R|_{\tilde{Y}_I^R} = 1 \quad \textit{and} \quad \tilde{Z}_I(\tilde{\rho}_I^R) = \tilde{\rho}_I^R.$$

(3) $\tilde{\rho}_I^R$ is linear along the rays emanating from $\tau\boldsymbol{\lambda}_I$, converging to zero at $\tau\boldsymbol{\lambda}_I$.

Proof. By [BSV22, Lemma 4.8], hypersurfaces $\tilde{Y}_I^R \subset \mathbb{R}^I$ with the following properties:

(1) $\tilde{Y}_I^R$ is contained in the region $V_{I,\geq 0} := V_I \cap \mathbb{R}^I_{\geq 0}$.
(2) Every flowline of $\tilde{Z}_I$ in V_I crosses $\tilde{Y}_I^R$ transversely at a unique point.
(3) If $\tilde{\nu}_I^R : \tilde{Y}_I^R \to \mathbb{R}^I$ is a normal vector field (pointing towards the component containing $\tau\boldsymbol{\lambda}$), then $\tilde{\nu}_{I,i}^R \geq 0$ for all i, where $\tilde{\nu}_{I,i}^R$ is the ith component of $\tilde{\nu}_I^R$.
(4) For any $J \subseteq I$, $\tilde{Y}_I^R$ coincides with $\tilde{Y}_J^R \times \mathbb{R}^{I \setminus J}$ over the region $\cap_{i \in I \setminus J}\{\tilde{\nu}_{I,i}^R = 0\}$.
(5) Let $P_I : V_I \to \tilde{Y}_I^0$ be the projection-from-$\tau\boldsymbol{\lambda}_I$ map, then

$$P_I^{-1}\left(\{r_i > R/2\}\right) \subseteq \{\tilde{\nu}_{I,i}^R = 0\}$$

Then one can check that our ρ^R satisfies all the desired conditions following the arguments in [BSV22, Section 4.5]. ∎

Then there is a well-defined function ρ^R so that on each $\rho^R = \mathring{U}D_I^{max}$ with $\rho^R \equiv 0$ on L, and ρ^R is the "radius" that we need.

4.2. **Action and index of canonically capped orbits.** Let $\rho = \rho^R$ where R is a sufficiently small positive number, and $h : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

(1) $h'(r) > 0$ and $h''(r) \leq 0$,
(2) $h(r) = \lambda r + C$ when $r \in [0, 1/2)$ for some non-integer λ and constant C,

then $h \circ \rho$ is a continuous function on M and smooth away from L.

To study the dynamic property of the above function $h \circ r$, we first compute the differential of ρ.

Proposition 4.5. [BSV22, Lemma 4.14] *There are smooth functions $w_i : M \backslash L \to \mathbb{R}_{\geq 0}$ supported in $\mathring{U}D_i^{max}$ satisfying*

$$d\rho = -\sum_i w_i \cdot dr_i^{max}. \tag{4.1}$$

We write $w : M\backslash L \to \mathbb{R}^N$ be the smooth map defined by $w = (w_1, \ldots, w_N)$.

There is a smooth extension of the map $h'(\rho) \cdot w : M\backslash L \to \mathbb{R}^N$ to M, we write the extension as $w_h = (w_{h,1}, \ldots, w_{h,N})$. A key observation is that on any orbit γ of $h \circ \rho$, w_h is constant, thus a vector $w_h(\gamma) \in \mathbb{R}^N$ is well-defined.

Remark 4.6. Each coordinate $w_{h,i}(\gamma)$ of $w_h(\gamma)$ can be regarded as a wrapping number of γ around the hypersurface D_i. Also note that if $h'(\rho) < 0$, then $w_{h,i}(\gamma) \leq 0$.

For each orbit γ, we define a subset of $[N]$ by

$$I(\gamma) = \{i \mid w_{h,i}(\gamma) \neq 0\}.$$

For any orbit γ of $h \circ r$, we give a canonical capping of γ in the sense of Section 3.3. The precise description is as follow.

(1) If γ is constant, then let u_{can} be the constant capping.
(2) If γ is contained in $UD_{I(\gamma)}$ then it is contained in an admissible standard chart, and we define u_{can} to be the canonical cap contained in that chart given by Proposition 3.10.
(3) If γ does not fall into the above two cases, then it must be not contained in D. Then we define u_{can} to be the union of the cylinder swept by γ along the Liouville flow taking it into $UD_{I(\gamma)}$, with the canonical cap in an admissible chart.

First we need to compute the action and index of the canonically capped orbits in order to estimate the spectral invariants latter.

Theorem 4.7. *For the canonically capped orbit (γ, u_{can}) of $h \circ \rho$, one has*

(1) The action is

$$\mathcal{A}_{h\circ\rho}(\gamma, u_{can}) = h(\rho(\gamma)) + h'(\rho(\gamma)) + \omega_h(\gamma) \cdot \boldsymbol{\lambda}. \tag{4.2}$$

(2) Let $J := \{j \in I(\gamma) : r_j^{max}(\gamma) \neq 0\}$. Define the $|J| \times |J|$ matrix

$$\text{Hess}_\gamma := \left(\frac{\partial^2 (h \circ \tilde{\rho}_I)}{\partial r_i \partial r_j} (r_I(\gamma)) \right)_{i,j \in J},$$

then the Conley-Zehnder index is

$$\text{CZ}(\gamma, u_{can}) = 2 \sum_i \lceil w_{h,i}(\gamma) \rceil + \frac{1}{2} \text{sign}\left(\text{Hess}_\gamma\right). \tag{4.3}$$

Proof. (1) Without loss of generality we let γ be contained in $\mathring{U}D_I^{max}$ for some $I \subseteq [N]$. Then u_{can} can be decomposed into two parts $u_{can,i} (i = 1, 2)$, where $u_{can,1}$ lies in an admissible chart in UD_I, and $u_{can,2} = \bigcup_{t\in[0,T]} \varphi_t(\gamma)$ is swept out by the Liouville flow.

Assume that the boundary of $u_{can,1}$ is contained in $r_I^{-1}((a_i)_{i\in I})$. We consider the embedding of the admissible chart into $\mathbb{C}^I \times \mathbb{C}^{n-|I|}$ and let M be the fiber over $(a_i)_{i\in I}$ with respect to the moment map

$$\mu : \mathbb{C}^I \times \mathbb{C}^{n-|I|} \to \mathbb{R}^I$$

Then $\int_{u_{can,1}} \omega$ is equal to the symplectic area of any cap of a 1-periodic orbit of $X_{\tilde{f}}$ contained inside M, where $\tilde{f} = f \circ r_I^{max}$ and

$$f : \mathbb{R}^I \to \mathbb{R}, r \mapsto \sum_{i\in I} -w_{h,i}(\gamma) \cdot r_i.$$

Then a direct computation yields

$$\int_{u_{can,1}} \omega = \sum_{i\in I} w_{h,i}(\gamma) \cdot a_i.$$

And it is from [BSV22, Lemma 4.19] that

$$\int_{u_{can,2}} \omega = \sum_{i\in I} w_{h,i}(\gamma) \cdot (r_i^{max}(\gamma) - a_i).$$

So in summarize we have shown that

$$\omega(u_{can}) = \sum_i w_{h,i}(\gamma) \cdot r_i^{max}(\gamma).$$

This along with the fact that

$$\mathcal{A}_{h\circ\rho}(\gamma, u_{can}) = h(\rho(\gamma)) + \omega(u_{can})$$

yield

$$\mathcal{A}_{h\circ\rho}(\gamma, u_{can}) = h(\rho(\gamma)) + \sum_i w_{h,i}(\gamma) \cdot r_i^{max}(\gamma).$$

Then result is obtained as follow.

$$\begin{aligned}
\mathcal{A}_{h\circ\rho}(\gamma, u_{can}) - w_h(\gamma) \cdot \boldsymbol{\lambda} &= h(\rho(\gamma)) + \sum_i w_{h,i}(\gamma) \cdot r_i^{max}(\gamma) - w_h(\gamma) \cdot \boldsymbol{\lambda} \\
&= h(\rho(\gamma)) + h'(\rho(\gamma)) \sum_i w_i(\gamma) \cdot (r_i(\gamma) - \lambda_i) \\
&= h(\rho(\gamma)) + h'(\rho(\gamma)) \cdot \tilde{Z}_I \left(\tilde{\rho}_I\right)_{r_I^{max}(\gamma)} \\
&= h(\rho(\gamma)) + h'(\rho(\gamma)) \cdot \rho(\gamma).
\end{aligned}$$

Recall that by our construction there is $\tilde{\rho}_I \circ r_I^{max} = \rho$ and $\tilde{\rho}_I$ is invariant under $\tilde{Z}_I$.
(2) The result is due to [BSV22, Lemma 4.22], using the computation techniques in [Oan04, Section 3].

□

Remark 4.8. Given (γ, u_{can}), we define its associated fractional cap(also called inner cap therein) by

$$u_{can}^{'} = u_{can} - w_h(\gamma) \cdot \boldsymbol{\lambda}.$$

Then there is

$$\mathcal{A}_{h\circ\rho}(\gamma, u_{can}^{'}) = h(\rho(\gamma)) + h'(\rho(\gamma)) \cdot \rho(\gamma).$$

The above computation result of the action can be seen as a generalization of Viterbo's y-intersection formula [Vit99].

5. Proof of Main Results

5.1. **Proof of Theorem 1.2.** Now we give the proof of Theorem 1.2.

We fix a smoothing parameter R throughout the proof and omit it from the notations.

By our construction of ρ, it suffices to show that $\{\rho \leq a\}$ is superheavy for any $a \in (0, \frac{1}{2})$.

Let $b = a + \varepsilon$ where $a > 0$ and $\varepsilon > 0$ is sufficiently small and choose any non-positive function H on M that is zero on $\{\rho \leq b\}$, by Lemma 2.9, it suffices to show that $\zeta(H) = 0$. Our goal is to show that there is some constant S so that

$$c(1, kH) \leq S \text{ for any } k \in \mathbb{N}.$$

Consider any smooth function g on M with $g'(\rho) < 0$. Recall $w_i : M \backslash L \to \mathbb{R}_{\geq 0}$ defined as in (4.1) is a smooth function and $w_{g,i}(\gamma) = w_i(\gamma) \cdot g'(\rho)$.

For any orbit γ of $g \circ \rho$ contained in $\{b - \frac{\varepsilon}{2} \leq \rho \leq 1\}$, since γ is not contained in some D_i, by [BSV22, Lemma 4.16], $w_{g,i}(\gamma)$ is a non-zero integer. Since $g'(\rho) < 0$ and $w_i \in \mathbb{R}_{\geq 0}$, there is $w_{g,i}(\gamma) \leq -1$.

By the definition of $w_{g,i}$, there is a number $\kappa > 0$ so that g is linear in the region $\{b - \frac{\varepsilon}{2} \leq \rho \leq 1\}$, if the absolute value of the slope of g in this region is larger than κ, one has

$$w_{g,i}(\gamma) \leq -2n.$$

Fix a smooth function f on M so that

(1) f only depends on ρ when $\rho \in (a, 1)$ with $f'(\rho) < 0, f''(\rho) \geq 0$.
(2) f is a negative C^2-small Morse function when $\rho \leq a$.
(3) f is linear in ρ when $\rho \in (b - \varepsilon/2, 1)$, with a slope in $(-\kappa - 1, -\kappa)$.

We also choose g with

(1) g only depends on ρ when $\rho \in (a, 1)$ with $g'(\rho) < 0, g''(\rho) \geq 0$.
(2) $g = f$ when $\rho \leq b - \frac{\varepsilon}{2}$.
(3) g is linear in ρ when $\rho \in (b, 1)$, with a slope in $(-\kappa - 2, -\kappa - 1)$.

By [BSV22, Lemma 4.17], there is a non-degenerate perturbation F of $f \circ \rho$, such that for any capped orbit (γ, u) of $f \circ \rho$, there is a capped orbit $(\bar{\gamma}, \bar{u})$ of F so that

(1) $|\operatorname{CZ}(\gamma, u) - \operatorname{CZ}(\bar{\gamma}, \bar{u})| \leq n$.
(2) $|\mathcal{A}_F(\bar{\gamma}, \bar{u}) - \mathcal{A}_{f \circ \rho}(\gamma, u)|$ is sufficiently small.

Similarly, there is a non-degenerate perturbation G of $g \circ \rho$ satisfying the properties above. Moreover, we can require that

$$F = G \text{ for } \rho \in (0, b - \frac{\varepsilon}{2}).$$

Then we only need to consider the orbits in the area $\{b - \frac{\varepsilon}{2} \leq \rho \leq 1\}$.

For any orbit γ of G in the above area, there is an orbit $\bar{\gamma}$ of $g \circ \rho$ in the same region. If $\bar{\gamma}$ is non-constant, then $\rho(\bar{\gamma})$ is a constant $\rho_0 \in (b - \frac{\varepsilon}{2}, b)$, thus by the action formula (4.2), one has

$$\begin{aligned}\mathcal{A}_{g\circ\rho}([\bar{\gamma}, u_{can}]) &= g(\rho_0) + \sum_i \omega_{g,i}(\gamma) \cdot r_i^{max}(\gamma) \\ &= g(\rho_0) + g'(\rho_0)\rho_0 + \omega_g(\gamma) \cdot \boldsymbol{\lambda} \\ &< g(\rho_0) + g'(\rho_0)\rho_0 \\ &\leq f(\rho_0) + f'(\rho_0)\rho_0 \\ &= f(2\rho_0) < f(2b - \varepsilon).\end{aligned}$$

And the index is computed as in (4.3)

$$\begin{aligned}\mathrm{CZ}([\bar{\gamma}, u_{can}]) &= 2\sum_i \lceil w_{g,i}(\gamma) \rceil + \frac{1}{2}\mathrm{sign}\,(\mathrm{Hess}_\gamma) \\ &\leq \sum_i 2w_{g,i}(\gamma) + n \\ &\leq -3n.\end{aligned}$$

Then by the condition on the perturbations, there is a capping u of γ with

$$\mathcal{A}_G([\gamma, u]) \leq f(2b - \varepsilon)$$

and

$$\begin{aligned}\deg([\gamma, u]) &= \mathrm{CZ}([\gamma, u]) + n \\ &\leq \mathrm{CZ}([\bar{\gamma}, u_{can}]) + 2n \leq -n.\end{aligned} \tag{5.1}$$

If $\bar{\gamma}$ is constant, the above result is trivial since $f(\rho_0) > g(\rho_0)$.

Now since (M, ω) is negatively monotone, the group

$$\{\omega(A) \mid A \in \pi_2(M)\} \subset \mathbb{R}$$

is discrete and we write the positive generator of this group as ε_0.

Then for any capping $u + A$ of γ with $\deg([\gamma, u + A]) = 0$, by (5.1),

$$c_1(A) = -\frac{1}{2}\big(\deg([\gamma, u]) - \deg([\gamma, u + A])\big) > 0.$$

So

$$\omega(A) = 2\tau c_1(A) < 0,$$

hence by the definition of ε_0, it follows

$$\mathcal{A}_G([\gamma, u + A]) \leq f(2b - \varepsilon) - \varepsilon_0.$$

To conclude, we have obtained that, for some $0 < \varepsilon < \varepsilon_0$, the following equality of two discrete subset of $\mathbb{R}$

$$\mathrm{Spec}_0(G) \cap (f(2b - \varepsilon) - \varepsilon_0, +\infty) = \mathrm{Spec}_0(F) \cap (f(2b - \varepsilon) - \varepsilon_0, +\infty)$$

Now using the exactly same arguments as above, we can construct a monotone sequence

$$G = G_0 \leq G_1 \leq \cdots \leq G_l = F$$

such that

(1) For any i, G_i is non-degenerate.
(2) With respect to any capping, any orbit of G_i in $\{b - \frac{\varepsilon}{2} \leq \rho \leq 1\}$ has degree smaller than $-n$.
(3) With respect to any capping, any orbit of G_i in $\{b - \frac{\varepsilon}{2} \leq \rho \leq 1\}$ has action smaller than $f(2b - \varepsilon)$.
(4) For any i, there is

$$\int_{S^1} \max_M (G_{i+1} - G_i) < \frac{\varepsilon'}{3},$$

where ε' is chosen(by the spectrality of spectral invariants and rational conditions) so that

$$(c(1, F) - \varepsilon', c(1, F) + \varepsilon') \cap \mathrm{Spec}_0(G) \cap (f(2b - \varepsilon) - \varepsilon', +\infty) = \{c(1, F)\}.$$

Clearly G_i is independent of i on the region where $F = G$. By the Lipschitz property of spectral invariants and (4), we have

$$|c(1, G_{l-1}) - c(1, G_l)| < \frac{\varepsilon'}{3}.$$

Then by the choice of ε' one has

$$c(1, G_{l-1}) = c(1, G_l) = c(1, F).$$

Combining conditions (2),(3) and the same computation as for G in above, one has that

$$\mathrm{Spec}_0(G_i) \cap (f(2b - \varepsilon) - \varepsilon', +\infty)$$

is independent of i. So by repeating this process for i inductively, we obtain that $c(1, G_i)$ is independent of i, which particularly yields $c(1, F) = c(1, G)$.

Using induction one has that $c(1, G^{(m)}) = c(1, F)$ for any $G^{(m)}$ defined in the above fashion with slope of the linear part lies in $(-\kappa - m, -\kappa - m + 1)$.

Finally, choose any non-positive smooth function H on M that is zero on $\{\rho \leq b\}$, for any positive integer k, there exists some m such that $G^{(m)} \leq kH$. Then by the monotone property of spectral invariant, $c(1, kH)$ is uniformly bounded in k, which implies that $\zeta(H) = 0$, so the result follows. □

References

[AMS21] Mohammed Abouzaid, Mark McLean, and Ivan Smith, *Complex cobordism, Hamiltonian loops and global Kuranishi charts*, arXiv preprint 2110.14320 (2021).

[BC02] Paul Biran and Kai Cieliebak, *Lagrangian embeddings into subcritical Stein manifolds*, Israel J. Math. **127** (2002), 221–244. MR 1900700

[BSV22] Matthew Strom Borman, Nick Sheridan, and Umut Varolgunes, *Quantum cohomology as a deformation of symplectic cohomology*, J. Fixed Point Theory Appl. **24** (2022), no. 2, Paper No. 48, 77. MR 4437454

[BX22] Shaoyun Bai and Guangbo Xu, *Arnold conjecture over integers*, arXiv preprint 2209.08599 (2022).

[CZ83] C. C. Conley and E. Zehnder, *The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold*, Invent. Math. **73** (1983), no. 1, 33–49. MR 707347

[DGPZ24] Adi Dickstein, Yaniv Ganor, Leonid Polterovich, and Frol Zapolsky, *Symplectic topology and ideal-valued measures*, Selecta Math. (N.S.) **30** (2024), no. 5, Paper No. 88, 92. MR 4808384

[EP06] Michael Entov and Leonid Polterovich, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), no. 1, 75–99. MR 2208798

[EP09] ———, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826. MR 2507748

[Flo88] Andreas Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), no. 6, 775–813. MR 948771

[FO99] Kenji Fukaya and Kaoru Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048. MR 1688434

[FOOO19] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Spectral invariants with bulk, quasi-morphisms and Lagrangian Floer theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1254, x+266. MR 3986938

[FZ25] Qi Feng and Jun Zhang, *Spectrally-large scale geometry via set-heaviness*, arXiv preprint 2503.14961v2 (2025).

[Gat24] Elliot Gathercole, *Superheavy skeleta for non-normal crossings divisors*, arXiv preprint 2408.13187 (2024).

[GS84] Victor Guillemin and Shlomo Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984. MR 770935

[KO21] Morimichi Kawasaki and Ryuma Orita, *Disjoint superheavy subsets and fragmentation norms*, J. Topol. Anal. **13** (2021), no. 2, 443–468. MR 4284615

[LT98] Gang Liu and Gang Tian, *Floer homology and Arnold conjecture*, J. Differential Geom. **49** (1998), no. 1, 1–74. MR 1642105

[McL12] Mark McLean, *The growth rate of symplectic homology and affine varieties*, Geom. Funct. Anal. **22** (2012), no. 2, 369–442. MR 2929069

[McL16] ———, *Reeb orbits and the minimal discrepancy of an isolated singularity*, Invent. Math. **204** (2016), no. 2, 505–594. MR 3489704

[MS12] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, second ed., American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2012. MR 2954391

[MSV24] Cheuk Yu Mak, Yuhan Sun, and Umut Varolgunes, *A characterization of heaviness in terms of relative symplectic cohomology*, J. Topol. **17** (2024), no. 1, Paper No. e12327, 26. MR 4821222

[Oan04] Alexandru Oancea, *A survey of Floer homology for manifolds with contact type boundary or symplectic homology*, Symplectic geometry and Floer homology. A survey of the Floer homology for manifolds with contact type boundary or symplectic homology, Ensaios Mat., vol. 7, Soc. Brasil. Mat., Rio de Janeiro, 2004, pp. 51–91. MR 2100955

[Par16] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. **20** (2016), no. 2, 779–1034. MR 3493097

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and symplectic geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge Univ. Press, Cambridge, 1996, pp. 171–200. MR 1432464

[Rez22] Semon Rezchikov, *Integral Arnold conjecture*, arXiv preprint 2209.08599 (2022).

[RSTZ14] Helge Ruddat, Nicolò Sibilla, David Treumann, and Eric Zaslow, *Skeleta of affine hypersurfaces*, Geom. Topol. **18** (2014), no. 3, 1343–1395. MR 3228454

[Sun24] Yuhan Sun, *Spectral diameter of negatively monotone manifolds*, arXiv preprint 2410.21416v2 (2024).

[TMZ18] Mohammad F. Tehrani, Mark McLean, and Aleksey Zinger, *Normal crossings singularities for symplectic topology*, Adv. Math. **339** (2018), 672–748. MR 3866910

[TV23] Dmitry Tonkonog and Umut Varolgunes, *Super-rigidity of certain skeleta using relative symplectic cohomology*, J. Topol. Anal. **15** (2023), no. 1, 57–105. MR 4548628

[Var21] Umut Varolgunes, *Mayer-Vietoris property for relative symplectic cohomology*, Geom. Topol. **25** (2021), no. 2, 547–642. MR 4251433

[Vit99] C. Viterbo, *Functors and computations in Floer homology with applications. I*, Geom. Funct. Anal. **9** (1999), no. 5, 985–1033. MR 1726235

[Zho20] Peng Zhou, *Lagrangian skeleta of hypersurfaces in* $(\mathbb{C}^*)^n$, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 26, 33. MR 4087022

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